NOTE ON RELATIONS AMONG MULTIPLE ZETA(-STAR) VALUES

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Abstract. In the present paper, we shall show that various relations among multiple zeta(-star) values and their multivariable extensions can be derived from the hypergeometric identities of G. E. Andrews, C. Krattenthaler and T. Rivoal. The results in the present paper give us various identities for multiple Hurwitz zeta values also.
Keywords: Multiple zeta value, multiple zeta-star value, hypergeometric series, multiple Hurwitz zeta value.

1. Introduction

The multiple zeta value $\zeta(k_1, \ldots, k_n)$ (MZV for short) and the multiple zeta-star value $\zeta^\star(k_1, \ldots, k_n)$ (MZSV for short) are defined by the multiple series

$$\zeta(k_1, \ldots, k_n) := \sum_{0<m_1<\cdots<m_n<\infty} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$
$$\zeta^\star(k_1, \ldots, k_n) := \sum_{1\le m_1\le\cdots\le m_n<\infty} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

where $n \in \mathbb{Z}_{\geq 1}$, $k_1, \ldots, k_{n-1} \in \mathbb{Z}_{\geq 1}$, $k_n \in \mathbb{Z}_{\geq 2}$ (Euler [11], Hoffman [15], Zagier [30]). MZVs can be expressed by $\mathbb{Z}$-linear combinations of MZSVs and vice versa. In recent years, many researchers have been studying MZ(S)Vs and the study of MZ(S)Vs has become one of the active research areas of mathematics. One of the interesting properties of MZ(S)Vs is that MZ(S)Vs satisfy various relations. In fact, there are many results on this property. In the present paper, we shall also study relations among MZ(S)Vs.

We recall the definition of the generalized hypergeometric series:

$${}_{p+1}F_p\left(\begin{matrix} a_1, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} ; z \right) := \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_{p+1})_m}{(b_1)_m \cdots (b_p)_m} \frac{z^m}{m!},$$

where $p \in \mathbb{Z}_{\geq 1}$, z, $a_i \in \mathbb{C}$, $b_i \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $(a)_m$ denotes the Pochhammer symbol defined by $(a)_m := a(a+1)\cdots(a+m-1)$ $(m \in \mathbb{Z}_{\geq 1})$, $(a)_0 := 1$. The

Pochhammer symbol $(a)_m$ can be expressed as $(a)_m = \Gamma(a+m)/\Gamma(a)$ by using the gamma function $\Gamma(z)$. The above power series converges absolutely for all $z \in \mathbb{C}$ such that $|z| = 1$ provided $\mathrm{Re}\,(\sum_{i=1}^{p} b_i - \sum_{i=1}^{p+1} a_i) > 0$.

In [24], Krattenthaler and Rivoal proved the following hypergeometric identities by taking a limit in the basic hypergeometric identity of Andrews [1, Theorem 4]:

Theorem A (Krattenthaler and Rivoal [24, Proposition 1 (i) and (ii)]). (*i*) *Let s be a positive integer, and let a, b_i, c_i $(i = 1, \ldots, s+1)$ be complex numbers. Suppose that the complex numbers a, b_i, c_i $(i = 1, \ldots, s+1)$ satisfy the conditions* $1 + a - b_i,\ 1 + a - c_i \notin \mathbb{Z}_{\leq 0}$ $(i = 1, \ldots, s+1)$,

$$\mathrm{Re}\left((2s+1)(a+1) - 2\sum_{i=1}^{s+1}(b_i + c_i)\right) > 0,$$

$$\mathrm{Re}\left(\sum_{i=r}^{s+1} A_i(1 + a - b_i - c_i)\right) > 0 \quad (r = 2, \ldots, s+1)$$

for all possible choices of $A_i = 1$ or 2 $(i = 2, \ldots, s)$, $A_{s+1} = 1$. (For the details of the choices of A_i, see [24].) Then the following identity holds:

$$\begin{aligned}
&{}_{2s+4}F_{2s+3}\left(\begin{matrix} a, \frac{a}{2}+1, b_1, c_1, \ldots, b_{s+1}, c_{s+1} \\ \frac{a}{2}, 1+a-b_1, 1+a-c_1, \ldots, 1+a-b_{s+1}, 1+a-c_{s+1} \end{matrix}; -1\right) \\
&= \frac{\Gamma(1+a-b_{s+1})\Gamma(1+a-c_{s+1})}{\Gamma(1+a)\Gamma(1+a-b_{s+1}-c_{s+1})} \\
&\quad \times \sum_{l_1, \ldots, l_s = 0}^{\infty} \prod_{i=1}^{s} \frac{(1+a-b_i-c_i)_{l_i}(b_{i+1})_{l_1+\cdots+l_i}(c_{i+1})_{l_1+\cdots+l_i}}{l_i!(1+a-b_i)_{l_1+\cdots+l_i}(1+a-c_i)_{l_1+\cdots+l_i}}.
\end{aligned}$$

(*ii*) *Let s be a positive integer, and let a, b_i, c_i $(i = 1, \ldots, s)$, c_0 be complex numbers. Suppose that the complex numbers a, b_i, c_i $(i = 1, \ldots, s)$, c_0 satisfy the conditions* $1 + a - b_i,\ 1 + a - c_j \notin \mathbb{Z}_{\leq 0}$ $(i = 1, \ldots, s; j = 0, 1, \ldots, s)$,

$$\mathrm{Re}\left(2s(a+1) - 2c_0 - 2\sum_{i=1}^{s}(b_i + c_i)\right) > 0,$$

$$\mathrm{Re}\left(-c_0 + \sum_{i=1}^{s} A_i(1 + a - b_i - c_i)\right) > 0,$$

$$\mathrm{Re}\left(\sum_{i=r}^{s} A_i(1 + a - b_i - c_i)\right) > 0 \quad (r = 2, \ldots, s)$$

for all possible choices of $A_i = 1$ *or* 2 $(i = 1, \ldots, s-1)$, $A_s = 1$. (*For the details of the choices of* A_i, *see* [*24*].) *Then the following identity holds*:

$$
\begin{aligned}
&{}_{2s+3}F_{2s+2}\left(\begin{matrix} a, \frac{a}{2}+1, c_0, b_1, c_1, \ldots, b_s, c_s \\ \frac{a}{2}, 1+a-c_0, 1+a-b_1, 1+a-c_1, \ldots, 1+a-b_s, 1+a-c_s \end{matrix}; 1\right) \\
=&\frac{\Gamma(1+a-b_s)\Gamma(1+a-c_s)}{\Gamma(1+a)\Gamma(1+a-b_s-c_s)} \\
&\times \sum_{l_1,\ldots,l_s=0}^{\infty} \frac{(b_1)_{l_1}(c_1)_{l_1}}{l_1!(1+a-c_0)_{l_1}} \left\{ \prod_{i=2}^{s} \frac{(1+a-b_{i-1}-c_{i-1})_{l_i}(b_i)_{l_1+\cdots+l_i}(c_i)_{l_1+\cdots+l_i}}{l_i!(1+a-b_{i-1})_{l_1+\cdots+l_i}(1+a-c_{i-1})_{l_1+\cdots+l_i}} \right\}.
\end{aligned}
$$

In [24], Krattenthaler and Rivoal used the above hypergeometric identities to give an alternative proof of the identity of Zudilin [33, Theorem 5], which is an identity between very-well-poised hypergeometric series and multiple integrals related to the construction of $\mathbb{Q}$-linear forms in the Riemann zeta values. The hypergeometric identities in Theorem A are limiting cases of the basic hypergeometric identity of Andrews [1, Theorem 4], which is a multiple series generalization of Watson's q-analogue of Whipple's hypergeometric identity (see Watson [29]). In [2], Andrews proved a further multiple series generalization of the Watson's q-analogue also (see [2, Theorem 1]). In [1], [2] and [3], Andrews derived various multiple q-series identities from his basic hypergeometric identities [1, Theorem 4], [2, Theorem 1].

In the present paper, we show that various relations among MZ(S)Vs and their multivariable extensions can be derived from the hypergeometric identities in Theorem A by taking a specialization and further consideration. See also Remark 1 (i) below. By virtue of the multivariable extensions, most of the results in the present paper can be regarded as relations among the same multiple series. The hypergeometric identities of Andrews, Krattenthaler and Rivoal were useful for us to find multivariable extensions of MZ(S)Vs which satisfy various relations. In fact, by studying Theorem A, we found some multivariable extensions of MZ(S)Vs and their relations (see Section 2). Other hypergeometric identities (e.g. Andrews [2, Theorem 1]) may also be useful for this kind of study in extensions of MZ(S)Vs. In the present paper, to prove the relations, we express the partial derivatives of the hypergeometric series by linear combinations of multiple zetas. This is done by using some calculational techniques for the Pochhammer symbols $(a)_m$, which are based on the techniques used in [20]. In particular, we avoid calculating the products of finite multiple harmonic sums by appropriate calculations for the Pochhammer symbols $(a)_m$. We remark that the calculational techniques used in the present paper can be applied to deriving relations among multiple zetas from other hypergeometric identities. For the study of the multiple zeta expression for hypergeometric series, see also [19, Remark 2] and the references therein. We hope that the results in the present paper are useful for the study of relations among multiple zetas and the study of the interaction between multiple zetas and (multiple) hypergeometric series. In the present paper, we prove various relations among the multiple series (1), (22) and (41) below, which are the main objects of study in [21] and in the present paper also. We remark that many of those relations are written in compact forms. The compactness comes from appropriate calculations for the Pochhammer symbols $(a)_m$ and appropriate choices of special cases of the multiple series (1), (22) and (41) below. From the results in the present paper and their proofs, we think that to make further research on relations among the multiple series (1), (22) and (41) below is interesting and the hypergeometric method is useful for the research.

Remark 1. (i) By examining the proof of [24, Proposition 1], we see that the results in the present paper can be derived from the identity of Andrews [1, Theorem 4]; therefore various relations among multiple zetas can be derived from "one" hypergeometric identity. This fact also means that the present research is an application of the basic hypergeometric identity of Andrews [1, Theorem 4] to the study of relations among multiple zetas.

(ii) Many of the results in the present paper can be regarded as generalizations of the following two identities for MZSVs:

$$\zeta^\star(\underbrace{2,\dots,2}_{s}) = 2\sum_{m=0}^{\infty}(-1)^m(m+1)^{-2s} = 2(1-2^{1-2s})\zeta(2s),$$
$$\zeta^\star(1,\underbrace{2,\dots,2}_{s}) = 2\zeta(2s+1)$$

fo all $s \in \mathbb{Z}_{\geq 1}$ (Aoki–Ohno [4, Theorem 1], Vasil'ev [28, Theorem], Zlobin [32]). These two identities can also be derived from the identity of Andrews [1, Theorem 4] by following the proof of [24, Proposition 1].

The present research is motivated by [17, Remarks 2.6 and 2.7]. The present paper is an expanded version of [19], [20], and a revised version of [21]. I proved the results in the present paper in 2013–2016 and modified several of them in 2016–2017. (I proved (29) and (30) below in 2011.)

2. Applications of the hypergeometric identities of Andrews, Krattenthaler and Rivoal

2.1 Notations and definitions

We use the notations

$$\{a\}^n := \underbrace{a,\dots,a}_{n}, \quad \{a_{i1},\dots,a_{in}\}_{i=1}^m := a_{11},\dots,a_{1n},\dots,a_{m1},\dots,a_{mn},$$

$$\sum\nolimits^{(k,l)} := \sum_{i=0}^{k+l} \sum_{\substack{k_1+\cdots+k_{i+1}=k+1 \\ l_1+\cdots+l_{i+1}=l+1 \\ k_j,l_j\in\mathbb{Z}_{\geq 0},k_j+l_j\geq 1 \\ (j=1,\dots,i);\,k_{i+1},l_{i+1}\in\mathbb{Z}_{\geq 1}}}$$

($k, l \in \mathbb{Z}_{\geq 0}$, $m, n \in \mathbb{Z}_{\geq 1}$). We regard $\{a\}^0$, $\{a_{i1}, \ldots, a_{in}\}_{i=1}^0$ as the empty set $\emptyset$. The summation $\sum^{(k,l)}$ has the symmetry $\sum^{(k,l)} = \sum^{(l,k)}$. For brevity, we frequently use the following notations for the partial differential operators:

$$\partial^{(k)}(x)\Big|_{x=\alpha} := \frac{1}{k!}\frac{\partial^k}{\partial x^k}\Big|_{x=\alpha}, \quad \partial^{(k,l)}(x,y)\Big|_{\substack{x=\alpha\\ y=\beta}} := \frac{1}{k!l!}\frac{\partial^{k+l}}{\partial x^k \partial y^l}\Big|_{\substack{x=\alpha\\ y=\beta}}$$

($k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$). We define the symbol $e(n)$ by

$$e(n) = \begin{cases} 1, & \text{if } n \geq 1, \\ 0, & \text{if } n = 0. \end{cases}$$

The symbols $\lesssim$, $\lesssim_i$ ($i \in \mathbb{Z}_{\geq 0}$) mean $<$ or $\leq$ (cf. Fischler–Rivoal [12]). We consider the following special value of the multiple Hurwitz zeta function:

$$\begin{aligned} &\zeta^{\lesssim}_{n,\pm}(\{a_{i1}\}_{i=1}^n | \ldots | \{a_{ik}\}_{i=1}^n ; (\{\alpha_j\}_{j=1}^k)) \\ &:= \sum_{0 \leq m_1 \lesssim \cdots \lesssim m_n < \infty} (\pm 1)^{m_n} \left\{ \prod_{i=1}^n \prod_{j=1}^k \frac{1}{(m_i + \alpha_j)^{a_{ij}}} \right\}, \end{aligned}$$

where $k, n \in \mathbb{Z}_{\geq 1}$, $\alpha_j \in \mathbb{C} \backslash \mathbb{Z}_{\leq 0}$, $a_{ij} \in \mathbb{Z}$ such that $\sum_{j=1}^k a_{ij} \geq 1$ ($i = 1, \ldots, n-1$), $\sum_{j=1}^k a_{nj} \geq 2$. We call this special value the multiple Hurwitz zeta value (MHZV for short). For brevity, we put

$$\zeta_-(\{k_i\}_{i=1}^n) := \zeta^{<}_{n,-}(\{a_{i1}\}_{i=1}^n | \ldots | \{a_{ik}\}_{i=1}^n ; (\{1\}^k)),$$

where $k_1, \ldots, k_{n-1} \in \mathbb{Z}_{\geq 1}$, $k_n \in \mathbb{Z}_{\geq 2}$. This alternating multiple series is a special value of multiple polylogarithms. (For the study of special values of multiple polylogarithms, see e.g. [5], [6], [7].) The results in the present paper give us various identities for MHZVs.

2.2 Application 1

We consider the multiple series

$$\begin{aligned} &\sum_{\substack{0 = m_0 \leq = \lesssim_0 m_1 \lesssim_1 \cdots \lesssim_{p-1} m_p \\ m_p \lesssim_p \cdots \lesssim_{p+q-1} m_{p+q} \lesssim_{p+q} = < \infty}} \frac{(\alpha)_{m_p}}{m_p!} \frac{(\beta)_{m_p}}{(\gamma)_{m_p}} \frac{m_{p+q}!}{(\alpha)_{m_{p+q}}} \frac{(\gamma)_{m_{p+q}}}{(\beta)_{m_{p+q}}} \\ &\times \left\{ \prod_{i=1}^{p+q} \frac{(2m_i + \alpha + \beta)^{s_i} (2m_i + \gamma + \delta)^{t_i}}{(m_i + \alpha)^{a_i} (m_i + \beta)^{b_i} (m_i + \gamma)^{c_i} (m_i + \delta)^{d_i}} \right\}, \end{aligned} \tag{1}$$

where $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q \geq 1$; $a_i, b_i, c_i, d_i, s_i, t_i \in \mathbb{Z}$ such that $a_i + b_i + c_i + d_i - s_i - t_i \geq 1$ ($i = 1, \ldots, p+q-1$), $a_{p+q} + b_{p+q} + c_{p+q} + d_{p+q} - s_{p+q} - t_{p+q} \geq 2$; $\alpha, \beta, \gamma, \delta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ such that $\mathrm{Re}\,(\alpha + \beta - \gamma) > 0$ and $(\alpha+\beta)/2, (\gamma+\delta)/2 \notin \mathbb{Z}_{\leq 0}$ in case of $s_i, t_i < 0$ for some i. The symbol $(a)_m$ is the Pochhammer symbol.

The case $\alpha = \beta = \gamma = \delta = 1$, $s_i = t_i = 0$ $(i = 1, \ldots, p+q)$ of (1) is the extension of MZV considered by Fischler and Rivoal [12], Kawashima [23]. By considering the multiple series with the summation in (1), many of the identities in the present paper can be written in compact forms. The case $q = 0$ of (1) is a MHZV; therefore the multiple series (1) gives us an extension of MHZV. In this subsection, we study relations among the multiple series (1) by using Theorem A.

