ON GENERALIZATION OF DIVISION NEAR-RINGS

Sanja Jancic Rasovic*
Department of Mathematics
Faculty of Natural Science and Mathematics
University of Montenegro
Podgorica, Montenegro
sabu@t-com.me

Vucic Dasic
Department of Mathematics
Faculty of Natural Science and Mathematics
University of Montenegro
Podgorica, Montenegro
mihajlo0107@gmail.com

Abstract. In this paper we introduce the class of D-division near-rings as a subclass of near-rings with a defect D and that one of division near-rings. We introduce the notion of D-division near-ring and we state necessary and sufficient condition under which a near-ring with defect of distributivity D is a D-division near-ring.
Keywords: near-ring, division near-ring.

1. Introduction and preliminaries

The interest in near-rings and near-fields started at the beginning of the 20^{th} century when Dickson wanted to know if the list of axioms for skew fields id redundant. He found in [3] that there do exist "near-fields" which fulfill all axioms for skew fields except one distributive law. Since 1950, the theory of near-rings had applications to several domains, for instance in area of dynamical systems, graphs, homological algebra, universal algebra, category theory, geometry and so on.

A comprehensive review of the theory of near-rings and its applications appears in Pilz [10], Meldrun [8], Clay [1], Wahling [14], Scot [12], Ferrero [4], Vukovic [13], and Satyanarayana and Prasad [11].

Let $(R, +, \cdot)$ be a left near-ring, i.e. $(R, +)$ is a group (not necessarily commutative) with the unit element 0, $(R, \cdot)$ is a semigroup and the left distributivity holds: $x \cdot (y + z) = x \cdot y + x \cdot z$ for any $x, y, z \in R$. It is clear that $x \cdot 0 = 0$, for any $x \in R$, while it might exists $y \in R$ such that $0 \cdot y \neq 0$. If 0 is a bilaterally absorbing element, that is $0 \cdot x = x \cdot 0 = 0$, for any $x \in R$, then R is called a zero-symmetric near-ring. Obviously, if $(R, +, \cdot)$ is a left near-ring then $x \cdot (-y) = -(xy)$ for any $x, y \in R$.

*. Corresponding author

A normal subgroup I of $(R, +)$ is called an ideal of a near-ring $(R, +, \cdot)$ if:
1) $RI = \{r \cdot i | r \in R, \quad i \in I\} \subseteq I$.
2) $(r + i)r' - r \cdot r' \in I$, for all $r, r' \in R$ and $i \in I$.

Obviously, if I is an ideal of zero-symmetric near-ring R, then $IR \subseteq I$ and $RI \subseteq I$. In particular, if $(R, +, \cdot)$ is a left near-ring that contains a multiplicative semigroup S, whose elements generate $(R, +)$ and satisfy $(x+y) \cdot s = x \cdot s + y \cdot s$, for all $x, y \in R$ and $s \in S$, then we say that R is a distributively generated near-ring (d.g. near-ring). Regarding the classical example of a near-ring, that one represented by the set of the functions from an additive group G into itself with the pointwise addition and the natural composition of functions, if S is the multiplicative semigroup of the endomorphisms of G and R' is the subnear-ring generated by S, then R' is a d.g. near-ring. Other examples of d.g. near-rings may be found in [5]. A near-ring containing more than one element is called a division near-ring, if the set $R \backslash \{0\}$ is a multiplicative group [7]. Several examples of division near-rings are given in [5]. It is well known that every division ring is a division near-ring, while there are division nearrings which are not division rings.

Ligh [7] give necessary and sufficient condition for a d.g. near-ring to be a division ring.

Lemma 1.1 ([7]). *If R is a d.g. near-ring, then $0 \cdot x = 0$, for all $x \in R$.*

Theorem 1.1 ([7]). *A necessary and sufficient condition for a d.g. near-ring with more than one element to be division ring is that for all non-zero elements $a \in R$, it holds $a \cdot R = R$.*

Lemma 1.2 ([7]). *The additive group $(R, +)$ of a division near-ring R is abelian.*

Another example of division ring is given by the following result.