We first prove the following identity, which is frequently used in the present paper:

Lemma 2.1. *The identity*

$$\frac{(X)_m}{(Z)_{m+1}}\frac{(Y)_m}{(W)_{m+1}} = \frac{(a)_m}{(c)_m}\frac{(b)_m}{(d)_m}\frac{1}{(Z+m)(W+m)}$$
$$\times \sum_{i=0}^{m} \sum_{0\le m_1<\cdots<m_i<m} \prod_{j=1}^{i} \left(\frac{(X-a)(Y+m_j)(c+m_j)(d+m_j)}{(a+m_j)(b+m_j)(Z+m_j)(W+m_j)} \right.$$
$$\left. + \frac{(Y-b)(c+m_j)(d+m_j)}{(b+m_j)(Z+m_j)(W+m_j)} - \frac{(Z-c)(d+m_j)}{(Z+m_j)(W+m_j)} - \frac{W-d}{W+m_j} \right)$$

holds for all $m \in \mathbb{Z}_{\ge 0}$, $X, Y \in \mathbb{C}$, $a, b, c, d, Z, W \in \mathbb{C} \setminus \mathbb{Z}_{\le 0}$.

Proof. By direct calculation, we get the identities

$$\frac{(X)_m}{(Z)_{m+1}}\frac{(Y)_m}{(W)_{m+1}} = \frac{(X)_m}{(Z)_{m+1}}\frac{(Y)_m}{(W)_{m+1}}\frac{(c)_m}{(a)_m}\frac{(d)_m}{(b)_m}\frac{(a)_m}{(c)_m}\frac{(b)_m}{(d)_m}$$
$$= \frac{(a)_m}{(c)_m}\frac{(b)_m}{(d)_m}\frac{1}{(Z+m)(W+m)}\prod_{i=0}^{m-1}\frac{(X+i)(Y+i)(c+i)(d+i)}{(a+i)(b+i)(Z+i)(W+i)}$$
$$= \frac{(a)_m}{(c)_m}\frac{(b)_m}{(d)_m}\frac{1}{(Z+m)(W+m)}\prod_{i=0}^{m-1}\left(1 + \frac{(X-a)(Y+i)(c+i)(d+i)}{(a+i)(b+i)(Z+i)(W+i)} \right.$$
$$\left. + \frac{(Y-b)(c+i)(d+i)}{(b+i)(Z+i)(W+i)} - \frac{(Z-c)(d+i)}{(Z+i)(W+i)} - \frac{W-d}{W+i}\right).$$

Expanding the last product, we get Lemma 2.1. $\square$

We consider the following special case of (1):

$$\Phi^{(p,q)}_{(\{r_i\}_{i=1}^{p+q})}(\{\{k_{ij}\}_{j=1}^{r_i}, K_i\}_{i=1}^{p+q}; (\alpha, \beta, \gamma))$$
$$:= \sum_{\substack{0=m_0\le m_{11}<\cdots<m_{1r_1}<m_1 \\ \vdots \\ m_{p-1}\le m_{p1}<\cdots<m_{pr_p}<m_p \\ \vdots \\ m_{p+q-1}\le m_{p+q1}<\cdots<m_{p+qr_{p+q}}<m_{p+q}<\infty}} \frac{(\alpha)_{m_p}}{m_p!}\frac{(\beta)_{m_p}}{(\gamma)_{m_p}}\frac{m_{p+q}!}{(\alpha)_{m_{p+q}}}\frac{(\gamma)_{m_{p+q}}}{(\beta)_{m_{p+q}}}$$

$$\times \left\{ \prod_{i=1}^{p} \frac{1}{(m_i+\gamma)^{K_i}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij}+\gamma)^{k_{ij}}} \right\}$$
$$\times \left\{ \prod_{i=p+1}^{p+q} \frac{1}{(m_i+\alpha)(m_i+\beta)(m_i+\gamma)^{K_i-2}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij}+\gamma)^{k_{ij}}} \right\},$$

where $p, q \in \mathbb{Z}_{\geq 0}$ such that $p+q \geq 1$; $r_i \in \mathbb{Z}_{\geq 0}$, $k_{ij} \in \mathbb{Z}_{\geq 1}$; $K_1, \dots, K_{p+q-1} \in \mathbb{Z}_{\geq 1}$, $K_{p+q} \in \mathbb{Z}_{\geq 2}$.

Theorem 2.2. *The identity*

$$\sum_{\substack{r_1+\cdots+r_{k+s}=l \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi^{(k+1,s-1)}_{(\{r_i\}_{i=1}^{k+s})}(\{1\}^{\sum_{i=1}^{k+1}(r_i+1)}, \{\{1\}^{r_i}, 2\}_{i=k+2}^{k+s}; (\alpha,\beta,\gamma))$$
$$(2) \qquad = \sum^{(k,l)} \phi_{i+1}\left(\begin{matrix} \{e(k_j)+e(l_j)\}_{j=1}^{i}, 1 \\ \{k_j+e(l_j)\}_{j=1}^{i}, k_{i+1} | \{l_j+e(k_j)\}_{j=1}^{i}, l_{i+1} | \{0\}^i, s-1 | \{0\}^i, s-1 \end{matrix} ; (\gamma, \alpha+\beta-\gamma, \alpha, \beta) \right)$$

holds for all $k, l \in \mathbb{Z}_{\geq 0}$*,* $s \in \mathbb{Z}_{\geq 2}$*,* $\alpha, \beta, \gamma \in \mathbb{C}$ *such that* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$*,* $\operatorname{Re}(\alpha+\beta) > \operatorname{Re}\gamma > 0$*, where*

$$\phi_p\left(\begin{matrix} \{s_i\}_{i=1}^{p} \\ \{a_i\}_{i=1}^{p} | \{b_i\}_{i=1}^{p} | \{c_i\}_{i=1}^{p} | \{d_i\}_{i=1}^{p} \end{matrix} ; (x,y,z,w) \right)$$
$$:= \sum_{0 \leq m_1 < \cdots < m_p < \infty} \prod_{i=1}^{p} \frac{(2m_i+x+y)^{s_i}}{(m_i+x)^{a_i}(m_i+y)^{b_i}(m_i+z)^{c_i}(m_i+w)^{d_i}},$$

which is also a special case of (1)*.*

Proof. Taking $a = \alpha+\beta$, $c_0 = \alpha+\beta-z$, $b_i = \alpha$, $c_i = \beta$ $(i = 1, \dots, s-1)$, $b_s = 1$, $c_s = w$, where $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta, z, w \in \mathbb{C}$ such that $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, $\operatorname{Re}(\alpha+\beta) > \operatorname{Re} z > 0$, $\operatorname{Re}(\alpha+\beta) > \operatorname{Re} w > 0$, $s-1 > \operatorname{Re}(w-z)$, in Theorem A (ii) and multiplying both sides of the result by z^{-1}, we get the identity

$$(3) \qquad \sum_{0 \leq m_1 \leq \cdots \leq m_s < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{(\beta)_{m_1}}{(z)_{m_1+1}} \frac{m_s!}{(\alpha)_{m_s}} \frac{(w)_{m_s}}{(\beta)_{m_s}} \left\{ \prod_{i=2}^{s} \frac{1}{(m_i+\alpha)(m_i+\beta)} \right\}$$
$$= \sum_{m=0}^{\infty} \frac{(\alpha+\beta-z)_m}{(z)_{m+1}} \frac{(w)_m}{(\alpha+\beta-w)_{m+1}} \frac{2m+\alpha+\beta}{(m+\alpha)^{s-1}(m+\beta)^{s-1}}.$$

Using the identity

$$(4) \qquad \frac{(-1)^r}{r!} \frac{\mathrm{d}^r}{\mathrm{d}z^r} \left(\frac{1}{(z)_{m+1}} \right) = \frac{1}{(z)_{m+1}} \sum_{0 \leq m_1 \leq \cdots \leq m_r \leq m} \prod_{i=1}^{r} \frac{1}{m_i+z}$$

$(m, r \in \mathbb{Z}_{\geq 0})$, we get the identity

$$\begin{aligned}
&\frac{(-1)^k}{k!}\frac{\partial^k}{\partial z^k}\left(\sum_{0\leq m_1\leq\cdots\leq m_s<\infty}\frac{(\alpha)_{m_1}}{m_1!}\frac{(\beta)_{m_1}}{(z)_{m_1+1}}\frac{m_s!}{(\alpha)_{m_s}}\frac{(w)_{m_s}}{(\beta)_{m_s}}\right.\\
&\qquad\left.\times\left\{\prod_{i=2}^{s}\frac{1}{(m_i+\alpha)(m_i+\beta)}\right\}\right)\\
&=\sum_{\substack{0\leq m_1\leq\cdots\leq m_{k+1}\\ m_{k+1}\leq\cdots\leq m_{k+s}<\infty}}\left\{\prod_{i=1}^{k+1}\frac{1}{m_i+z}\right\}\frac{(\alpha)_{m_{k+1}}}{m_{k+1}!}\frac{(\beta)_{m_{k+1}}}{(z)_{m_{k+1}}}\frac{m_{k+s}!}{(\alpha)_{m_{k+s}}}\frac{(w)_{m_{k+s}}}{(\beta)_{m_{k+s}}}\\
&\qquad\times\left\{\prod_{i=k+2}^{k+s}\frac{1}{(m_i+\alpha)(m_i+\beta)}\right\}
\end{aligned}\tag{5}$$

$(k \in \mathbb{Z}_{\geq 0})$. Using the identity

$$(w)_{m_{k+s}} = \prod_{i=1}^{k+s}\frac{(w)_{m_i}}{(w)_{m_{i-1}}},$$

where we regard m_0 as 0, we get the identities

$$\begin{aligned}
\frac{1}{l!}\frac{\mathrm{d}^l}{\mathrm{d}w^l}(w)_{m_{k+s}}&=\sum_{\substack{r_1+\cdots+r_{k+s}=l\\ r_i\in\mathbb{Z}_{\geq 0}}}\prod_{i=1}^{k+s}\frac{(w)_{m_i}}{(w)_{m_{i-1}}}\sum_{m_{i-1}\leq m_{i1}<\cdots<m_{ir_i}<m_i}\prod_{j=1}^{r_i}\frac{1}{m_{ij}+w}\\
&=(w)_{m_{k+s}}\sum_{\substack{r_1+\cdots+r_{k+s}=l\\ r_i\in\mathbb{Z}_{\geq 0}}}\sum_{\substack{0=m_0\leq m_{11}<\cdots<m_{1r_1}<m_1\\ \vdots\\ m_{k+s-1}\leq m_{k+s1}<\cdots<m_{k+sr_{k+s}}<m_{k+s}}}\prod_{i=1}^{k+s}\prod_{j=1}^{r_i}\frac{1}{m_{ij}+w}
\end{aligned}\tag{6}$$

for $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$ and $m_1, \ldots, m_{k+s} \in \mathbb{Z}$ such that $0 \leq m_1 \leq \cdots \leq m_{k+s}$. Thus, applying the operator $(-1)^k\partial^{(l,k)}(w, z)|_{z=\gamma}^{w=\gamma}$ $(k, l \in \mathbb{Z}_{\geq 0}$; $\gamma \in \mathbb{C}$ such that $\mathrm{Re}\,(\alpha+\beta) > \mathrm{Re}\,\gamma > 0$, $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0)$ to the left-hand side of (3) with $s \in \mathbb{Z}_{\geq 2}$ and using (5) and (6), we get the left-hand side of (2). The right-hand side of (2) can be proved as follows: Taking $a = \alpha+\beta-\gamma$, $b = \delta$, $c = \gamma$, $d = \alpha+\beta-\delta$, $X = \alpha+\beta-z$, $Y = w$, $Z = z$, $W = \alpha+\beta-w$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$\begin{aligned}
&\frac{(\alpha+\beta-z)_m}{(z)_{m+1}}\frac{(w)_m}{(\alpha+\beta-w)_{m+1}}\\
&=\frac{(\alpha+\beta-\gamma)_m}{(\gamma)_m}\frac{(\delta)_m}{(\alpha+\beta-\delta)_m}\frac{1}{(z+m)(\alpha+\beta-w+m)}
\end{aligned}$$

$$\times \sum_{i=0}^{m} \sum_{0\le m_1<\cdots<m_i<m} \prod_{j=1}^{i} \Biggl(-\frac{(z-\gamma)(w-\delta)(\gamma+m_j)(\alpha+\beta-\delta+m_j)}{(z+m_j)(\alpha+\beta-w+m_j)(\alpha+\beta-\gamma+m_j)(\delta+m_j)}$$
$$-\frac{(z-\gamma)(\alpha+\beta-\delta+m_j)(2m_j+\alpha+\beta)}{(z+m_j)(\alpha+\beta-w+m_j)(\alpha+\beta-\gamma+m_j)} + \frac{(w-\delta)(\gamma+m_j)(\alpha+\beta-\delta+m_j)}{(\alpha+\beta-w+m_j)(z+m_j)(\delta+m_j)}$$
$$+\frac{w-\delta}{\alpha+\beta-w+m_j}\Biggr).$$

Using this identity, we get the identity

$$\begin{aligned}
&(-1)^k \partial^{(l,k)}(w,z) \left(\frac{(\alpha+\beta-z)_m}{(z)_{m+1}} \frac{(w)_m}{(\alpha+\beta-w)_{m+1}} \right) \Big|_{\substack{w=\delta\\ z=\gamma}} \\
&= \sum_{i=0}^{m} \sum_{\substack{k_1+\cdots+k_{i+1}=k+1\\ l_1+\cdots+l_{i+1}=l+1\\ k_j,l_j\in\mathbb{Z}_{\ge0},k_j+l_j\ge1\\ (j=1,\ldots,i);\, k_{i+1},l_{i+1}\in\mathbb{Z}_{\ge1}}} \sum_{0\le m_1<\cdots<m_i<m} \\
&\left\{ \prod_{j=1}^{i} \frac{(2m_j+\alpha+\beta)^{e(k_j)}(2m_j+\alpha+\beta)^{e(l_j)}}{(m_j+\gamma)^{k_j}(m_j+\alpha+\beta-\gamma)^{e(k_j)}(m_j+\alpha+\beta-\delta)^{l_j}(m_j+\delta)^{e(l_j)}} \right\} \\
&\times \frac{1}{(m+\gamma)^{k_{i+1}}(m+\alpha+\beta-\delta)^{l_{i+1}}} \frac{(\alpha+\beta-\gamma)_m}{(\gamma)_m} \frac{(\delta)_m}{(\alpha+\beta-\delta)_m}
\end{aligned} \tag{7}$$

for $k, l, m \in \mathbb{Z}_{\ge0}$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\gamma, \delta, \alpha+\beta-\gamma, \alpha+\beta-\delta \notin \mathbb{Z}_{\le0}$. Thus, applying $(-1)^k \partial^{(l,k)}(w,z)|_{z=\gamma}^{w=\gamma}$ to the right-hand side of (3) with $s \in \mathbb{Z}_{\ge2}$ and using the case $\gamma = \delta$ of (7), we get the right-hand side of (2). □

Here we put

$$\Phi^{\star}_{p,q}(\{K_i\}_{i=1}^{p+q}; (\alpha,\beta,\gamma)) := \Phi^{(p,q)}_{(\{0\}^{p+q})}(\{K_i\}_{i=1}^{p+q}; (\alpha,\beta,\gamma)).$$

Using (2), we can prove the following identities:

Corollary 2.3. *The following two identities hold*:

(*i*)

$$\begin{aligned}
&\Phi^{\star}_{k+1,s-1}(\{1\}^{k+1}, \{2\}^{s-1}; (\alpha,\beta,\gamma)) \\
&= \sum\nolimits^{(k,0)} \phi_{i+1} \left(\begin{matrix} \{1\}^{i+1} \\ \{k_j\}_{j=1}^{i+1} | \{1\}^{i+1} | \{0\}^i, s-1 | \{0\}^i, s-1 \end{matrix} ; (\gamma, \alpha+\beta-\gamma, \alpha, \beta) \right)
\end{aligned} \tag{8}$$

for all $k \in \mathbb{Z}_{\ge0}$, $s \in \mathbb{Z}_{\ge2}$, $\alpha, \beta, \gamma \in \mathbb{C}$ *such that* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, $\operatorname{Re}(\alpha+\beta) > \operatorname{Re}\gamma > 0$.

(ii)

$$\begin{aligned}(9)\qquad &\sum_{\substack{r_1+\cdots+r_{k+s}=l\\ r_i\in\mathbb{Z}_{\geq 0}}} \Phi^{(k+1,s-1)}_{(\{r_i\}_{i=1}^{k+s})}(\{1\}^{\sum_{i=1}^{k+1}(r_i+1)}, \{\{1\}^{r_i},2\}_{i=k+2}^{k+s};(\alpha,\beta,\gamma))\\ &= \sum_{\substack{r_1+\cdots+r_{l+s}=k\\ r_i\in\mathbb{Z}_{\geq 0}}} \Phi^{(l+1,s-1)}_{(\{r_i\}_{i=1}^{l+s})}(\{1\}^{\sum_{i=1}^{l+1}(r_i+1)}, \{\{1\}^{r_i},2\}_{i=l+2}^{l+s};(\alpha,\beta,\alpha+\beta-\gamma))\end{aligned}$$

for all $k,l\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 2}$, $\alpha,\beta,\gamma\in\mathbb{C}$ *such that* $\operatorname{Re}\alpha,\operatorname{Re}\beta>0$, $\operatorname{Re}(\alpha+\beta)>\operatorname{Re}\gamma>0$.

Proof. Taking $l=0$ in (2), we get (8). The identity (9) can be proved as follows: By the replacements $k\leftrightarrow l$ and $\gamma\leftrightarrow\alpha+\beta-\gamma$ in the right-hand side of (2), we see that it has a symmetry with respect to k, l, γ, $\alpha+\beta-\gamma$. By this fact and (2), we get (9). □

Remark 2. Taking $l=0$ in (9), we get the following another expression for the multiple series on the left-hand side of (8):

$$\begin{aligned}(10)\qquad &\Phi^{\star}_{k+1,s-1}(\{1\}^{k+1},\{2\}^{s-1};(\alpha,\beta,\gamma))\\ &= \sum_{\substack{r_1+\cdots+r_s=k\\ r_i\in\mathbb{Z}_{\geq 0}}} \Phi^{(1,s-1)}_{(\{r_i\}_{i=1}^{s})}(\{1\}^{r_1+1},\{\{1\}^{r_i},2\}_{i=2}^{s};(\alpha,\beta,\alpha+\beta-\gamma))\end{aligned}$$

for all $k\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 2}$, $\alpha,\beta,\gamma\in\mathbb{C}$ such that $\operatorname{Re}\alpha,\operatorname{Re}\beta>0$, $\operatorname{Re}(\alpha+\beta)>\operatorname{Re}\gamma>0$. The symmetry of (9) with respect to α,β,γ comes from the symmetry of the left-hand side of the identity in Theorem A (ii) with respect to $b_1,\dots,b_s$, $c_0,\dots,c_s$.

Here we consider the multiple series

$$\begin{aligned}(11)\qquad &\Phi^{(p,q),\lesssim^{(p+q-1)}}_{A^{(p+q)},B^{(p+q)}}\left(\begin{matrix}\{s_i\}_{i=1}^{p+q}\\ \{a_i\}_{i=1}^{p+q}|\{b_i\}_{i=1}^{p+q}|\{c_i\}_{i=1}^{p+q}\end{matrix};(\alpha,\beta,\gamma,\delta)\right)\\ &:= \sum_{\substack{0=m_0\leq=\lesssim_0 m_1\lesssim_1\cdots\lesssim_{p-1}m_p\\ m_p\lesssim_p\cdots\lesssim_{p+q-1}m_{p+q}\lesssim_{p+q}=<\infty}} \frac{(\alpha)_{m_p}}{m_p!}\frac{(\beta)_{m_p}}{(\gamma)_{m_p}}\frac{m_{p+q}!}{(\alpha)_{m_{p+q}}}\frac{(\gamma)_{m_{p+q}}}{(\beta)_{m_{p+q}}}\\ &\quad\times\left\{\prod_{i=1}^{p}\frac{A_i(m_i+\gamma)^{s_i}+B_i(m_i+\delta)^{s_i}}{(m_i+\gamma)^{a_i}(m_i+\delta)^{b_i}(2m_i+\gamma+\delta)^{c_i}}\right\}\\ &\quad\times\left\{\prod_{i=p+1}^{p+q}\frac{A_i(m_i+\alpha)^{s_i}+B_i(m_i+\beta)^{s_i}}{(m_i+\alpha)^{a_i}(m_i+\beta)^{b_i}(2m_i+\alpha+\beta)^{c_i}}\right\},\end{aligned}$$

where $p,q,\alpha,\beta,\gamma,\delta$ are the same as those in the definition of (1); $a_i,b_i,c_i,s_i\in\mathbb{Z}$ such that $a_i+b_i+c_i-s_i\geq 1$ $(i=1,\dots,p+q-1)$, $a_{p+q}+b_{p+q}+c_{p+q}-s_{p+q}\geq 2$;

$A_i, B_i \in \mathbb{C}$; $A^{(p+q)} := \{A_i\}_{i=1}^{p+q}$, $B^{(p+q)} := \{B_i\}_{i=1}^{p+q}$; $\lesssim^{(p+q-1)} := \{\lesssim_i\}_{i=1}^{p+q-1}$. The multiple series (11) is a one-multiple series expression for a linear combination of the multiple series (1). We denote the case $q = 0$, $\lesssim_1 = \cdots = \lesssim_{p-1} = \lesssim$ $(\in \{<, \leq\})$ of (11) by

$$\phi^{p,\lesssim}_{A^{(p)},B^{(p)}}\left(\begin{matrix} \{s_i\}_{i=1}^{p} \\ \{a_i\}_{i=1}^{p} | \{b_i\}_{i=1}^{p} | \{c_i\}_{i=1}^{p} \end{matrix} ; (\gamma, \delta) \right). \tag{12}$$

Differentiating both sides of (8) with respect to α, β and γ, we can derive the following identities:

Corollary 2.4. *The following identities hold:*

(i)

$$\begin{aligned} &\sum_{\substack{r_1+\cdots+r_{s-1}=r \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi^{(k+1,r+s-1),\leq}(\{1\}^{k+r+s} | \{0\}^{k+1}, \{\{0\}^{r_i}, 1\}_{i=1}^{s-1}; (\alpha, \beta, \gamma)) \\ &= \sum\nolimits^{(k,0)} \sum_{\substack{r_1+\cdots+r_{i+2}=r \\ r_j \in \mathbb{Z}_{\geq 0}}} \binom{s-2+r_{i+2}}{r_{i+2}} \\ &\quad \times \phi_{i+1}\left(\begin{matrix} \{1-e(r_j)\}_{j=1}^{i+1} \\ \{k_j - e(r_j)\}_{j=1}^{i+1} | \{1+r_j\}_{j=1}^{i+1} | \{0\}^i, s-1+r_{i+2} | \{0\}^i, s-1 \end{matrix} ; \right. \\ &\qquad\qquad \left. (\gamma, \alpha+\beta-\gamma, \alpha, \beta) \right) \end{aligned} \tag{13}$$

for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re}\gamma > 0$, *where $\Phi^{(p,q),\leq}(\{a_i\}_{i=1}^{p+q} | \{b_i\}_{i=1}^{p+q}; (\alpha, \beta, \gamma))$ denotes the case $\delta = \gamma$, $\lesssim_i = \leq$ $(i = 1, \ldots, p+q-1)$, $c_i = s_i = A_i = B_i = 1$ $(i = 1, \ldots, p+q)$ of* (11).

(ii)

$$\begin{aligned} &\sum_{\substack{r_1+\cdots+r_{k+s}=r \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi^{\star}_{k+1,s-1}(\{r_i+1\}_{i=1}^{k+1}, \{r_i+2\}_{i=k+2}^{k+s}; (\alpha, \beta, \beta)) \\ &= \sum\nolimits^{(k,0)} \sum_{\substack{r_1+\cdots+r_{i+1}=r \\ r_j \in \mathbb{Z}_{\geq 0}}} \\ &\quad \phi^{i+1,<}_{A^{(i+1)},B^{(i+1)}}\left(\begin{matrix} \{1\}^{i+1} \\ \{1\}^i, s | \{k_j + r_j\}_{j=1}^{i}, k_{i+1} + r_{i+1} + s - 1 | \{0\}^{i+1} \end{matrix} ; \right. \\ &\qquad\qquad \left. (\alpha, \beta) \right) \end{aligned} \tag{14}$$

for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re}\alpha > 0$, $\beta \notin \mathbb{Z}_{\leq 0}$, *where* $A^{(i+1)} = \left\{\binom{k_j+r_j-1}{r_j}\right\}_{j=1}^{i}, \binom{k_{i+1}+r_{i+1}+s-2}{r_{i+1}}$; $B^{(i+1)} = \left\{\binom{k_j+r_j-2}{r_j}\right\}_{j=1}^{i}, \binom{k_{i+1}+r_{i+1}+s-3}{r_{i+1}}$.

The right-hand side of (14) *can be written as a linear combination of* $\zeta^{<}_{i+1,+}(\{a_j\}_{j=1}^{i+1}|\{b_j\}_{j=1}^{i+1};(\alpha,\beta))$, *explicitly.*

(*iii*)

$$\sum_{\substack{u_1+\cdots+u_{k+1}\\+r_1+\cdots+r_{s-1}=r\\r_i,u_i\in\mathbb{Z}_{\geq 0}}}(-1)^{\sum_{i=1}^{k+1}u_i}\Phi^{(k+1,s-1)}_{(\{0\}^{k+1},\{r_i\}_{i=1}^{s-1})}(\{1+u_i\}_{i=1}^{k+1},\{\{1\}^{r_i},2\}_{i=1}^{s-1};(\alpha,\beta,\gamma))$$

$$\begin{aligned}&=\sum\nolimits^{(k,0)}\sum_{\substack{u_1+\cdots+u_{i+1}\\+r_1+\cdots+r_{i+1}=r\\r_j,u_j\in\mathbb{Z}_{\geq 0}}}\left\{\prod_{j=1}^{i+1}(-1)^{u_j}\binom{k_j-1+u_j}{u_j}\right\}\\&\quad\times\phi_{i+1}\Bigg(\begin{matrix}\{1\}^{i+1}\\\{k_j+u_j\}_{j=1}^{i+1}|\{1+r_j\}_{j=1}^{i+1}|\{0\}^i,s-1|\{0\}^i,s-1\end{matrix};\\&\qquad(\gamma,\alpha+\beta-\gamma,\alpha,\beta)\Bigg)\end{aligned}\tag{15}$$

for all $k,r\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 2}$, $\alpha,\beta,\gamma\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,(\alpha+\beta)>\mathrm{Re}\,\gamma>0$. *The right-hand side of* (15) *can be written as a linear combination of* $\zeta^{<}_{i+1,+}(\{a_j\}_{j=1}^{i+1}|\{b_j\}_{j=1}^{i+1}|\{0\}^i,s-1|\{0\}^i,s-1;(\gamma,\alpha+\beta-\gamma,\alpha,\beta))$, *explicitly. In particular, the right-hand side of the case* $k=0$ *of* (15) *can be written as*

$$\zeta^{<}_{1,+}(r+1|s-1|s-1;(\alpha+\beta-\gamma,\alpha,\beta))+(-1)^r\zeta^{<}_{1,+}(r+1|s-1|s-1;(\gamma,\alpha,\beta)).\tag{16}$$

Proof. Using the identity

$$\frac{(\alpha)_{m_p}}{(\alpha)_{m_{p+q}}}\left(\prod_{i=1}^{p+q}\frac{1}{(m_i+\alpha)^{a_i}}\right)=\left(\prod_{i=1}^{p}\frac{1}{(m_i+\alpha)^{a_i}}\right)\left(\prod_{i=p+1}^{p+q}\frac{(\alpha)_{m_{i-1}}}{(\alpha)_{m_i+1}}\frac{1}{(m_i+\alpha)^{a_i-1}}\right),$$

we get the identities

$$\begin{aligned}&\frac{(-1)^r}{r!}\frac{\mathrm{d}^r}{\mathrm{d}\alpha^r}\left\{\frac{(\alpha)_{m_p}}{(\alpha)_{m_{p+q}}}\left(\prod_{i=1}^{p+q}\frac{1}{(m_i+\alpha)^{a_i}}\right)\right\}\\&=\sum_{\substack{u_1+\cdots+u_{p+q}\\+r_1+\cdots+r_q=r\\r_i,u_i\in\mathbb{Z}_{\geq 0}}}\left\{\prod_{i=1}^{p}\binom{a_i-1+u_i}{u_i}\frac{1}{(m_i+\alpha)^{a_i+u_i}}\right\}\\&\quad\times\left\{\prod_{i=p+1}^{p+q}\frac{(\alpha)_{m_{i-1}}}{(\alpha)_{m_i+1}}\binom{a_i-2+u_i}{u_i}\frac{1}{(m_i+\alpha)^{a_i-1+u_i}}\right.\\&\quad\left.\times\sum_{m_{i-1}\leq n_{i1}\leq\cdots\leq n_{ir_{i-p}}\leq m_i}\prod_{j=1}^{r_{i-p}}\frac{1}{n_{ij}+\alpha}\right\}\end{aligned}\tag{17}$$

$$= \frac{(\alpha)_{m_p}}{(\alpha)_{m_{p+q}}} \sum_{\substack{u_1+\cdots+u_{p+q}\\ +r_1+\cdots+r_q=r\\ r_i,u_i\in\mathbb{Z}_{\geq 0}}} \left\{\prod_{i=1}^{p} \binom{a_i-1+u_i}{u_i} \frac{1}{(m_i+\alpha)^{a_i+u_i}}\right\}$$

$$\times \left\{\prod_{i=p+1}^{p+q} \binom{a_i-2+u_i}{u_i} \frac{1}{(m_i+\alpha)^{a_i+u_i}}\right\}$$

$$\times \sum_{\substack{m_p\leq m_{11}\leq\cdots\leq m_{1r_1}\leq m_{p+1}\\ \vdots \\ m_{p+q-1}\leq m_{q1}\leq\cdots\leq m_{qr_q}\leq m_{p+q}}} \prod_{i=1}^{q}\prod_{j=1}^{r_i} \frac{1}{m_{ij}+\alpha}$$

for $p,q,r\in\mathbb{Z}_{\geq 0}$ and $m_p,\ldots,m_{p+q}\in\mathbb{Z}$ such that $0\leq m_p\leq\cdots\leq m_{p+q}$, where we regard m_0 as 0. Differentiating the left-hand side of (8) r times with respect to α and using the case $p=k+1$, $q=s-1$, $a_i=0$ $(i=1,\ldots,k+1)$, $a_i=1$ $(i=k+2,\ldots,k+s)$ of (17), we get the left-hand side of (13). On the other hand, using the expression

$$\phi_n\left(\begin{matrix} \{1\}^n \\ \{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n|\{0\}^{n-1},c|\{0\}^{n-1},d \end{matrix}; (x,y,z,w)\right)$$

$$= \sum_{0\leq m_1<\cdots<m_n<\infty} \left\{\prod_{i=1}^{n}\left(\frac{1}{(m_i+x)^{a_i-1}(m_i+y)^{b_i}} + \frac{1}{(m_i+x)^{a_i}(m_i+y)^{b_i-1}}\right)\right\} \tag{18}$$

$$\times \frac{1}{(m_n+z)^c(m_n+w)^d}$$

and differentiating the right-hand side of (8) r times with respect to α, we can get the right-hand side of (13).

The identity (14) can be proved as follows: Using (18) and differentiating both sides of the case $\gamma=\beta$ of (8) r times with respect to β, we get the identity

$$\sum_{\substack{r_1+\cdots+r_{k+s}=r\\ r_i\in\mathbb{Z}_{\geq 0}}} \Phi^{\star}_{k+1,s-1}(\{r_i+1\}_{i=1}^{k+1},\{r_i+2\}_{i=k+2}^{k+s};(\alpha,\beta,\beta))$$

$$= \sum^{(k,0)} \sum_{\substack{r_1+\cdots+r_{i+1}=r\\ r_j\in\mathbb{Z}_{\geq 0}}} \sum_{0\leq m_1<\cdots<m_{i+1}<\infty} \tag{19}$$

$$\left\{\prod_{j=1}^{i}\left(\frac{\binom{k_j+r_j-1}{r_j}}{(m_j+\beta)^{k_j+r_j}} + \frac{\binom{k_j+r_j-2}{r_j}}{(m_j+\beta)^{k_j+r_j-1}(m_j+\alpha)}\right)\right\}$$

$$\times \left(\frac{\binom{k_{i+1}+r_{i+1}+s-2}{r_{i+1}}}{(m_{i+1}+\beta)^{k_{i+1}+r_{i+1}+s-1}(m_{i+1}+\alpha)^{s-1}} + \frac{\binom{k_{i+1}+r_{i+1}+s-3}{r_{i+1}}}{(m_{i+1}+\beta)^{k_{i+1}+r_{i+1}+s-2}(m_{i+1}+\alpha)^{s}}\right)$$

for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re}\alpha > 0$, $\beta \notin \mathbb{Z}_{\leq 0}$. The right-hand side of (19) is the same as that of (14). Further, applying the identity

$$\prod_{i=1}^{n}(X_i + Y_i) = \sum_{\substack{s_i+t_i=1 \\ s_i,t_i\in\{0,1\} \\ (i=1,\ldots,n)}} \prod_{i=1}^{n} X_i^{s_i} Y_i^{t_i} \tag{20}$$

($n \in \mathbb{Z}_{\geq 1}$) to the products on the right-hand side of (19), we can write the right-hand side of (19) as a linear combination of $\zeta^{<}_{i+1,+}(\{a_j\}_{j=1}^{i+1}|\{b_j\}_{j=1}^{i+1};(\alpha,\beta))$, explicitly.