Lemma 1.3. *Every d.g. division near-ring R is a division ring.*

Proof. By Lemma 1.2, the additive group $(R, +)$ of a division near-ring is abelian. It follows ([5], p.93) that every element of R is right distributive, i.e. $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$. Thereby, if R is d.g. near-ring, then R is a division near-ring if and only if R is a division ring. $\square$

In [2] Dasic introduced the notion of a near-ring with defect of distributivity as a generalization of d.g. near-ring.

Definition 1.1 ([2]). *Let R be a zero-symmetric (left) near-ring. A set S of generators of R is a multiplicative subsemigroup $(S, \cdot)$ of the semigroup $(R, \cdot)$, whose elements generate $(R, +)$. The normal subgroup D of the group $(R, +)$ which is generated by the set $D_S = \{d \in R | d = -(x \cdot s + y \cdot s) + (x + y) \cdot s, x, y \in R, s \in S\}$ is called the defect of distributivity of the near-ring R.*

In other words, if $s \in S$, then for all $x, y \in R$, there exists $d \in D$ such that $(x + y) \cdot s = x \cdot s + y \cdot s + d$. This expresses the fact that the elements of S are distributive with the defect D. When we want to stress the set S of generators, we will denote the near-ring by the couple (R, S). In particular, if $D = 0$, then R is a distributively generated near-ring. The following lemma is easy to verify.

Lemma 1.4. *Let (R, S) be a near-ring with the defect D.*

i) If $s \in S$ and $x \in R$, then there exists $d \in D$ such that $(-x)s = -(xs) + d$.

ii) If $s \in S$, and $x, y \in R$,then there exists $d \in D$ such that that $(x - y) \cdot s = x \cdot s - y \cdot s + d$.

The main properties of this kind of near-rings are summarized in the following results [2].

Theorem 1.2. *i) Every homomorphic image of a near-ring with the defect D is a near-ring with the defect $f(D)$, when f is a homomorphism of near-rings.*

ii) Every direct sum of a family of near-rings R_i with the defects D_i, respectively, is a near-ring whose defect is a direct sum of the defects D_i.

iii) The defect D of the near-ring R is an ideal of R.

iv) Let R be a near-ring with the defect D and A be an ideal of R. The quotient near-ring R/A has the defect $D = \{d + A | d \in D\}$. Moreover, R/A is distributively generated if and only if $D \subseteq A$.

Following this idea, Jancic Rasovic and Cristea [6], introduce the concept of hypernear-ring with a defect of distributivity, and present several properties of this class of hypernear-rings, in connection with their direct product, hyperhomomorphisms, or factor hypernear-rings.

In this paper we introduce the class of $D-$division near-rings as a subclass of near-rings with a defect D and that one of division near-rings. Then we state necessary and sufficient condition under which a near-ring with defect of distributivity D is a $D-$ division near-ring. On the end, we show that Ligh's theorem proved for distributively generated near-rings is a corollary of our result.

2. D-division near-rings

Definition 2.1. *Let (R, S) be a near-ring with the defect of distributivity $D \neq R$. The structure $(R \backslash D, \cdot)$ is a $D-$ multiplicative group of the near -ring R if:*

i) The set $R \backslash D$ is closed under the multiplication.

ii) There exists $e \in R \backslash D$ such that, for each $x \in R$ it holds $x \cdot e = x + d_1$ and $e \cdot x = x + d_2$, for some $d_1, d_2 \in D$. A such element e is called the identity element.

iii) For each $x \in R \backslash D$ there exists $x' \in R \backslash D$ and $d_1, d_2 \in D$, such that: $x \cdot x' = e + d_1$ and $x' \cdot x = e + d_2$.

Definition 2.2. *Let (R, S) be a near-ring with the defect of distributivity $D \neq R$. We say that R is a $D-$division near-ring (a near-ring of $D-$ fractions) if $(R \backslash D, \cdot)$ is a $D-$multiplicative group.*

Obviously, if (R, S) is a near-ring with defect of distributivity $D \neq R$, such that $(R \backslash D, \cdot)$ is a multiplicative group, then $(R \backslash D, \cdot)$ is a $D-$multiplicative group. Also, if R is a distributively generated such that R is a division near-ring, then R is an example of $D-$division near ring with defect of distributivity $D = \{0\}$.

Now we present another examples of $D-$division near-rings.

Example 2.1. Let $(R, +) = (Z_6, +)$, be the additive group of integers modulo 6, and define on R the multiplication as follows:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	5	4	3	2	1
2	0	1	2	3	4	5
3	0	0	0	0	0	0
4	0	5	4	3	2	1
5	0	1	2	3	4	5

It is simple to check that the multiplication is associative, so $(R, \cdot)$ is a semigroup, having 0 as two-sided absorbing element. Moreover, the multiplication distributes over addition, so for any $x, y, z \in R$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$ (we let these part to the reader as a simple exercice). For example, $1 \cdot (4 + 2) = 1 \cdot 0 = 0(0 = 2 + 4 = 1 \cdot 4 + 1 \cdot 2.)$ Take $S = \{0, 2, 3\}$ a system of generators of the hypergroup $(R, +)$. We also notice that $(S, \cdot)$ is a subsemigroup of $(R, \cdot)$. Now we determine the set $D_S : D_S = \{d \in R | d = -(x \cdot s + y \cdot s) + (x + y) \cdot s, x, y \in R, s \in S\} = \{-(x \cdot 0 + y \cdot 0) + (x + y) \cdot 0 | x, y \in R\} \cup \{-(x \cdot 2 + y \cdot 2) + (x + y) \cdot 2 | x, y \in R\} \cup \{-(x \cdot 3 + y \cdot 3) + (x + y) \cdot 3 | x, y \in R\} = \{0\} \cup \{0\} \cup \{0, 3\} = \{0, 3\}$. The table of the hypercomposition $x \cdot 3 + y \cdot 3$ is the following one:

	0	1	2	3	4	5
0	0	3	3	0	3	3
1	3	0	0	3	0	0
2	3	0	0	3	0	0
3	0	3	3	0	3	3
4	3	0	0	3	0	0
5	3	0	0	3	0	0

It follows that the table of $-(x \cdot 3 + y \cdot 3)$ is:

	0	1	2	3	4	5
0	0	3	3	0	3	3
1	3	0	0	3	0	0
2	3	0	0	3	0	0
3	0	3	3	0	3	3
4	3	0	0	3	0	0
5	3	0	0	3	0	0

Similarly, the table of the hypercomposition $(x + y) \cdot 3$ is:

	0	1	2	3	4	5
0	0	3	3	0	3	3
1	3	3	0	3	3	0
2	3	0	3	3	0	3
3	0	3	3	0	3	3
4	3	3	0	3	3	0
5	3	0	3	3	0	3

We obtain that $A = \{-(x \cdot 3 + y \cdot 3) + (x + y) \cdot 3 | x, y \in R\} = \{0, 3\}$.

It follows that the defect of distributivity of the near-ring R is $D = \{0, 3\}$.

It can be easily verified that $(R \backslash D, \cdot)$ is a $D-$multiplicative group. Indeed, $R \backslash D = \{1, 2, 4, 5\}$ is closed under the multiplication. Moreover, $e = 2$ is the identity element. Finally, for any $a \in R \backslash D$, there exists $d \in D$ such that $a \cdot a = 2 + d$, meaning that the inverse of each element $a \in R \backslash D$ is a itself. So (R, S) is a $D-$division near-ring.

Example 2.2. Let $(R, +) = (Z_4, +)$ be the additive group of the integers modulo 4 and define on R the multiplication as follows:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	0	0	0
3	0	3	2	1

Then, $(R, \cdot)$ is a semigroup, having 0 as a bilaterally absorbing element. It can be veried that, for any $x, y, z \in R$, it holds $x \cdot (y + z) = x \cdot y + x \cdot z$, meaning that $(R, +, \cdot)$ is a near-ring. Take $S = \{1\}$. Obviously, S is a subsemigroup of $(R, \cdot)$ and it generates $(R, +)$. Since the set $D_S = \{-(x \cdot 1 + y \cdot 1) + (x + y) \cdot 1 | x, y \in R\} = \{0, 2\}$, we conclude that the the defect of distributivity of the near-ring R is $D = \{0, 2\}$.