The identity (15) can be proved in a way similar to that of proving (13). Indeed, using the identity

$$\frac{(\gamma)_{m_{p+q}}}{(\gamma)_{m_p}} = \prod_{i=p}^{p+q-1} \frac{(\gamma)_{m_{i+1}}}{(\gamma)_{m_i}},$$

we get the identities

$$\begin{aligned}
&\frac{1}{r!}\frac{\mathrm{d}^r}{\mathrm{d}\gamma^r}\left\{\frac{(\gamma)_{m_{p+q}}}{(\gamma)_{m_p}}\left(\prod_{i=1}^{p+q}\frac{1}{(m_i+\gamma)^{c_i}}\right)\right\} \\
&= \sum_{\substack{u_1+\cdots+u_{p+q} \\ +r_1+\cdots+r_q=r \\ r_i,u_i\in\mathbb{Z}_{\geq 0}}} \left\{\prod_{i=1}^{p+q}\binom{c_i-1+u_i}{u_i}\frac{(-1)^{u_i}}{(m_i+\gamma)^{c_i+u_i}}\right\} \\
&\quad \times \left\{\prod_{i=p}^{p+q-1}\frac{(\gamma)_{m_{i+1}}}{(\gamma)_{m_i}} \sum_{m_i\leq n_{i1}<\cdots<n_{ir_{i-p+1}}<m_{i+1}} \prod_{j=1}^{r_{i-p+1}}\frac{1}{n_{ij}+\gamma}\right\} \\
&= \frac{(\gamma)_{m_{p+q}}}{(\gamma)_{m_p}} \sum_{\substack{u_1+\cdots+u_{p+q} \\ +r_1+\cdots+r_q=r \\ r_i,u_i\in\mathbb{Z}_{\geq 0}}} \left\{\prod_{i=1}^{p+q}\binom{c_i-1+u_i}{u_i}\frac{(-1)^{u_i}}{(m_i+\gamma)^{c_i+u_i}}\right\} \\
&\quad \times \sum_{\substack{m_p\leq m_{11}<\cdots<m_{1r_1}<m_{p+1} \\ \vdots \\ m_{p+q-1}\leq m_{q1}<\cdots<m_{qr_q}<m_{p+q}}} \prod_{i=1}^{q}\prod_{j=1}^{r_i}\frac{1}{m_{ij}+\gamma}
\end{aligned} \tag{21}$$

for $p, q, r \in \mathbb{Z}_{\geq 0}$ and $m_p, \ldots, m_{p+q} \in \mathbb{Z}$ such that $0 \leq m_p \leq \cdots \leq m_{p+q}$, where we regard m_0 as 0. Differentiating both sides of (8) r times with respect to γ and using the case $p = k+1$, $q = s-1$, $c_i = 1$ $(i = 1, \ldots, k+1)$, $c_i = 0$ $(i = k+2, \ldots, k+s)$ of (21), we get (15). Using (18) and (20), we can write the right-hand side of (15) as a linear combination of $\zeta^{<}_{i+1,+}(\{a_j\}_{j=1}^{i+1}|\{b_j\}_{j=1}^{i+1}|\{0\}^i, s-1|\{0\}^i, s-1;(\gamma,\alpha+\beta-\gamma,\alpha,\beta))$, explicitly. The expression (16) can be proved by differentiating both sides of the case $n = 1$, $a_1 = b_1 = 1$, $c = d = s - 1$, $x = \gamma$, $y = \alpha + \beta - \gamma$, $z = \alpha$, $w = \beta$ of (18) r times with respect to γ. $\square$

Here we consider the multiple series

(22)
$$\sum_{0 \leq m_1 \lesssim \cdots \lesssim m_n < \infty} (\pm 1)^{m_n} \frac{(y)_{m_n}}{(x)_{m_n}} \left\{ \prod_{i=1}^{n} \frac{(2m_i + x + y)^{s_i}}{(m_i + x)^{a_i} (m_i + y)^{b_i} (m_i + z)^{c_i} (m_i + w)^{d_i}} \right\},$$

where $n \in \mathbb{Z}_{\geq 1}$, $x, y, z, w \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $a_i, b_i, c_i, d_i, s_i \in \mathbb{Z}$ such that $a_i + b_i + c_i + d_i - s_i \geq 1$ $(i = 1, \ldots, n-1)$, $a_n + b_n + c_n + d_n - s_n + \mathrm{Re}\,(x - y) > 1$ and $(x+y)/2 \notin \mathbb{Z}_{\leq 0}$ in case of $s_i < 0$ for some i. The symbol $(a)_m$ is the Pochhammer symbol. We denote the case $\lesssim = <$ of (22) by

$$\psi_{n,\pm} \left(\begin{matrix} \{s_i\}_{i=1}^n \\ \{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{c_i\}_{i=1}^n | \{d_i\}_{i=1}^n \end{matrix} ; (x, y, z, w) \right).$$

Remark 3. Coppo [8], Coppo and Candelpergher [9], Émery [10], Hasse [14] studied relations between special values of the multiple series

(23)
$$\sum_{0 \leq m_1 \lesssim \cdots \lesssim m_n < \infty} z^{m_n} \frac{m_n!}{(\alpha)_{m_n}} \left\{ \prod_{i=1}^{n} \frac{1}{(m_i + \alpha)^{a_i} (m_i + 1)^{b_i}} \right\}$$

and special values of the Hurwitz–Lerch zeta function $\sum_{m=0}^{\infty} z^{m+1} (m + \alpha)^{-s}$. See also the part **(R3)** of Section 3, Subsection 2.3, and [16]. The multiple series (22) and (41) below give us extensions of the multiple series (23) with $z = \pm 1$.

We can prove the following identity, which is similar to (2) with $\gamma = \alpha + \beta - 1$:

Theorem 2.5. *The identity*

(24)
$$\sum_{\substack{r_1 + \cdots + r_s = l \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi^{(1, s-1)}_{(\{r_i\}_{i=1}^s)} (\{1\}^{r_1}, k + 2, \{\{1\}^{r_i}, 2\}_{i=2}^s; (\alpha, \beta, \alpha + \beta - 1))$$
$$= \sum^{(k,l)} \psi_{i+1,-} \left(\begin{matrix} \{e(k_j) + e(l_j)\}_{j=1}^i, 1 \\ \{k_j + e(l_j)\}_{j=1}^i, k_{i+1} + 1 | \{l_j + e(k_j)\}_{j=1}^i, l_{i+1} | \{0\}^i, s - 1 | \{0\}^i, s - 1 \end{matrix} ; (\alpha + \beta - 1, 1, \alpha, \beta) \right)$$

holds for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ *such that* $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,(\alpha + \beta) > 1$.

Proof. Taking $a = \alpha + \beta$, $b_1 = \alpha + \beta - z$, $b_i = \alpha$ $(i = 2, \ldots, s)$, $b_{s+1} = w$, $c_1 = 1$, $c_i = \beta$ $(i = 2, \ldots, s)$, $c_{s+1} = 1$, where $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta, z, w \in \mathbb{C}$ such that

$\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,(\alpha+\beta) > 1$, $\mathrm{Re}\,(\alpha+\beta) > \mathrm{Re}\,z > 0$, $\mathrm{Re}\,(\alpha+\beta) > \mathrm{Re}\,w > 0$, $2s-3 > \mathrm{Re}\,(2(w-z)-\alpha-\beta)$, in Theorem A (i) and multiplying both sides of the result by $(\alpha+\beta-1)^{-1}$, we get the identity

$$\begin{aligned}
&\sum_{0\le m_1\le\cdots\le m_s<\infty} \frac{(\alpha)_{m_1}}{m_1!}\frac{(\beta)_{m_1}}{(\alpha+\beta-1)_{m_1}}\frac{m_s!}{(\alpha)_{m_s}}\frac{(w)_{m_s}}{(\beta)_{m_s}} \\
&\quad\times \frac{1}{(m_1+\alpha+\beta-1)(m_1+z)}\left\{\prod_{i=2}^{s}\frac{1}{(m_i+\alpha)(m_i+\beta)}\right\} \\
&= \sum_{m=0}^{\infty}\frac{(\alpha+\beta-z)_m}{(z)_{m+1}}\frac{(w)_m}{(\alpha+\beta-w)_{m+1}} \\
&\quad\times \frac{m!}{(\alpha+\beta-1)_{m+1}}\frac{2m+\alpha+\beta}{(m+\alpha)^{s-1}(m+\beta)^{s-1}}(-1)^m.
\end{aligned} \tag{25}$$

Applying the operator $(-1)^k\partial^{(l,k)}(w,z)|^{w=\alpha+\beta-1}_{z=\alpha+\beta-1}$ ($k,l\in\mathbb{Z}_{\ge0}$; $\alpha,\beta\in\mathbb{C}$ such that $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,(\alpha+\beta) > 1$) to the left-hand side of (25) and using (6), we get the left-hand side of (24). The right-hand side of (24) can be proved by applying $(-1)^k\partial^{(l,k)}(w,z)|^{w=\alpha+\beta-1}_{z=\alpha+\beta-1}$ to the right-hand side of (25) and using the case $\gamma=\delta=\alpha+\beta-1$ of (7). □

We can derive the following identities from (24):

Corollary 2.6. *The following identities hold:*

(*i*)

$$\begin{aligned}
&\Phi^{\star}_{1,s-1}(k+2,\{2\}^{s-1};(\alpha,\beta,\alpha+\beta-1)) \\
&=\sum\nolimits^{(k,0)}\psi_{i+1,-}\left(\begin{matrix}\{1\}^{i+1}\\ \{k_j\}_{j=1}^{i},k_{i+1}+1|\{1\}^{i+1}|\{0\}^{i},s-1|\{0\}^{i},s-1\end{matrix};\right. \\
&\qquad\qquad\left.(\alpha+\beta-1,1,\alpha,\beta)\right)
\end{aligned} \tag{26}$$

for all $k\in\mathbb{Z}_{\ge0}$, $s\in\mathbb{Z}_{\ge1}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,(\alpha+\beta) > 1$.

(*ii*)

$$\begin{aligned}
&\zeta^{<}_{l+1,+}(\{1\}^l,k+2;\alpha) \\
&=\sum\nolimits^{(k,l)}\psi_{i+1,-}\left(\begin{matrix}\{e(k_j)+e(l_j)\}_{j=1}^{i},1\\ \{k_j+e(l_j)\}_{j=1}^{i},k_{i+1}+1|\{l_j+e(k_j)\}_{j=1}^{i},l_{i+1}\end{matrix};(\alpha,1)\right)
\end{aligned} \tag{27}$$

for all $k,l\in\mathbb{Z}_{\ge0}$, $\alpha\in\mathbb{C}$ *with* $\mathrm{Re}\,\alpha>0$, *where*

$$\psi_{n,-}\left(\begin{matrix}\{s_i\}_{i=1}^{n}\\ \{a_i\}_{i=1}^{n}|\{b_i\}_{i=1}^{n}\end{matrix};(x,y)\right) := \psi_{n,-}\left(\begin{matrix}\{s_i\}_{i=1}^{n}\\ \{a_i\}_{i=1}^{n}|\{b_i\}_{i=1}^{n}|\{0\}^n|\{0\}^n\end{matrix};(x,y,z,w)\right).$$

(*iii*)

$$\zeta(\{1\}^l,k+2)=\sum\nolimits^{(k,l)}2^{\sum_{j=1}^{i}\{e(k_j)+e(l_j)\}+1}\zeta_{-}(\{k_j+l_j\}_{j=1}^{i+1}) \tag{28}$$

for all $k,l\in\mathbb{Z}_{\ge0}$.

Proof. Taking $l = 0$ in (24), we get (26). Taking $\beta = s = 1$ in (24) and $\alpha = \beta = s = 1$ in (24), we get (27) and (28), respectively. □

Taking $\alpha = \beta = \gamma = 1$ in (8) and (26), we get (29) and (30) below, respectively:

Corollary 2.7. *The following two identities hold*:

(*i*)

$$\zeta^\star(\{1\}^{r+1}, \{2\}^{s-1}) = \sum_{i=0}^{r} \sum_{\substack{r_1+\cdots+r_{i+1}=r+1 \\ r_j \in \mathbb{Z}_{\geq 1}}} 2^{i+1} \zeta(\{r_j\}_{j=1}^{i}, r_{i+1} + 2s - 2) \tag{29}$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$.

(*ii*)

$$\zeta^\star(r+2, \{2\}^{s-1}) = \sum_{i=0}^{r} \sum_{\substack{r_1+\cdots+r_{i+1}=r+1 \\ r_j \in \mathbb{Z}_{\geq 1}}} 2^{i+1} \zeta_-(\{r_j\}_{j=1}^{i}, r_{i+1} + 2s - 1) \tag{30}$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$.

Remark 4. The identities (29) and (30) were proved in [19]. The identity (29) is the two-one formula for $\zeta^\star(\{1\}^{s_1}, \{2\}^{s_2})$ $(s_1, s_2 \in \mathbb{Z}_{\geq 1})$. The two-one formula for MZSVs was conjectured by Ohno and Zudilin in [26, pp. 327–328]. The general case of the two-one formula was proved by Zhao in [31, Theorem 1.1]. The identity (30) seems to give a solution to the project of Ohno and Zudilin stated in [26, p. 326]. In [19], I proved (30) to solve my observation for $\zeta^\star(s_1 + 2, \{2\}^{s_2})$ $(s_1, s_2 \in \mathbb{Z}_{\geq 0})$ stated in [19, (**A4**)]. I remark that the observation for $\zeta^\star(s_1 + 2, \{2\}^{s_2})$ comes from the similarity between (31) and (32) below, and my observation for the right-hand side of (31) below stated in [19, (**A3**)]:

$$\zeta^\star(\{1\}^{r+1}, \{2\}^{s}) = 2 \sum_{m=0}^{\infty} \frac{1}{(m+1)^{2s+1}} \left(\sum_{i=0}^{r} S_m^{<}(\{1\}^{r-i}) S_m^{\leq}(\{1\}^{i}) \right), \tag{31}$$

$$\zeta^\star(r+2, \{2\}^{s-1}) = 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^{2s}} \left(\sum_{i=0}^{r} S_m^{<}(\{1\}^{r-i}) S_m^{\leq}(\{1\}^{i}) \right) \tag{32}$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, where

$$S_m^{\lesssim}(\{k_i\}_{i=1}^{n}) := \sum_{0 \leq m_1 \lesssim \cdots \lesssim m_n \lesssim m} \prod_{i=1}^{n} \frac{1}{(m_i+1)^{k_i}}$$

$(n \in \mathbb{Z}_{\geq 1};\ k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1})$, $S_m^{\lesssim}(\emptyset) := 1$. The identities (31) and (32) were also derived in [19] by using Theorem A.

We can prove the following two identities for (11) also:

Theorem 2.8. *The following two identities hold*:

(*i*)

$$\sum_{i=0}^{k}\sum_{\substack{k_1+\cdots+k_{i+1}=k+1\\ k_j\in\mathbb{Z}_{\geq 1}(j=1,\ldots,i+1)}} \Phi^{(i+1,s-1),\lesssim^{(i+s-1)}}_{A^{(i+s)},B^{(i+s)}}\left(\begin{matrix}\{k_j+1\}_{j=1}^{i},k_{i+1},\{1\}^{s-1}\\ \{k_j\}_{j=1}^{i+1},\{1\}^{s-1}|\{k_j\}_{j=1}^{i+1},\{1\}^{s-1}|\{1\}^{i+s}\end{matrix};(\alpha,\beta,\alpha+\beta-1,1)\right)$$
$$=\psi_{1,-}(1|k+1|s-1|s-1;(1,\alpha+\beta-1,\alpha,\beta)) + (-1)^k\psi_{1,-}(k+2|0|s-1|s-1;(1,\alpha+\beta-1,\alpha,\beta)) \tag{33}$$

for all $k\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $2s+1>\mathrm{Re}\,(\alpha+\beta)>1$, *where* $A^{(i+s)}=\{(-1)^{k_j}\}_{j=1}^{i},(-1)^{k_{i+1}-1},\{1\}^{s-1}$; $B^{(i+s)}=\{1\}^{i+s}$; $\lesssim^{(i+s-1)}=\{<\}^{i},\{\leq\}^{s-1}$ *and*

$$\psi_{n,\pm}(\{a_i\}_{i=1}^{n}|\{b_i\}_{i=1}^{n}|\{c_i\}_{i=1}^{n}|\{d_i\}_{i=1}^{n};(x,y,z,w)) :=\psi_{n,\pm}\left(\begin{matrix}\{0\}^{n}\\ \{a_i\}_{i=1}^{n}|\{b_i\}_{i=1}^{n}|\{c_i\}_{i=1}^{n}|\{d_i\}_{i=1}^{n}\end{matrix};(x,y,z,w)\right).$$

(*ii*)

$$\sum_{i=0}^{k}\sum_{\substack{k_1+\cdots+k_{i+1}=k+1\\ k_j\in\mathbb{Z}_{\geq 1}(j=1,\ldots,i+1)}} \Phi^{(1,i+s-1),\lesssim^{(i+s-1)}}_{A^{(i+s)},B^{(i+s)}}\left(\begin{matrix}1,\{k_j+1\}_{j=1}^{i},k_{i+1},\{1\}^{s-2}\\ 1,\{k_j\}_{j=1}^{i+1},\{1\}^{s-2}|0,\{k_j\}_{j=1}^{i+1},\{1\}^{s-2}|\{1\}^{i+s}\end{matrix};(\alpha,\beta,\gamma,\gamma)\right)$$
$$=\zeta^{<}_{1,+}(k+s-1|s-2|1|1;(\alpha,\beta,\gamma,\alpha+\beta-\gamma)) + (-1)^k\zeta^{<}_{1,+}(s-2|k+s-1|1|1;(\alpha,\beta,\gamma,\alpha+\beta-\gamma)) \tag{34}$$

for all $k\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 2}$, $\alpha,\beta,\gamma\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,(\alpha+\beta)>\mathrm{Re}\,\gamma>0$, *where* $A^{(i+s)}=1,\{(-1)^{k_j}\}_{j=1}^{i},(-1)^{k_{i+1}-1},\{1\}^{s-2}$; $B^{(i+s)}=\{1\}^{i+s}$; $\lesssim^{(i+s-1)}=\leq,\{<\}^{i},\{\leq\}^{s-2}$.