We can see that the multiplicative structure$(R \backslash D, \cdot)$ is a group, so $R \backslash D$ is a $D-$multiplicative group, i.e. (R, S) is a $D-$division near-ring.

Definition 2.3. *Let (R, S) be a near-ring with the defect of distributivity D. We say that (R, S) is a near-ring without $D-$ divisors if, for all $x, y \in R, x \cdot y \in D$ implies that $x \in D$ or $y \in D$. Otherwise, we say that R has $D-$divisors if there exist $x, y \in R \backslash D$ such that $x \cdot y \in D$.*

Proposition 2.1. *Let (R, S) be a near-ring with the defect of distributivity $D \neq R$. If $a \cdot (R \backslash D) + D = R \backslash D + D$, for all $a \in R \backslash D$, then R is a near-ring without $D-$divisors.*

Proof. Suppose there exist $x, y \in R \backslash D$ such that $x \cdot y \in D$. Since $x \in R \backslash D \subseteq R \backslash D + D = x \cdot (R \backslash D) + D$, it follows that there exists $x' \in R \backslash D$ and $d_1 \in D$ such that $x = x \cdot x' + d_1$. Moreover, from $x' \in R \backslash D \subseteq R \backslash D + D = y \cdot (R \backslash D) + D$, it follows that there exist $y' \in R \backslash D$ and $d_2 \in D$ such that $x' = y \cdot y' + d_2$. Therefore, $x = x \cdot (y \cdot y' + d_2) + d_1 = x \cdot y \cdot y' + x \cdot d_2 + d_1$. Since D is an ideal of R, and R

is a zero-symmetric near-ring, then $(x \cdot y) \cdot y' \in D$ as $x \cdot y \in D$, and $x \cdot d_2 \in D$, as $d_2 \in D$. It follows that $(x \cdot y) \cdot y' + x \cdot d_2 + d_1 \in D$, i.e. $x \in D$. It contradicts the initial assumption. Therefore, R is a near-ring without $D-$divisors. $\square$

Corollary 2.1. *If (R, S) is a near-ring with the defect of distributivity $D \neq R$, such that $a \cdot (R \backslash D) + D = R \backslash D + D$, for all $a \in R \backslash D$, then the set $R \backslash D$ is closed under the multiplication.*

Proof. It follows immediately from the previous proposition. $\square$

Theorem 2.1. *Let (R, S) be a near-ring with the defect $D \neq R$. A necessary and sufficient condition for the near-ring R to be a $D-$division near-ring is that $a \cdot (R \backslash D) + D = R \backslash D + D$, for all $a \in R \backslash D$.*

Proof. Sufficiency. Let $a \cdot (R \backslash D) + D = R \backslash D + D$, for all $a \in R \backslash D$. By Corollary 2.1, it follows that the set $R \backslash D$ is closed under the multiplication. Note that there exists $s \in R \backslash D$ such that $s \in S$. To the contrary, if $S \subseteq D$, then $R = \langle S \rangle \subseteq D$, meaning that $R = D$, which contradicts our assumption. Thus, let $s \in R \backslash D$ such that $s \in S$. Since $s \in (R \backslash D) + D = s \cdot (R \backslash D) + D$, it follows that there exists $e \in R \backslash D$ and $d_1 \in D$ such that $s = s \cdot e + d_1$. Hence, $s \cdot (e \cdot s - s) = (s \cdot e) \cdot s - s \cdot s = (s - d_1) \cdot s - s \cdot s \in D$, since D is an ideal in R. By Proposition 2.1, R is a near-ring without $D-$divisors and since $s \in R \backslash D$, we get $e \cdot s - s \in D$, i.e. $e \cdot s \in D + s = s + D$, and so there exists $d_2 \in D$ such that $es = s + d_2$. If $x \in R \backslash D$, then for some $d_3 \in D$ it holds: $(x \cdot e - x) \cdot s = x \cdot (e \cdot s) - x \cdot s + d_3 = x \cdot (s + d_2) - x \cdot s + d_3 = x \cdot s + x \cdot d_2 - x \cdot s + d_3 \in x \cdot s + D - x \cdot s + D \subseteq D + D = D$. Since $s \notin D$, we have $x \cdot e - x \in D$, meaning that $x \cdot e \in D + x = x + D$. So, there exists $d_4 \in D$ such that $xe = x + d_4$. Besides, $s \cdot (e \cdot x - x) = (s \cdot e) \cdot x - s \cdot x = (s - d_1) \cdot x - s \cdot x \in D$, since D is an ideal. Again, since $s \notin D$, we obtain $e \cdot x - x \in D$, implying that $e \cdot x \in D + x = x + D$. Thus, there exists $d_5 \in D$ such that $ex = x + d_5$ .Thereby e is the identity element.

Suppose now that $a \in R \backslash D$. Since $e \in R \backslash D \subseteq R \backslash D + D = a \cdot (R \backslash D) + D$, then there exist $a' \in R \backslash D$ and $d \in D$ such that $e = a \cdot a' + d$. Besides, $a \cdot (a' \cdot a - e) = (a \cdot a') \cdot a - a \cdot e = (e - d) \cdot a - (a + d_1)$, for some $d_1 \in D$. Since D is an ideal of R, we have $(e - d) \cdot a - e \cdot a \in D$, i.e. $(e - d) \cdot a \in D + e \cdot a = e \cdot a + D$. Therefore, $a \cdot (a' \cdot a - e) \in e \cdot a + D - (a + d_1) = e \cdot a + D - d_1 - a$. Besides, $e \cdot a = a + d_2$, for some $d_2 \in D$ and thus $a \cdot (a' \cdot a - e) \in a + d_2 + D - d_1 - a \subseteq a + D - a \subseteq D$. Since $a \notin D$, it follows that $a' \cdot a - e \in D$, meaning that $a' \cdot a \in D + e = e + D$ i.e $a' \cdot a = e + d_4$ for some $d_4 \in D$. Hence, we have shown that $R \backslash D$ is a $D-$multiplicative group, implying that (R, S) is a $D-$division near-ring.

Necessity. Let $R \backslash D$ be a $D-$multiplicative group with the identity element e. Let $a \in R \backslash D$. Obviously, $a \cdot (R \backslash D) + D \subseteq R \backslash D + D$. We prove now the other inclusion $R \backslash D + D \subseteq a \cdot (R \backslash D) + D$. Suppose $x \in R \backslash D$. Since $R \backslash D$ is a $D-$multiplicative group, it follows that there exist $a' \in R \backslash D$ and $d_1 \in D$ such that $a \cdot a' = e + d_1$. Besides there exists $d_2 \in D$ such that $x = e \cdot x + d_2 = (a \cdot a' - d_1) \cdot x + d_2$. Since D is an ideal of R, we have $(aa' - d_1) \cdot x - (a \cdot a') \cdot x \in D$, and therefore $(a\cdot a'-d_1)\cdot x=(aa'-d_1)\cdot x-(a\cdot a')\cdot x+(a\cdot a')\cdot x\in D+(a\cdot a')\cdot x$. It follows that $x\in D+a\cdot(a'\cdot x)+d_2=a\cdot(a'\cdot x)+D\subseteq a\cdot(R\backslash D)+D$. Therefore, $R\backslash D\subseteq a\cdot(R\backslash D)+D$, i.e. $R\backslash D+D\subseteq a\cdot(R\backslash D)+D$. □

Now we will show that Theorem 1.1 [7] follows from the previous theorem.