Proof. Taking $a=\alpha+\beta$, $b_1=\alpha+\beta-z$, $c_1=z$, $b_i=\alpha$, $c_i=\beta$ $(i=2,\ldots,s)$, $b_{s+1}=1$, $c_{s+1}=\alpha+\beta-1$, where $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta,z\in\mathbb{C}$ such that $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$,

$2s+1 > \mathrm{Re}\,(\alpha+\beta) > 1$, $\mathrm{Re}\,(\alpha+\beta) > \mathrm{Re}\, z > 0$, in Theorem A (i), we get the identity

$$\begin{aligned}
&\sum_{0\le m_1\le\cdots\le m_s<\infty} \frac{m_1!}{(z)_{m_1+1}} \frac{(\alpha+\beta-1)_{m_1}}{(\alpha+\beta-z)_{m_1+1}} \frac{(\alpha)_{m_1}}{m_1!} \frac{(\beta)_{m_1}}{(\alpha+\beta-1)_{m_1}} \\
&\times \frac{m_s!}{(\alpha)_{m_s}} \frac{(\alpha+\beta-1)_{m_s}}{(\beta)_{m_s}} \left\{ \prod_{i=2}^{s} \frac{1}{(m_i+\alpha)(m_i+\beta)} \right\} \\
&= \sum_{m=0}^{\infty} \frac{(\alpha+\beta-1)_m}{(m+1)!} \frac{(-1)^m}{(m+\alpha)^{s-1}(m+\beta)^{s-1}} \left(\frac{1}{m+z} + \frac{1}{m+\alpha+\beta-z} \right).
\end{aligned} \tag{35}$$

Taking $a = X$, $b = Y$, $c = 1$, $d = \alpha+\beta-1$, $Z = \alpha+\beta-z$, $W = z$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$\begin{aligned}
&\frac{m_1!}{(z)_{m_1+1}} \frac{(\alpha+\beta-1)_{m_1}}{(\alpha+\beta-z)_{m_1+1}} \\
&= \left(\frac{1}{(z+m_1)(\alpha+\beta+2m_1)} + \frac{1}{(\alpha+\beta-z+m_1)(\alpha+\beta+2m_1)} \right) \\
&\times \sum_{i=0}^{m_1} \sum_{0\le n_1<\cdots<n_i<m_1} \prod_{j=1}^{i} \left(-\frac{(z-\alpha-\beta+1)(1+n_j)}{(z+n_j)(\alpha+\beta+2n_j)} \right. \\
&\left. + \frac{(z-\alpha-\beta+1)(\alpha+\beta-1+n_j)}{(\alpha+\beta-z+n_j)(\alpha+\beta+2n_j)} \right).
\end{aligned}$$

Using this identity, we get the identity

$$\begin{aligned}
&(-1)^k \partial^{(k)}(z) \left(\frac{m_1!}{(z)_{m_1+1}} \frac{(\alpha+\beta-1)_{m_1}}{(\alpha+\beta-z)_{m_1+1}} \right) \Big|_{z=\alpha+\beta-1} \\
&= \sum_{i=0}^{m_1} \sum_{\substack{k_1+\cdots+k_{i+1}=k \\ k_j\in\mathbb{Z}_{\ge1} (j=1,\ldots,i); \\ k_{i+1}\in\mathbb{Z}_{\ge0}}} \sum_{0\le n_1<\cdots<n_i<m_1} \\
&\left\{ \prod_{j=1}^{i} \left(\frac{1+n_j}{(\alpha+\beta-1+n_j)^{k_j}(\alpha+\beta+2n_j)} + \frac{(-1)^{k_j}(\alpha+\beta-1+n_j)}{(1+n_j)^{k_j}(\alpha+\beta+2n_j)} \right) \right\} \\
&\times \left(\frac{1}{(\alpha+\beta-1+m_1)^{k_{i+1}+1}(\alpha+\beta+2m_1)} + \frac{(-1)^{k_{i+1}}}{(1+m_1)^{k_{i+1}+1}(\alpha+\beta+2m_1)} \right)
\end{aligned} \tag{36}$$

for $k, m_1 \in \mathbb{Z}_{\ge0}$, $\alpha, \beta \in \mathbb{C}$ such that $(\alpha+\beta)/2, \alpha+\beta-1 \notin \mathbb{Z}_{\le0}$. Thus, applying the operator $(-1)^k\partial^{(k)}(z)|_{z=\alpha+\beta-1}$ ($k \in \mathbb{Z}_{\ge0}$; $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$, $2s+1 > \mathrm{Re}\,(\alpha+\beta) > 1$) to the left-hand side of (35) and using (36), we get the left-hand side of (33). The right-hand side of (33) can be proved by applying $(-1)^k\partial^{(k)}(z)|_{z=\alpha+\beta-1}$ to the right-hand side of (35).

The identity (34) can be proved in a way similar to the above. Indeed, taking $a = \alpha+\beta$, $c_0 = \alpha+\beta-\gamma$, $b_1 = z$, $c_1 = \alpha+\beta-z$, $b_i = \alpha$, $c_i = \beta$

$(i = 2, \dots, s-1)$, $b_s = 1$, $c_s = \gamma$, where $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma, z \in \mathbb{C}$ such that $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, $\operatorname{Re}(\alpha+\beta) > \operatorname{Re}\gamma > 0$, $\operatorname{Re}(\alpha+\beta) > \operatorname{Re} z > 0$, in Theorem A (ii), we get the identity

$$\begin{aligned}
&\sum_{0 \leq m_1 \leq \cdots \leq m_s < \infty} \frac{(z)_{m_1}}{(z)_{m_2+1}} \frac{(\alpha+\beta-z)_{m_1}}{(\alpha+\beta-z)_{m_2+1}} \frac{(\alpha)_{m_2}}{(\alpha)_{m_1}} \frac{(\beta)_{m_2}}{(\beta)_{m_1}} \\
&\quad \times \frac{(\alpha)_{m_1}}{m_1!} \frac{(\beta)_{m_1}}{(\gamma)_{m_1+1}} \frac{m_s!}{(\alpha)_{m_s}} \frac{(\gamma)_{m_s}}{(\beta)_{m_s}} \left\{ \prod_{i=3}^{s} \frac{1}{(m_i+\alpha)(m_i+\beta)} \right\} \\
&= \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{s-2}(m+\beta)^{s-2}(m+\gamma)(m+\alpha+\beta-\gamma)} \\
&\quad \times \left(\frac{1}{m+z} + \frac{1}{m+\alpha+\beta-z} \right).
\end{aligned} \tag{37}$$

Using the case $m = m_2 - m_1$, $a = c = X = \beta + m_1$, $b = Y$, $d = \alpha + m_1$, $Z = \alpha + \beta - z + m_1$, $W = z + m_1$ of Lemma 2.1, where $m_1, m_2 \in \mathbb{Z}$ such that $0 \leq m_1 \leq m_2$, we get the identities

$$\begin{aligned}
&\frac{(z)_{m_1}}{(z)_{m_2+1}} \frac{(\alpha+\beta-z)_{m_1}}{(\alpha+\beta-z)_{m_2+1}} \frac{(\alpha)_{m_2}}{(\alpha)_{m_1}} \frac{(\beta)_{m_2}}{(\beta)_{m_1}} \\
&= \frac{(\alpha+m_1)_{m_2-m_1}}{(z+m_1)_{m_2-m_1+1}} \frac{(\beta+m_1)_{m_2-m_1}}{(\alpha+\beta-z+m_1)_{m_2-m_1+1}} \\
&= \left(\frac{1}{(z+m_2)(\alpha+\beta+2m_2)} + \frac{1}{(\alpha+\beta-z+m_2)(\alpha+\beta+2m_2)} \right) \\
&\quad \times \sum_{i=0}^{m_2-m_1} \sum_{m_1 \leq n_1 < \cdots < n_i < m_2} \prod_{j=1}^{i} \left(-\frac{(z-\alpha)(\beta+n_j)}{(z+n_j)(\alpha+\beta+2n_j)} \right. \\
&\quad \left. + \frac{(z-\alpha)(\alpha+n_j)}{(\alpha+\beta-z+n_j)(\alpha+\beta+2n_j)} \right).
\end{aligned} \tag{38}$$

Using (38), we get the identity

$$\begin{aligned}
&(-1)^k \partial^{(k)}(z) \left(\frac{(z)_{m_1}}{(z)_{m_2+1}} \frac{(\alpha+\beta-z)_{m_1}}{(\alpha+\beta-z)_{m_2+1}} \frac{(\alpha)_{m_2}}{(\alpha)_{m_1}} \frac{(\beta)_{m_2}}{(\beta)_{m_1}} \right) \Big|_{z=\alpha} \\
&= \sum_{i=0}^{m_2-m_1} \sum_{\substack{k_1+\cdots+k_{i+1}=k \\ k_j \in \mathbb{Z}_{\geq 1} (j=1,\dots,i); \\ k_{i+1} \in \mathbb{Z}_{\geq 0}}} \sum_{m_1 \leq n_1 < \cdots < n_i < m_2}
\end{aligned} \tag{39}$$

$$\begin{aligned}
&\left\{ \prod_{j=1}^{i} \left(\frac{\beta+n_j}{(\alpha+n_j)^{k_j}(\alpha+\beta+2n_j)} + \frac{(-1)^{k_j}(\alpha+n_j)}{(\beta+n_j)^{k_j}(\alpha+\beta+2n_j)} \right) \right\} \\
&\times \left(\frac{1}{(\alpha+m_2)^{k_{i+1}+1}(\alpha+\beta+2m_2)} + \frac{(-1)^{k_{i+1}}}{(\beta+m_2)^{k_{i+1}+1}(\alpha+\beta+2m_2)} \right)
\end{aligned}$$

for $k, m_1, m_2 \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ such that $0 \le m_1 \le m_2$ and $\alpha, \beta, (\alpha+\beta)/2 \notin \mathbb{Z}_{\le 0}$. Thus, applying the operator $(-1)^k \partial^{(k)}(z)|_{z=\alpha}$ ($k \in \mathbb{Z}_{\ge 0}$; $\alpha \in \mathbb{C}$ such that $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,(\alpha+\beta) > \mathrm{Re}\,\gamma > 0$) to the left-hand side of (37) and using (39), we get the left-hand side of (34). The right-hand side of (34) can be proved by applying $(-1)^k \partial^{(k)}(z)|_{z=\alpha}$ to the right-hand side of (37). $\square$

Remark 5. We consider the following special case of (1):

$$\begin{aligned}
&\Phi^{(p,q)}_{\mathbf{2},(\{r_i\}_{i=1}^{p+q})}(\{\{a_{ij}\}_{j=1}^{r_i}, A_i\}_{i=1}^{p+q} | \{\{b_{ij}\}_{j=1}^{r_i}, B_i\}_{i=1}^{p+q}; (\alpha,\beta,\gamma,\delta)) \\
&:= \sum_{\substack{0=m_0 \le m_{11} < \cdots < m_{1r_1} < m_1 \\ \vdots \\ m_{p-1} \le m_{p1} < \cdots < m_{pr_p} < m_p \\ \vdots \\ m_{p+q-1} \le m_{p+q1} < \cdots < m_{p+qr_{p+q}} < m_{p+q} < \infty}} \frac{(\alpha)_{m_p}}{m_p!} \frac{(\beta)_{m_p}}{(\gamma)_{m_p}} \frac{m_{p+q}!}{(\alpha)_{m_{p+q}}} \frac{(\gamma)_{m_{p+q}}}{(\beta)_{m_{p+q}}} \\
&\quad \times \left\{ \prod_{i=1}^{p} \frac{1}{(m_i+\gamma)^{A_i}(m_i+\delta)^{B_i}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij}+\gamma)^{a_{ij}}(m_{ij}+\delta)^{b_{ij}}} \right\} \\
&\quad \times \left\{ \prod_{i=p+1}^{p+q} \frac{1}{(m_i+\alpha)^{A_i}(m_i+\beta)^{B_i}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij}+\gamma)^{a_{ij}}(m_{ij}+\delta)^{b_{ij}}} \right\},
\end{aligned}$$

where $p, q \in \mathbb{Z}_{\ge 0}$ such that $p+q \ge 1$; $r_i \in \mathbb{Z}_{\ge 0}$, $a_{ij}, b_{ij}, A_i, B_i \in \mathbb{Z}$. We remark that the left-hand sides of the identities (2), (13)–(15), (24) can be rewritten in this multiple series. Using Theorem A (i), we can prove the following identity also:

$$\begin{aligned}
&\sum_{i=0}^{k} \sum_{\substack{k_1+\cdots+k_{i+1}=k+1 \\ l_1+\cdots+l_{i+1}=l \\ k_j \in \mathbb{Z}_{\ge 1}, l_j \in \mathbb{Z}_{\ge 0} \\ (j=1,\ldots,i+1)}} (-1)^i (\alpha+\beta-2)^{\sum_{j=1}^{i}\{1-e(l_j)\}} \\
&\times \Phi^{(1,s-1)}_{\mathbf{2},(i,\{0\}^{s-1})}(\{k_j - e(l_j)\}_{j=1}^{i}, k_{i+1}, \{1\}^{s-1} | \{l_j+1\}_{j=1}^{i+1}, \{1\}^{s-1}; \\
&\qquad (\alpha,\beta,\alpha+\beta-1,1))
\end{aligned} \tag{40}$$

$$\begin{aligned}
&= \sum_{i=0}^{k+l} \sum_{\substack{k_1+\cdots+k_{i+1}=k+1 \\ l_1+\cdots+l_i+r_2+\cdots+r_s=l \\ k_j \in \mathbb{Z}_{\ge 0}, l_j \in \{0,1\}, k_j+l_j \ge 1 \\ (j=1,\ldots,i); k_{i+1} \in \mathbb{Z}_{\ge 1}, \\ r_2,\ldots,r_s \in \mathbb{Z}_{\ge 0}}} (2-\alpha-\beta)^{\sum_{j=1}^{i} e(k_j)} \\
&\times \Phi^{(1,s-1)}_{\mathbf{2},(i,\{r_j\}_{j=2}^{s})}(\{k_j+e(l_j)\}_{j=1}^{i}, k_{i+1}, \{1\}^{\sum_{j=2}^{s}(r_j+1)} | \\
&\{e(k_j)\}_{j=1}^{i+1}, \{\{0\}^{r_j}, 1\}_{j=2}^{s}; (\alpha,\beta,\alpha+\beta-1,1))
\end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$, $2s+1 > \mathrm{Re}\,(\alpha+\beta) > 1$. The identity (40) can be proved in a way similar to the proof of Theorem 2.15 below. Taking $k = 0$ in (40), we get the following identity, which is similar to (10):

$$\begin{aligned} &\Phi^{(1,s-1)}_{\mathbf{2},(\{0\}^s)}(\{1\}^s | l+1, \{1\}^{s-1}; (\alpha, \beta, \alpha+\beta-1, 1)) \\ &= \sum_{\substack{r_1+\cdots+r_s=l \\ r_j \in \mathbb{Z}_{\geq 0}}} \Phi^{(1,s-1)}_{\mathbf{2},(\{r_j\}_{j=1}^s)}(\{1\}^{\sum_{j=1}^s (r_j+1)} | \{\{0\}^{r_j}, 1\}_{j=1}^s; (\alpha, \beta, \alpha+\beta-1, 1)) \end{aligned}$$

for all $l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$, $2s+1 > \mathrm{Re}\,(\alpha+\beta) > 1$.

2.3 Application 2

We consider the multiple series

$$\begin{aligned} &\sum_{0 \leq = \lesseqgtr_0 m_1 \lesseqgtr_1 \cdots \lesseqgtr_{n-1} m_n \lesseqgtr_n = < \infty} \frac{(y)_{m_n}}{(x)_{m_n}} \frac{(w)_{m_n}}{(z)_{m_n}} \\ &\times \left\{ \prod_{i=1}^n \frac{(2m_i+x+y)^{s_i}(2m_i+z+w)^{t_i}}{(m_i+x)^{a_i}(m_i+y)^{b_i}(m_i+z)^{c_i}(m_i+w)^{d_i}} \right\}, \end{aligned} \tag{41}$$

where $n \in \mathbb{Z}_{\geq 1}$, $x, y, z, w \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $a_i, b_i, c_i, d_i, s_i, t_i \in \mathbb{Z}$ such that $a_i + b_i + c_i + d_i - s_i - t_i \geq 1$ $(i = 1, \ldots, n-1)$, $a_n + b_n + c_n + d_n - s_n - t_n + \mathrm{Re}\,(x-y+z-w) > 1$ and $(x+y)/2, (z+w)/2 \notin \mathbb{Z}_{\leq 0}$ in case of $s_i, t_i < 0$ for some i. The symbol $(a)_m$ is the Pochhammer symbol. The multiple series (41) also gives us extensions of MZV and MHZV. In this subsection, we study relations among the multiple series (41) by using Theorem A.

We consider the following special case of (41):

$$\begin{aligned} &\sum_{\substack{0 = m_0 \leq m_{11} < \cdots < m_{1r_1} < m_1 \\ \vdots \\ m_{p-1} \leq m_{p1} < \cdots < m_{pr_p} < m_p \\ \vdots \\ m_{p+q-1} \leq m_{p+q1} < \cdots < m_{p+qr_{p+q}} < m_{p+q} < \infty}} \frac{(y)_{m_{p+q}}}{(x)_{m_{p+q}}} \frac{m_{p+q}!}{(w)_{m_{p+q}}} \\ &\times \left\{ \prod_{i=1}^p \frac{1}{(m_i+x)(m_i+y)^{K_i-1}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij}+y)^{k_{ij}}} \right\} \\ &\times \left\{ \prod_{i=p+1}^{p+q} \frac{1}{(m_i+x)(m_i+y)^{K_i-2}(m_i+z)} \prod_{j=1}^{r_i} \frac{1}{(m_{ij}+y)^{k_{ij}}} \right\}, \end{aligned} \tag{42}$$

where $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q \geq 1$; $r_i \in \mathbb{Z}_{\geq 0}$, $k_{ij}, K_i \in \mathbb{Z}_{\geq 1}$. We denote the case $w = 1$ of (42) by $\Psi^{(p,q)}_{(\{r_i\}_{i=1}^{p+q})}(\{\{k_{ij}\}_{j=1}^{r_i}, K_i\}_{i=1}^{p+q}; (x, y, z))$. We can prove the following two identities, which are similar to (2) and (24):

Theorem 2.9. *The following two identities hold*:

(*i*)

$$
\begin{aligned}
&\sum_{\substack{r_1+\cdots+r_{k+s}=l \\ r_i\in\mathbb{Z}_{\geq 0}}} \Psi^{(k+1,s-1)}_{(\{r_i\}_{i=1}^{k+s})}(\{1\}^{\sum_{i=1}^{k+1}(r_i+1)},\{\{1\}^{r_i},2\}_{i=k+2}^{k+s};(\alpha,\beta,1)) \\
&=\sum\nolimits^{(k,l)} \\
&\Psi_{i+1}\left(\begin{array}{c} \{e(k_j)\}_{j=1}^{i},1|\{e(l_j)\}_{j=1}^{i},0 \\ \{k_j\}_{j=1}^{i},k_{i+1}+s-1|\{e(k_j)\}_{j=1}^{i},s-1|\{l_j\}_{j=1}^{i+1}|\{e(l_j)\}_{j=1}^{i},0 \end{array};\right. \\
&\qquad\qquad\left.(\alpha,1,\alpha-\beta+1,\beta)\right)
\end{aligned}
\tag{43}
$$

for all $k,l\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,(\alpha-\beta)>\max\{-1,-s+1\}$, *where*

$$\Psi_n\left(\begin{array}{c} \{s_i\}_{i=1}^{n}|\{t_i\}_{i=1}^{n} \\ \{a_i\}_{i=1}^{n}|\{b_i\}_{i=1}^{n}|\{c_i\}_{i=1}^{n}|\{d_i\}_{i=1}^{n} \end{array};(x,y,z,w)\right)$$

denotes the case $\lesssim_i=<$ $(i=1,\ldots,n-1)$ *of* (41).

(*ii*)

$$
\begin{aligned}
&\sum_{\substack{r_1+\cdots+r_{s}=l \\ r_i\in\mathbb{Z}_{\geq 0}}} \Psi^{(1,s-1)}_{(\{r_i\}_{i=1}^{s})}(\{1\}^{r_1},k+2,\{\{1\}^{r_i},2\}_{i=2}^{s};(\alpha,\beta,1)) \\
&=\sum\nolimits^{(k,l)} \\
&\psi_{i+1,-}\left(\begin{array}{c} \{e(k_j)+e(l_j)\}_{j=1}^{i},1 \\ \{0\}^{i},s|\{0\}^{i},s-1|\{k_j+e(l_j)\}_{j=1}^{i},k_{i+1}|\{l_j+e(k_j)\}_{j=1}^{i},l_{i+1} \end{array};\right. \\
&\qquad\qquad\left.(\alpha,1,\beta,\alpha-\beta+1)\right)
\end{aligned}
\tag{44}
$$

for all $k,l\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha>0$, $\mathrm{Re}\,\alpha+1>\mathrm{Re}\,\beta>0$.

Proof. The proofs of (43) and (44) are similar to those of (2) and (24), respectively. Indeed the identity (43) can be proved by applying the operator $(-1)^k\partial^{(l,k)}(w,z)|_{z=\alpha}^{w=\beta}$ ($k,l\in\mathbb{Z}_{\geq 0}$; $\alpha,\beta\in\mathbb{C}$ such that $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,(\alpha-\beta)>\max\{-1,-s+1\}$, $s\in\mathbb{Z}_{\geq 1}$) to both sides of the case $\beta=1$ of (3) and using (5), (6) and the case $\beta=1$, $\gamma=\alpha$, $\delta=\beta$ of (7). The identity (44) can be proved by applying $(-1)^k\partial^{(l,k)}(w,z)|_{z=\beta}^{w=\beta}$ ($k,l\in\mathbb{Z}_{\geq 0}$; $\beta\in\mathbb{C}$ such that $\mathrm{Re}\,\alpha+1>\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,\alpha>0$) to both sides of the case $\beta=1$ of (25) and using (6) and the case $\beta=1$, $\gamma=\delta=\beta$ of (7). □

Here we put

$$\Psi^{\star}_{p,q}(\{K_i\}_{i=1}^{p+q};(x,y,z)):=\Psi^{(p,q)}_{(\{0\}^{p+q})}(\{K_i\}_{i=1}^{p+q};(x,y,z)).$$

Taking $l=0$ in (43) and $s=1$ in (44), we get (45) and (46) below, respectively:

Corollary 2.10. *The following two identities hold*:

(*i*)

$$
\begin{aligned}
&\Psi^{\star}_{k+1,s-1}(\{1\}^{k+1},\{2\}^{s-1};(\alpha,\beta,1))\\
&=\sum\nolimits^{(k,0)}\Psi_{i+1}\left(\begin{array}{c}\{1\}^{i+1}|\{0\}^{i+1}\\ \{k_j\}_{j=1}^{i},k_{i+1}+s-1|\{1\}^{i},s-1|\{0\}^{i},1|\{0\}^{i+1}\end{array};\right.\\
&\qquad\qquad\left.(\alpha,1,\alpha-\beta+1,\beta)\right)
\end{aligned}\tag{45}
$$

for all $k\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,(\alpha-\beta)>\max\{-1,-s+1\}$.

(*ii*)

$$
\begin{aligned}
&\Psi_{l+1}(\{1\}^{l},k+2;(\alpha,\beta))\\
&=\sum\nolimits^{(k,l)}\\
&\psi_{i+1,-}\left(\begin{array}{c}\{e(k_j)+e(l_j)\}_{j=1}^{i},1\\ \{0\}^{i},1|\{0\}^{i+1}|\{k_j+e(l_j)\}_{j=1}^{i},k_{i+1}|\{l_j+e(k_j)\}_{j=1}^{i},l_{i+1}\end{array};\right.\\
&\qquad\qquad\left.(\alpha,1,\beta,\alpha-\beta+1)\right)
\end{aligned}\tag{46}
$$

for all $k,l\in\mathbb{Z}_{\geq 0}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha>0$, $\mathrm{Re}\,\alpha+1>\mathrm{Re}\,\beta>0$, *where*

$$
\Psi_{r_1+1}(\{k_{1j}\}_{j=1}^{r_1},K_1;(x,y)):=\Psi^{(1,0)}_{(r_1)}(\{k_{1j}\}_{j=1}^{r_1},K_1;(x,y,z)).
$$

We consider the following special case of (41):

$$
\begin{aligned}
&\sum_{\substack{0=m_0\leq m_{11}\leq\cdots\leq m_{1r_1}\leq m_1\\ \vdots\\ m_{p-1}\leq m_{p1}\leq\cdots\leq m_{pr_p}\leq m_p\\ \vdots\\ m_{p+q-1}\leq m_{p+q1}\leq\cdots\leq m_{p+qr_{p+q}}\leq m_{p+q}<\infty}}\frac{(y)_{m_{p+q}}}{(x)_{m_{p+q}}}\frac{m_{p+q}!}{(w)_{m_{p+q}}}\\
&\times\left\{\prod_{i=1}^{p}\frac{1}{(m_i+x)(m_i+y)^{K_i-1}}\prod_{j=1}^{r_i}\frac{1}{(m_{ij}+x)^{k_{ij}}}\right\}\\
&\times\left\{\prod_{i=p+1}^{p+q}\frac{1}{(m_i+x)(m_i+y)^{K_i-2}(m_i+z)}\prod_{j=1}^{r_i}\frac{1}{(m_{ij}+x)^{k_{ij}}}\right\},
\end{aligned}\tag{47}
$$

where p,q,r_i,k_{ij},K_i are the same as those for (42). We denote the case $w=1$ of (47) by $\Psi^{(p,q),\leq}_{(\{r_i\}_{i=1}^{p+q})}(\{\{k_{ij}\}_{j=1}^{r_i},K_i\}_{i=1}^{p+q};(x,y,z))$. We can prove the following identity:

Theorem 2.11. *The identity*

$$
\begin{aligned}
&\sum_{\substack{r_1+\cdots+r_s=l\\ r_i\in\mathbb{Z}_{\geq 0}}} \Psi^{(1,s-1),\leq}_{(\{r_i\}_{i=1}^s)}(\{1\}^{r_1},k+2,\{\{1\}^{r_i},2\}_{i=2}^s;(\alpha,\beta,1))\\
&=\sum\nolimits^{(k,l)} \sum_{\substack{0\leq n_j\leq e(k_j)(l_j+1)\\ (j=1,\dots,i);\\ 0\leq n_{i+1}\leq l_{i+1}}} \binom{s-1+l_{i+1}-n_{i+1}}{s-1}\left(\frac{l_{i+1}}{l_{i+1}+s-1}\right)^{1-e(n_{i+1})}\\
&\quad\times\psi_{i+1,-}(\{l_j-n_j+e(n_j)\}_{j=1}^i,l_{i+1}-n_{i+1}+e(n_{i+1})+s-1|\{0\}^i,s-1|\\
&\quad\{k_j-e(n_j)\}_{j=1}^{i+1}|\{n_j\}_{j=1}^{i+1};(\alpha,1,\beta,\alpha-\beta+1))
\end{aligned}\tag{48}
$$

holds for all $k,l\in\mathbb{Z}_{\geq 0}$*,* $s\in\mathbb{Z}_{\geq 1}$*,* $\alpha,\beta\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha>0$*,* $\mathrm{Re}\,\alpha+1>\mathrm{Re}\,\beta>0$*, where* $\psi_{n,-}(\{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n|\{c_i\}_{i=1}^n|\{d_i\}_{i=1}^n;(x,y,z,w))$ *is the same as that in Theorem* 2.8 *(i).*

Proof. Taking $\alpha=1$ in (25) and making the replacement $\beta\leftrightarrow w$ in the result, we get the identity

$$
\begin{aligned}
&\sum_{0\leq m_1\leq\cdots\leq m_s<\infty}\frac{(\beta)_{m_s}}{(w)_{m_s}}\frac{1}{(m_1+z)(m_1+w)}\left\{\prod_{i=2}^s\frac{1}{(m_i+w)(m_i+1)}\right\}\\
&=\sum_{m=0}^\infty(-1)^m\frac{m!}{(z)_{m+1}}\frac{(\beta)_m}{(w)_{m+1}}\frac{(w+1-z)_m}{(w+1-\beta)_{m+1}}\frac{w+2m+1}{(m+w)^{s-1}(m+1)^{s-1}}
\end{aligned}\tag{49}
$$

for all $s\in\mathbb{Z}_{\geq 1}$, $\beta,z,w\in\mathbb{C}$ such that $\mathrm{Re}\,w>0$, $\mathrm{Re}\,w+1>\mathrm{Re}\,z>0$, $\mathrm{Re}\,w+1>\mathrm{Re}\,\beta>0$, $2s-2>\mathrm{Re}\,(2(\beta-z)-w)$. Taking $a=w+1-\beta$, $b=\delta+1-\gamma$, $c=\gamma$, $d=\delta$, $X=\delta+1-\beta$, $Y=w+1-z$, $Z=z$, $W=w$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$
\begin{aligned}
&\frac{m!}{(z)_{m+1}}\frac{(\beta)_m}{(w)_{m+1}}\frac{(w+1-z)_m}{(w+1-\beta)_{m+1}}\frac{w+2m+1}{(w+m)^{s-1}(1+m)^{s-1}}\\
&=\frac{m!}{(\gamma)_m}\frac{(\beta)_m}{(\delta)_m}\frac{(\delta+1-\gamma)_m}{(\delta+1-\beta)_m}\frac{w+2m+1}{(z+m)(w+m)^s(w+1-\beta+m)(1+m)^{s-1}}\\
&\quad\times\sum_{i=0}^m\sum_{0\leq m_1<\cdots<m_i<m}\prod_{j=1}^i\Bigg(-\frac{(w-\delta)(\beta-\gamma)(\gamma+m_j)(\delta+m_j)}{(w+m_j)(w+1-\beta+m_j)(z+m_j)(\delta+1-\gamma+m_j)}\\
&\quad-\frac{(z-\gamma)(\gamma+m_j)(\delta+m_j)(\delta+1-\beta+m_j)}{(z+m_j)(w+m_j)(w+1-\beta+m_j)(\delta+1-\gamma+m_j)}\\
&\quad-\frac{(z-\gamma)(\delta+m_j)}{(z+m_j)(w+m_j)}-\frac{w-\delta}{w+m_j}\Bigg).
\end{aligned}\tag{50}
$$

We can calculate the partial differential coefficients of the factors on the right-hand side of (50) as follows:

$$
\begin{aligned}
&(-1)^{k+l}\partial^{(k,l)}(z,w)\left(\frac{w+2m+1}{(z+m)(w+m)^{s}(w+1-\beta+m)(1+m)^{s-1}}\right)\Big|_{\substack{z=\gamma\\ w=\delta}}\\
&=\sum_{n=0}^{l+1}\binom{s+l-n}{s-1}\left(\frac{l+1}{l+s}\right)^{1-e(n)}\\
&\quad\times\frac{(\beta+m)^{e(n)}}{(\gamma+m)^{k+1}(\delta+m)^{l-n+e(n)+s}(\delta+1-\beta+m)^{n}(1+m)^{s-1}}
\end{aligned}\tag{51}
$$

$(k,l,m\in\mathbb{Z}_{\geq 0},\ s\in\mathbb{Z}_{\geq 1},\ \gamma,\delta,\delta+1-\beta\notin\mathbb{Z}_{\leq 0})$ and

$$
\begin{aligned}
&(-1)^{k+l}\partial^{(k,l)}(z,w)\Bigg(-\frac{(w-\delta)(\beta-\gamma)(\gamma+m)(\delta+m)}{(w+m)(w+1-\beta+m)(z+m)(\delta+1-\gamma+m)}\\
&-\frac{(z-\gamma)(\gamma+m)(\delta+m)(\delta+1-\beta+m)}{(z+m)(w+m)(w+1-\beta+m)(\delta+1-\gamma+m)}\\
&-\frac{(z-\gamma)(\delta+m)}{(z+m)(w+m)}-\frac{w-\delta}{w+m}\Bigg)\Big|_{\substack{z=\gamma\\ w=\delta}}\\
&=\begin{cases}0, & \text{if } k=l=0,\\ \sum_{n=0}^{l+1}\frac{(\beta-\gamma)^{\{1-e(k)\}e(n)}(\beta+m)^{e(k)e(n)}}{(\gamma+m)^{k}(\delta+m)^{l-n+e(n)}(\delta+1-\beta+m)^{n-1}(\delta+1-\gamma+m)}, & \\ & \text{if } k,l\in\mathbb{Z}_{\geq 0}, k+l\geq 1\end{cases}
\end{aligned}\tag{52}
$$

$(m\in\mathbb{Z}_{\geq 0},\ \gamma,\delta,\delta+1-\beta,\delta+1-\gamma\notin\mathbb{Z}_{\leq 0})$. Using (50), (51) and (52), we get the identity

$$
\begin{aligned}
&(-1)^{k+l}\partial^{(k,l)}(z,w)\left(\frac{m!}{(z)_{m+1}}\frac{(\beta)_m}{(w)_{m+1}}\frac{(w+1-z)_m}{(w+1-\beta)_{m+1}}\frac{w+2m+1}{(m+w)^{s-1}(m+1)^{s-1}}\right)\Big|_{\substack{z=\gamma\\ w=\delta}}\\
&=\sum_{i=0}^{m}\sum_{\substack{k_1+\cdots+k_{i+1}=k+1\\ l_1+\cdots+l_{i+1}=l+1\\ k_j,l_j\in\mathbb{Z}_{\geq 0},\,k_j+l_j\geq 1\\ (j=1,\ldots,i);\,k_{i+1},l_{i+1}\in\mathbb{Z}_{\geq 1}}}\ \sum_{\substack{0\leq n_j\leq l_j+1\\ (j=1,\ldots,i);\\ 0\leq n_{i+1}\leq l_{i+1}}}\binom{s-1+l_{i+1}-n_{i+1}}{s-1}\left(\frac{l_{i+1}}{l_{i+1}+s-1}\right)^{1-e(n_{i+1})}\\
&\times\sum_{0\leq m_1<\cdots<m_i<m}\left\{\prod_{j=1}^{i}\frac{(\beta-\gamma)^{\{1-e(k_j)\}e(n_j)}(\beta+m_j)^{e(k_j)e(n_j)}}{(\gamma+m_j)^{k_j}(\delta+m_j)^{l_j-n_j+e(n_j)}(\delta+1-\beta+m_j)^{n_j-1}(\delta+1-\gamma+m_j)}\right\}\\
&\times\frac{(\beta+m)^{e(n_{i+1})}}{(\gamma+m)^{k_{i+1}}(\delta+m)^{l_{i+1}-n_{i+1}+e(n_{i+1})+s-1}(\delta+1-\beta+m)^{n_{i+1}}(1+m)^{s-1}}\\
&\times\frac{m!}{(\gamma)_m}\frac{(\beta)_m}{(\delta)_m}\frac{(\delta+1-\gamma)_m}{(\delta+1-\beta)_m}
\end{aligned}\tag{53}
$$

for $k,l,m\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\beta,\gamma,\delta\in\mathbb{C}$ such that $\gamma,\delta,\delta+1-\beta,\delta+1-\gamma\notin\mathbb{Z}_{\leq 0}$. We see that the right-hand side of the case $\gamma=\beta$, $\delta=\alpha$, $k,l\in\mathbb{Z}_{\geq 0}$, $k+l\geq 1$

of (52) can be written as

$$\sum_{n=0}^{e(k)(l+1)} \frac{1}{(\alpha+m)^{l-n+e(n)}(\beta+m)^{k-e(n)}(\alpha-\beta+1+m)^n}.$$