Corollary 2.2. *A necessary and sufficient condition for a d.g. near-ring with more than one element to be division ring is that for all non-zero elements $a\in R$, it holds $a\cdot R=R$.*

Proof. If R is a d.g. near-ring, then by Lemma 1.1, R is a zero symmetric near-ring, with the defect of disrtributivity $D=\{0\}$. From the previous theorem, it follows that a necessary and sucient condition for a d.g. near-ring R with more than one element to be a division near-ring is that $a\cdot(R\backslash\{0\})=R\backslash\{0\}$, for all $a\in R\backslash\{0\}$. Now we prove that if R is a d.g. near-ring with more than one element, then $a\cdot R=R$, for all $a\in R\backslash\{0\}$, if and only if $a\cdot(R\backslash\{0\})=R\backslash\{0\}$, for all $a\in R\backslash\{0\}$. Obviously, $a\cdot(R\backslash\{0\})=R\backslash\{0\}$, for all $a\in R\backslash\{0\}$ implies that $a\cdot R=R$, for all $a\in R\backslash\{0\}$.

Suppose now that we have $a\cdot R=R$, for all $a\in R\backslash\{0\}$. First we prove that $a\cdot R\backslash\{0\}\subseteq R\backslash\{0\}$, for $a\neq 0$. If there exist $a\neq 0, b\neq 0$, such that $a\cdot b=0$, then since $a\cdot R=R$ and $b\cdot R=R$ it follows that there exist $x,y\in R$ such that $a=a\cdot x$ and $x=b\cdot y$. Therefore, by Lemma 1.1, we have $0=0\cdot y=a\cdot b\cdot y=a\cdot x=a$, which is a contradiction. Thus $a\cdot R\backslash\{0\}\subseteq R\backslash\{0\}$. On the other side, for all $a\in R\backslash\{0\}$, it holds $R\backslash\{0\}\subseteq a\cdot R=R$ and since $a\cdot 0=0$ it follows that $R\backslash\{0\}\subseteq a\cdot(R\backslash\{0\})$. Therefore, $a\cdot(R\backslash\{0\})=R\backslash\{0\}$, for all $a\in R\backslash\{0\}$. Thus, from Lemma 1.3, we obtain Corollary 2.2. □

3. Conclusion and future work

In our future research we intend to extend to the case of hypernear-rings the notions that were studied in this paper. Jancic- Rasovic and Cristea have recently started [6] the study of hypernear-rings with a defect of distributivity D. Our aim is to continue in the same direction, introducing the class of $D-$division hypernear-rings as a subclass of hypernear-rings with a defect D, and that one of division hypernear-rings. Another aim is to state a necessary and sufficient condition under which a hypernear-ring with a defect of distributivity D is a $D-$division hypernear-ring.

References

[1] J. Clay, *Nearrings: Geneses and Application*, Oxford Univ. Press, Oxford, 1992.

[2] V. Dasic, *A defect of distributivity of the near-rings* , Math. Balkanica, 8 (1978), 63-75.

[3] L. Dickson, *Definitions of a group and a field by independent postulates*, Trans. Amer. Math. Soc., 6 (1905), 198-204.

[4] G. Ferrero, C. Ferrero-Cotti, *Nearrings. Some Developments Linked to Semigroups and Groups*, Kluwer, Dordrecht, 2002.

[5] A. Fronlich, *Distributively generated near-rings (I. Ideal Theory)*, Proc. London Math. Soc., (3) 8 (1958), 76-94.

[6] S. Jancic Rasovic, I. Cristea, *Hypernear-rings with a defect of distributivity*, submitted.

[7] S. Ligh, *On distributively generated near-rings*, Proc. Edinburg Math. Soc., 16, Issue 3 (1969), 239-242.

[8] J. Meldrum, *Near-Rings and their Links with Groups*, Pitman, London, 1985.

[9] B.H. Neumann, *On the commutativity of addition*, London Math. Soc. 15 (1940), 203-208.

[10] G. Pilz, *Near-rings*, North-Holland Publ.Co., rev.ed. 1983.

[11] B. Satyanarayana, K.S Prasad, *Near-Rings, Fuzzy Ideals, and Graph Theory*, CRC Press, New York, 2013.

[12] S. Scott, *Tame Theory*, Amo Publishing, Auckland, 1983.

[13] V. Vukovic, *Nonassociative Near-Rings*, Univ. of Kragujevac-Studio Plus, Belgrade, 1996.

[14] H. Wahling, *Theorie der Fastkörper*, Thales-Verlag, Essen, 1987.

Accepted: 24.01.2018