Thus, applying the operator $(-1)^{k+l}\partial^{(k,l)}(z,w)|_{w=\alpha}^{z=\beta}$ ($k,l\in\mathbb{Z}_{\geq 0}$; $\alpha,\beta\in\mathbb{C}$ such that $\operatorname{Re}\alpha>0$, $\operatorname{Re}\alpha+1>\operatorname{Re}\beta>0$) to the right-hand side of (49) and using the case $\gamma=\beta$, $\delta=\alpha$ of (53), we get the right-hand side of (48). The left-hand side of (48) can be proved by applying $(-1)^{k+l}\partial^{(k,l)}(z,w)|_{w=\alpha}^{z=\beta}$ to the left-hand side of (49) and using the case $p=0$, $q=s$, $a_i=1$ ($i=1,\dots,s$) of (17):

$$\begin{aligned}&\frac{(-1)^l}{l!}\frac{\mathrm{d}^l}{\mathrm{d}w^l}\left(\frac{1}{(w)_{m_s}}\prod_{i=1}^{s}\frac{1}{m_i+w}\right)\\ &=\frac{1}{(w)_{m_s}}\left(\prod_{i=1}^{s}\frac{1}{m_i+w}\right)\sum_{\substack{r_1+\cdots+r_s=l\\ r_i\in\mathbb{Z}_{\geq 0}}}\sum_{\substack{0=m_0\leq m_{11}\leq\cdots\leq m_{1r_1}\leq m_1\\ \vdots\\ m_{s-1}\leq m_{s1}\leq\cdots\leq m_{sr_s}\leq m_s}}\prod_{i=1}^{s}\prod_{j=1}^{r_i}\frac{1}{m_{ij}+w}\end{aligned}\tag{54}$$

for $l\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$ and $m_1,\dots,m_s\in\mathbb{Z}$ such that $0\leq m_1\leq\cdots\leq m_s$. □

Using (48), we can derive the following identities, which are similar to (26), (27) and (28):

Corollary 2.12. *The following identities hold*:

(i)

$$\begin{aligned}&\Psi^{\star}_{1,s-1}(k+2,\{2\}^{s-1};(\alpha,\beta,1))\\ &=\sum\nolimits^{(k,0)}\psi_{i+1,-}\Bigg(\begin{matrix}\{1\}^{i+1}\\ \{0\}^i,s|\{0\}^i,s-1|\{k_j\}_{j=1}^{i+1}|\{1\}^{i+1}\end{matrix};\\ &\qquad\qquad(\alpha,1,\beta,\alpha-\beta+1)\Bigg)\end{aligned}\tag{55}$$

for all $k,l\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\operatorname{Re}\alpha>0$, $\operatorname{Re}\alpha+1>\operatorname{Re}\beta>0$.

(ii)

$$\begin{aligned}&\Psi^{\star}_{l,1}(\{1\}^l,k+2;(\alpha,\beta,\beta))\\ &=\sum\nolimits^{(k,l)}\sum_{\substack{0\leq n_j\leq e(k_j)(l_j+1)\\ (j=1,\dots,i);\\ 0\leq n_{i+1}\leq l_{i+1}}}\psi_{i+1,-}(\{l_j-n_j+e(n_j)\}_{j=1}^{i+1}|\{0\}^{i+1}|\\ &\quad\{k_j-e(n_j)\}_{j=1}^{i+1}|\{n_j\}_{j=1}^{i+1};(\alpha,1,\beta,\alpha-\beta+1))\end{aligned}\tag{56}$$

for all $k,l\in\mathbb{Z}_{\geq 0}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\operatorname{Re}\alpha>0$, $\operatorname{Re}\alpha+1>\operatorname{Re}\beta>0$.

(iii)

$$\begin{aligned}&\zeta^{\le}_{l+1,+}(\{1\}^l,k+2;\alpha)\\ &=\sum\nolimits^{(k,l)}\sum_{\substack{0\le n_j\le e(k_j)(l_j+1)\\(j=1,\dots,i);\\0\le n_{i+1}\le l_{i+1}}}\psi_{i+1,-}(\{k_j+l_j-n_j\}_{j=1}^{i+1}|\{n_j\}_{j=1}^{i+1};(\alpha,1))\end{aligned}\tag{57}$$

for all $k,l\in\mathbb{Z}_{\ge0}$, $\alpha\in\mathbb{C}$ *with* $\operatorname{Re}\alpha>0$, *where*

$$\psi_{n,\pm}(\{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n;(x,y)):=\psi_{n,\pm}\left(\begin{matrix}\{0\}^n\\ \{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n|\{0\}^n|\{0\}^n\end{matrix};(x,y,z,w)\right).$$

(iv)

$$\zeta^\star(\{1\}^l,k+2)=\sum\nolimits^{(k,l)}\left\{\prod_{j=1}^{i}(l_j+2)^{e(k_j)}\right\}(l_{i+1}+1)\zeta_-(\{k_j+l_j\}_{j=1}^{i+1})\tag{58}$$

for all $k,l\in\mathbb{Z}_{\ge0}$.

Proof. For $n_j=0,1$ $(j=1,\dots,i+1)$, we see that

$$e(n_j)=n_j,\qquad \binom{s-n_{i+1}}{s-1}s^{e(n_{i+1})-1}=1.$$

By these facts and using (20), we get the expressions

$$\begin{aligned}&\sum_{\substack{0\le n_j\le1\\(j=1,\dots,i+1)}}\binom{s-n_{i+1}}{s-1}s^{e(n_{i+1})-1}\\ &\times\left\{\prod_{j=1}^{i+1}\frac{1}{(\alpha+m_j)^{-n_j+e(n_j)}(\beta+m_j)^{k_j-e(n_j)}(\alpha-\beta+1+m_j)^{n_j}}\right\}\\ &=\sum_{\substack{0\le n_j\le1\\(j=1,\dots,i+1)}}\prod_{j=1}^{i+1}\frac{1}{(\beta+m_j)^{k_j-n_j}(\alpha-\beta+1+m_j)^{n_j}}\\ &=\prod_{j=1}^{i+1}\left(\frac{1}{(\beta+m_j)^{k_j-1}(\alpha-\beta+1+m_j)}+\frac{1}{(\beta+m_j)^{k_j}}\right)\\ &=\prod_{j=1}^{i+1}\frac{2m_j+\alpha+1}{(\beta+m_j)^{k_j}(\alpha-\beta+1+m_j)}.\end{aligned}\tag{59}$$

Thus, taking $l=0$ in (48) and using (59), we get (55). Taking $s=1$ in (48) and $s=1$, $\alpha=\beta$ in (48), we get (56) and (57), respectively. The identity (58) can be proved by taking $s=\alpha=\beta=1$ in (48) and using the identities

$$\sum_{n=0}^{e(k)(l+1)} \frac{1}{(m+1)^n (m+1)^{l-n+e(n)} (m+1)^{k-e(n)}} = \left(\sum_{n=0}^{e(k)(l+1)} 1 \right) \frac{1}{(m+1)^{k+l}}$$
$$= (e(k)(l+1)+1)\, \frac{1}{(m+1)^{k+l}} = (l+2)^{e(k)} \frac{1}{(m+1)^{k+l}}$$

($k,l,m\in\mathbb{Z}_{\geq 0}$): the last identity comes from the definition of the symbol $e(k)$. $\square$

We denote the cases $w=z$ of (42) and (47) by $\Psi^{(p,q)}_{\mathbf{2},(\{r_i\}_{i=1}^{p+q})}(\{\{k_{ij}\}_{j=1}^{r_i}, K_i\}_{i=1}^{p+q}; (x,y,z))$ and $\Psi^{(p,q),\leq}_{\mathbf{2},(\{r_i\}_{i=1}^{p+q})}(\{\{k_{ij}\}_{j=1}^{r_i}, K_i\}_{i=1}^{p+q}; (x,y,z))$, respectively. We consider the following special case of (41) also:

$$\Psi_p^{\star}(\{k_i\}_{i=1}^p; (x,y)) := \sum_{0\leq m_1\leq\cdots\leq m_p<\infty} \frac{m_p!}{(x)_{m_p}} \left\{ \prod_{i=1}^p \frac{1}{(m_i+x)(m_i+y)^{k_i-1}} \right\},$$

where $p\in\mathbb{Z}_{\geq 1}$, $k_1,\ldots,k_{p-1}\in\mathbb{Z}_{\geq 1}$, $k_p\in\mathbb{Z}$, $x,y\in\mathbb{C}\backslash\mathbb{Z}_{\leq 0}$ such that $k_p+\mathrm{Re}\, x>2$.

Theorem 2.13. *The following identities hold*:

(*i*)

$$\begin{aligned}
&\sum_{\substack{r_1+\cdots+r_s=l\\ r_i\in\mathbb{Z}_{\geq 0}}} \Psi^{(0,s)}_{\mathbf{2},(\{r_i\}_{i=1}^s)}(\{\{1\}^{r_i}, 2\}_{i=1}^s; (\alpha,\gamma,\beta)) \\
&= \sum_{i=0}^{l} \sum_{\substack{l_1+\cdots+l_{i+1}=l+1\\ l_j\in\mathbb{Z}_{\geq 1}}} \psi_{i+1,-}\Bigg(\begin{array}{c} \{1\}^{i+1} \\ \{l_j\}_{j=1}^{i+1} | \{1\}^i, 0 | \{0\}^i, s | \{0\}^i, s \end{array}; \\
&\qquad\qquad (\alpha+\beta-\gamma,\gamma,\alpha,\beta)\Bigg)
\end{aligned} \tag{60}$$

for all $l\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta,\gamma\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta>0$, $\mathrm{Re}\,(\alpha+\beta)>\mathrm{Re}\,\gamma>0$, $2s-1>\mathrm{Re}\,(2\gamma-\alpha-\beta)$.

(*ii*)

$$\begin{aligned}
&\sum_{\substack{r_1+\cdots+r_s=r\\ r_i\in\mathbb{Z}_{\geq 0}}} \Psi^{(0,s),\leq}_{\mathbf{2},(\{r_i\}_{i=1}^s)}(\{\{1\}^{r_i}, 2\}_{i=1}^s; (\alpha,\gamma,\beta)) \\
&= \sum_{i=0}^{r} \binom{s-1+i}{i} \psi^{\leq}_{r-i+1,-}(\{1\}^{r-i+1}; s+i | \{0\}^{r-i+1}; s-1; \\
&\quad (\alpha+\beta-\gamma,\gamma,\alpha,\beta)) \\
&\quad + \sum_{i=0}^{r} \binom{s-2+i}{i} \psi^{\leq}_{r-i+1,-}(\{1\}^{r-i+1}; s-1+i | \{0\}^{r-i+1}; s; \\
&\quad (\alpha+\beta-\gamma,\gamma,\alpha,\beta))
\end{aligned} \tag{61}$$

for all $r\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta,\gamma\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,(\alpha+\beta)>\mathrm{Re}\,\gamma>0$, $2s-1>\mathrm{Re}\,(2\gamma-\alpha-\beta)$, *where*

$$\begin{aligned}&\psi^{\leq}_{n,-}(\{a_i\}_{i=1}^n;c|\{b_i\}_{i=1}^n;d;(x,y,z,w))\\ :=&\sum_{0\leq m_1\leq\cdots\leq m_n<\infty}(-1)^{m_n}\frac{(y)_{m_n}}{(x)_{m_n}}\left\{\prod_{i=1}^n\frac{1}{(m_i+x)^{a_i}(m_i+y)^{b_i}}\right\}\frac{1}{(m_n+z)^c(m_n+w)^d}.\end{aligned}$$

(*iii*)

$$\begin{aligned}&\sum_{\substack{r_1+\cdots+r_s=r\\ r_i\in\mathbb{Z}_{\geq 0}}}\Psi^\star_s(\{r_i+2\}_{i=1}^s;(\alpha,\beta))\\ &=\sum_{i=0}^r(-1)^{r-i}\binom{s-1+i}{i}\psi_{r-i+1,-}(\{0\}^{r-i},s|\{1\}^{r-i},s+i;(\alpha,\beta))\\ &\quad+\sum_{i=0}^r(-1)^{r-i}\binom{s-2+i}{i}\psi_{r-i+1,-}(\{0\}^{r-i},s+1|\{1\}^{r-i},s-1+i;(\alpha,\beta))\end{aligned}\tag{62}$$

for all $r\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $2s-1>\mathrm{Re}\,(\beta-\alpha)$, *where* $\psi_{n,-}(\{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n;(x,y))$ *is the same as that in Corollary* 2.12 (*iii*).

(*iv*)

$$\sum_{\substack{p_1+\cdots+p_s\\ +q_1+\cdots+q_s=r\\ p_i,q_i\in\mathbb{Z}_{\geq 0}}}\Psi^\star_{\sum_{i=1}^s p_i+s}(\{\{1\}^{p_i},q_i+2\}_{i=1}^s;(\alpha,\alpha))=2\binom{2s+r-1}{r}\zeta^{<}_{1,-}(2s+r;\alpha)\tag{63}$$

for all $r\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha\in\mathbb{C}$ *with* $\mathrm{Re}\,\alpha>0$.

Proof. Taking $a=\alpha+\beta$, $b_i=\alpha$, $c_i=\beta$ $(i=1,\ldots,s)$, $b_{s+1}=w$, $c_{s+1}=1$, where $s\in\mathbb{Z}_{\geq 1}$, $\alpha,\beta,w\in\mathbb{C}$ such that $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,(\alpha+\beta)>\mathrm{Re}\,w>0$, $2s-1>\mathrm{Re}\,(2w-\alpha-\beta)$, in Theorem A (i), we get the identity

$$\begin{aligned}&\sum_{0\leq m_1\leq\cdots\leq m_s<\infty}\frac{(w)_{m_s}}{(\alpha)_{m_s}}\frac{m_s!}{(\beta)_{m_s}}\left\{\prod_{i=1}^s\frac{1}{(m_i+\alpha)(m_i+\beta)}\right\}\\ &=\sum_{m=0}^\infty\frac{(w)_m}{(\alpha+\beta-w)_{m+1}}\frac{2m+\alpha+\beta}{(m+\alpha)^s(m+\beta)^s}(-1)^m.\end{aligned}\tag{64}$$

The identity (60) can be proved by applying the operator $\partial^{(l)}(w)|_{w=\gamma}$ ($l\in\mathbb{Z}_{\geq 0}$; $\gamma\in\mathbb{C}$ such that $\mathrm{Re}\,\alpha,\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,(\alpha+\beta)>\mathrm{Re}\,\gamma>0$, $2s-1>\mathrm{Re}\,(2\gamma-\alpha-\beta)$) to both sides of (64) and using (6) and the case $k=0$, $\delta=\gamma$ of (7). The identity (61) can be proved by differentiating both sides of (64) r times with respect to α and using (4) and (54). The identity (62) can be proved by differentiating both sides of the case $w = \beta$ of (64) r times with respect to β and using the identity

$$\frac{1}{r!}\frac{\mathrm{d}^r}{\mathrm{d}w^r}(w)_m = (w)_m \sum_{0\leq m_1<\cdots<m_r<m}\prod_{i=1}^{r}\frac{1}{m_i+w} \tag{65}$$

($m, r \in \mathbb{Z}_{\geq 0}$; see Hoffman [15, Proof of Corollary 4.2]). The identity (63) can be proved by differentiating both sides of the case $\alpha = \beta = w$ of (64) r times with respect to α and using (54). $\square$

Remark 6. (i) The identity (63) and the cases $\alpha = \beta = \gamma$ of the identities (60), (61), (62) give relations between the MHZVs $\zeta^{<}_{n,-}(\{a_i\}_{i=1}^{n};\alpha)$, $\zeta^{\leq}_{n,-}(\{a_i\}_{i=1}^{n};\alpha)$ and the multiple series (23) with $z = 1$. The cases $\beta = \gamma = 1$ of the identities (60), (61), (62) also give relations between the multiple series (23) with $z = \pm 1$.

(ii) Taking $\alpha = 1$ in (63) and using the identity $\zeta^{<}_{1,-}(a;1) = \sum_{m=0}^{\infty}(-1)^m(m+1)^{-a} = (1-2^{1-a})\zeta(a)$, we get the relation among MZSVs of Aoki and Ohno [4, Theorem 1]. We see that the sums on the left-hand sides of the cases $\alpha = \beta = \gamma$ of the identities (61) and (62) are partial sums of the sum on the left-hand side of (63).

We remark that the multiple series (42) and (47) are special cases of the multiple series (1) with $p = 0$; therefore the results in this subsection can be regarded as identities for the multiple series (1).

2.4 Application 3

In this subsection, we derive several identities for MHZVs by using Theorem A (i).

Theorem 2.14. *The following two identities hold*:

(*i*)

$$\begin{aligned}&\zeta^{<}_{l+1,+}(\{1\}^{l+1}|\{0\}^{l}, k+1; (\alpha,\beta))\\ &= \sum\nolimits^{(k,l)} \psi_{i+1,-}\left(\begin{array}{c}\{e(k_j)+e(l_j)\}_{j=1}^{i}, 1\\ \{k_j\}_{j=1}^{i+1}|\{e(k_j)\}_{j=1}^{i}, 0|\{e(l_j)\}_{j=1}^{i}, 1|\{l_j\}_{j=1}^{i+1}\end{array};\right.\\ &\qquad\qquad \left.(\beta, \alpha-\beta+1, \alpha, 1)\right)\end{aligned} \tag{66}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ *such that* $\operatorname{Re}\alpha + 1 > \operatorname{Re}\beta > \operatorname{Re}\alpha/2 > 0$.

(ii)

$$\begin{aligned}
&\zeta^{\leq}_{l+1,+}(\{1\}^{l+1}|\{0\}^{l},k+1;(\alpha,\beta))\\
&=\sum\nolimits^{(k,l)}\sum_{\substack{0\leq n_j\leq l_j+1\\(j=1,\dots,i);\\0\leq n_{i+1}\leq l_{i+1}}}(\alpha-\beta)^{\sum_{j=1}^{i}\mathbf{e}(k_j,n_j)}\psi_{i+1,-}(\{k_j\}_{j=1}^{i+1}|\{1\}^{i},0|\\
&\{l_j-n_j+\mathbf{e}(k_j,n_j)\}_{j=1}^{i},l_{i+1}-n_{i+1}|\\
&\{n_j-1\}_{j=1}^{i},n_{i+1};(\beta,\alpha-\beta+1,\alpha,1))
\end{aligned}\tag{67}$$

for all $k,l\in\mathbb{Z}_{\geq 0}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\operatorname{Re}\alpha+1>\operatorname{Re}\beta>\operatorname{Re}\alpha/2>0$, *where*

$$\mathbf{e}(k,n):=\{1-e(k)\}e(n)=\begin{cases}1, & \textit{if } k=0 \textit{ and } n\geq 1,\\ 0, & \textit{otherwise,}\end{cases}$$

and $\psi_{n,-}(\{a_i\}_{i=1}^{n}|\{b_i\}_{i=1}^{n}|\{c_i\}_{i=1}^{n}|\{d_i\}_{i=1}^{n};(x,y,z,w))$ *is the same as that in Theorem* 2.8 *(i).*

Proof. Taking $s=\beta=1$ in (25), we get the identity

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(w)_m}{(\alpha)_m}\frac{1}{(m+\alpha)(m+z)}\\
&=\sum_{m=0}^{\infty}\frac{(\alpha+1-z)_m}{(z)_{m+1}}\frac{(w)_m}{(\alpha+1-w)_{m+1}}\frac{m!}{(\alpha)_{m+1}}(2m+\alpha+1)(-1)^m
\end{aligned}\tag{68}$$

for all $\alpha,z,w\in\mathbb{C}$ such that $\operatorname{Re}\alpha>0$, $\operatorname{Re}\alpha+1>\operatorname{Re}z>0$, $\operatorname{Re}\alpha+1>\operatorname{Re}w>0$, $\operatorname{Re}(\alpha+2(z-w))>0$. Applying the operator $(-1)^k\partial^{(l,k)}(w,z)|_{z=\beta}^{w=\alpha}$ ($k,l\in\mathbb{Z}_{\geq 0}$; $\alpha,\beta\in\mathbb{C}$ such that $\operatorname{Re}\alpha+1>\operatorname{Re}\beta>\operatorname{Re}\alpha/2>0$) to the right-hand side of (68) and using the case $\beta=1$, $\gamma=\beta$, $\delta=\alpha$ of (7), we get the right-hand side of (66). The left-hand side of (66) can be proved by applying $(-1)^k\partial^{(l,k)}(w,z)|_{z=\beta}^{w=\alpha}$ to the left-hand side of (68) and using (65).

The identity (67) can be proved in a way similar to the above. Indeed, making the replacement $\alpha\leftrightarrow w$ in (68), we get the identity

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(\alpha)_m}{(w)_m}\frac{1}{(m+z)(m+w)}\\
&=\sum_{m=0}^{\infty}\frac{m!}{(z)_{m+1}}\frac{(\alpha)_m}{(w)_{m+1}}\frac{(w+1-z)_m}{(w+1-\alpha)_{m+1}}(2m+w+1)(-1)^m
\end{aligned}\tag{69}$$

for all $\alpha,z,w\in\mathbb{C}$ such that $\operatorname{Re}w>0$, $\operatorname{Re}w+1>\operatorname{Re}z>0$, $\operatorname{Re}w+1>\operatorname{Re}\alpha>0$, $\operatorname{Re}(w+2(z-\alpha))>0$. Applying the operator $(-1)^{k+l}\partial^{(k,l)}(z,w)|_{w=\alpha}^{z=\beta}$ ($k,l\in\mathbb{Z}_{\geq 0}$; $\alpha,\beta\in\mathbb{C}$ such that $\operatorname{Re}\alpha+1>\operatorname{Re}\beta>\operatorname{Re}\alpha/2>0$) to the right-hand side of (69) and using the case $s=1$, $\beta=\delta=\alpha$, $\gamma=\beta$ of (53), we get the right-hand side of (67). The left-hand side of (67) can be proved by applying $(-1)^{k+l}\partial^{(k,l)}(z,w)|_{w=\alpha}^{z=\beta}$ to the left-hand side of (69) and using the identity (4). $\square$

We can prove the following expressions for the MHZVs on the left-hand sides of the identities (66) and (67) also:

Theorem 2.15. *The following two identities hold*:

(*i*)

$$\begin{aligned}
&\zeta^{<}_{l+1,+}(\{1\}^{l+1}|\{0\}^{l},k+1;(\alpha,\beta))\\
&=\sum\nolimits^{(k,l)}(-1)^{i}(\alpha-\beta)^{\sum_{j=1}^{i}\{1-e(k_j)\}}(\alpha-1)^{\sum_{j=1}^{i}\{1-e(l_j)\}}\\
&\quad\times\psi_{i+1,+}(\{k_j\}_{j=1}^{i+1}|\{1\}^{i},0|\{1-e(k_j)-e(l_j)\}_{j=1}^{i},0|\{l_j\}_{j=1}^{i+1};\\
&\quad(\beta,\beta-\alpha+1,\alpha,1))
\end{aligned}\tag{70}$$

for all $k,l\in\mathbb{Z}_{\geq 0}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\operatorname{Re}\alpha,\operatorname{Re}\beta>0$, $\operatorname{Re}\alpha+1>\operatorname{Re}\beta>\max\{\operatorname{Re}\alpha/2,\operatorname{Re}\alpha-1\}$, *where* $\psi_{n,+}(\{a_i\}_{i=1}^{n}|\{b_i\}_{i=1}^{n}|\{c_i\}_{i=1}^{n}|\{d_i\}_{i=1}^{n};(x,y,z,w))$ *is the same as that in Theorem* 2.8 (*i*).

(*ii*)

$$\begin{aligned}
&\zeta^{\leq}_{l+1,+}(\{1\}^{l+1}|\{0\}^{l},k+1;(\alpha,\beta))\\
&=\sum_{i=k}^{k+l}\sum_{\substack{k_1+\cdots+k_i=k\\ l_1+\cdots+l_{i+2}=l+2\\ k_j\in\{0,1\},l_j\in\mathbb{Z}_{\geq 0},\,k_j+l_j\geq 1\\ (j=1,\dots,i);\,l_{i+1},l_{i+2}\in\mathbb{Z}_{\geq 1}}}(1-\beta)^{\sum_{j=1}^{i}\{1-e(k_j)\}}\psi_{i+1,+}(\{l_j\}_{j=1}^{i+1}|\{1\}^{i},0|\{0\}^{i},l_{i+2};\\
&(\alpha,\alpha-\beta+1,1))
\end{aligned}\tag{71}$$

for all $k,l\in\mathbb{Z}_{\geq 0}$, $\alpha,\beta\in\mathbb{C}$ *such that* $\operatorname{Re}\alpha+1>\operatorname{Re}\beta>\operatorname{Re}\alpha/2>0$, *where*

$$\begin{aligned}
&\psi_{n,+}(\{a_i\}_{i=1}^{n}|\{b_i\}_{i=1}^{n}|\{c_i\}_{i=1}^{n};(x,y,z))\\
&:=\psi_{n,+}\left(\begin{matrix}\{0\}^{n}\\ \{a_i\}_{i=1}^{n}|\{b_i\}_{i=1}^{n}|\{c_i\}_{i=1}^{n}|\{0\}^{n}\end{matrix};(x,y,z,w)\right).
\end{aligned}$$

Proof. Taking $s=1$, $a=\alpha+1$, $b_1=\alpha+1-z$, $c_1=w$, $b_2=c_2=1$, where $\alpha,z,w\in\mathbb{C}$ such that $\operatorname{Re}\alpha>0$, $\operatorname{Re}\alpha+1>\operatorname{Re}z>0$, $\operatorname{Re}\alpha+1>\operatorname{Re}w>0$, $\operatorname{Re}(\alpha+2(z-w))>0$, $\operatorname{Re}(z-w+1)>0$, in Theorem A (i) and multiplying both sides of the result by $\{z(\alpha+1-w)\}^{-1}$, we get the identity

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(z-w+1)_m}{(z)_{m+1}}\frac{m!}{(\alpha+1-w)_{m+1}}\\
&=\sum_{m=0}^{\infty}\frac{(\alpha+1-z)_m}{(z)_{m+1}}\frac{(w)_m}{(\alpha+1-w)_{m+1}}\frac{m!}{(\alpha)_{m+1}}(2m+\alpha+1)(-1)^{m}.
\end{aligned}\tag{72}$$

Taking $a=b=c=\beta-\alpha+1$, $d=\beta$, $X=z-w+1$, $Y=1$, $Z=\alpha+1-w$, $W=z$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$\begin{aligned}&\frac{(z-w+1)_m}{(z)_{m+1}}\frac{m!}{(\alpha+1-w)_{m+1}}\\&=\frac{(\beta-\alpha+1)_m}{(\beta)_m}\frac{1}{(z+m)(\alpha+1-w+m)}\\&\times\sum_{i=0}^{m}\sum_{0\le m_1<\cdots<m_i<m}\prod_{j=1}^{i}\biggl(-\frac{z-\beta}{z+m_j}+\frac{(z-\beta)(1+m_j)(\beta+m_j)}{(z+m_j)(\alpha+1-w+m_j)(\beta-\alpha+1+m_j)}\\&+\frac{(w-\alpha)(\beta-\alpha)(\beta+m_j)}{(z+m_j)(\alpha+1-w+m_j)(\beta-\alpha+1+m_j)}\biggr).\end{aligned}$$

Using this identity, we get the identity

(73)

$$\begin{aligned}&(-1)^k\partial^{(l,k)}(w,z)\left(\frac{(z-w+1)_m}{(z)_{m+1}}\frac{m!}{(\alpha+1-w)_{m+1}}\right)\Big|_{\substack{w=\alpha\\z=\beta}}\\&=\sum_{i=0}^{m}\sum_{\substack{k_1+\cdots+k_{i+1}=k+1\\l_1+\cdots+l_{i+1}=l+1\\k_j,l_j\in\mathbb{Z}_{\ge0},\,k_j+l_j\ge1\\(j=1,\dots,i);\,k_{i+1},l_{i+1}\in\mathbb{Z}_{\ge1}}}(-1)^i(\alpha-\beta)^{\sum_{j=1}^{i}\{1-e(k_j)\}}(\alpha-1)^{\sum_{j=1}^{i}\{1-e(l_j)\}}\\&\times\sum_{0\le m_1<\cdots<m_i<m}\left\{\prod_{j=1}^{i}\frac{1}{(\beta+m_j)^{k_j}(\beta-\alpha+1+m_j)(\alpha+m_j)^{1-e(k_j)-e(l_j)}(1+m_j)^{l_j}}\right\}\\&\times\frac{1}{(\beta+m)^{k_{i+1}}(1+m)^{l_{i+1}}}\frac{(\beta-\alpha+1)_m}{(\beta)_m}\end{aligned}$$

for $k,l,m\in\mathbb{Z}_{\ge0}$, $\alpha,\beta\in\mathbb{C}$ such that $\alpha,\beta,\beta-\alpha+1\notin\mathbb{Z}_{\le0}$. Thus, applying the operator $(-1)^k\partial^{(l,k)}(w,z)|_{\substack{w=\alpha\\z=\beta}}$ ($k,l\in\mathbb{Z}_{\ge0}$; $\alpha,\beta\in\mathbb{C}$ such that $\operatorname{Re}\alpha,\operatorname{Re}\beta>0$, $\operatorname{Re}\alpha+1>\operatorname{Re}\beta>\max\{\operatorname{Re}\alpha/2,\operatorname{Re}\alpha-1\}$) to the left-hand side of (72) and using (73), we get the right-hand side of (70). We note that the right-hand side of (72) is completely the same as that of (68). By this fact and using (68) and (72), we get (70).

The identity (71) can also be proved in a way similar to the above. Indeed, taking $s=1$, $a=w+1$, $b_1=\alpha$, $b_2=w+1-z$, $c_1=c_2=1$, where $\alpha,z,w\in\mathbb{C}$ such that $\operatorname{Re}w>0$, $\operatorname{Re}w+1>\operatorname{Re}z>0$, $\operatorname{Re}w+1>\operatorname{Re}\alpha>0$, $\operatorname{Re}(w+2(z-\alpha))>0$, in Theorem A (i) and multiplying both sides of the result by w^{-1}, we get the identity

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{(w+1-z)_m}{(w)_{m+1}}\frac{1}{w+1-\alpha+m}\\&=\sum_{m=0}^{\infty}\frac{m!}{(z)_{m+1}}\frac{(\alpha)_m}{(w)_{m+1}}\frac{(w+1-z)_m}{(w+1-\alpha)_{m+1}}(2m+w+1)(-1)^m.\end{aligned}\tag{74}$$

Taking $a = X$, $b = Y$, $c = w+1-z$, $d = \alpha$, $Z = \alpha-\beta+1$, $W = w$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$\begin{aligned}
&\frac{(w+1-z)_m}{(w)_{m+1}}\frac{1}{w+1-\alpha+m}\\
=&\frac{(\alpha-\beta+1)_m}{(\alpha)_m}\frac{1}{(w+m)(w+1-\alpha+m)}\\
&\times\sum_{i=0}^{m}\sum_{0\le m_1<\cdots<m_i<m}\prod_{j=1}^{i}\biggl(-\frac{(z-\beta)(\alpha+m_j)}{(w+m_j)(\alpha-\beta+1+m_j)}\\
&+\frac{(w-\alpha)(\beta-1)}{(w+m_j)(\alpha-\beta+1+m_j)}\biggr).
\end{aligned}\tag{75}$$

Using this identity, we get the identity

$$\begin{aligned}
&(-1)^{k+l}\partial^{(k,l)}(z,w)\left(\frac{(w+1-z)_m}{(w)_{m+1}}\frac{1}{w+1-\alpha+m}\right)\Big|_{\substack{z=\beta\\ w=\alpha}}\\
=&\sum_{i=0}^{m}\sum_{\substack{k_1+\cdots+k_i=k\\ l_1+\cdots+l_{i+2}=l+2\\ k_j\in\{0,1\},l_j\in\mathbb{Z}_{\ge0},\,k_j+l_j\ge1\\ (j=1,\dots,i);\,l_{i+1},l_{i+2}\in\mathbb{Z}_{\ge1}}}(1-\beta)^{\sum_{j=1}^{i}\{1-e(k_j)\}}\frac{(\alpha-\beta+1)_m}{(\alpha)_m}\\
&\times\sum_{0\le m_1<\cdots<m_i<m}\left\{\prod_{j=1}^{i}\frac{1}{(\alpha+m_j)^{l_j}(\alpha-\beta+1+m_j)}\right\}\\
&\times\frac{1}{(\alpha+m)^{l_{i+1}}(1+m)^{l_{i+2}}}
\end{aligned}\tag{76}$$

for $k,l,m\in\mathbb{Z}_{\ge0}$, $\alpha,\beta\in\mathbb{C}$ such that $\alpha,\alpha-\beta+1\notin\mathbb{Z}_{\le0}$. Since the right-hand side of (75) is a polynomial in terms of z, the summation $\sum_{i=0}^{m}$ on the right-hand side of (76) can be changed into $\sum_{i=k}^{m}$. Thus, applying the operator $(-1)^{k+l}\partial^{(k,l)}(z,w)|_{w=\alpha}^{z=\beta}$ ($k,l\in\mathbb{Z}_{\ge0}$; $\alpha,\beta\in\mathbb{C}$ such that $\mathrm{Re}\,\alpha+1>\mathrm{Re}\,\beta>\mathrm{Re}\,\alpha/2>0$) to the left-hand side of (74) and using (76), we get the right-hand side of (71). We note that the right-hand side of (74) is completely the same as that of (69). By this fact and using (69) and (74), we get (71). □

Remark 7. (i) Taking $\alpha=\beta=1$ in (70), we get the following identity for MZVs, which is similar to (28):

$$\zeta(\{1\}^l,k+2)=\sum_{i=0}^{\min\{k,l\}}\sum_{\substack{k_1+\cdots+k_{i+1}=k+1\\ l_1+\cdots+l_{i+1}=l+1\\ k_j,l_j\in\mathbb{Z}_{\ge1}}}(-1)^i\zeta(\{k_j+l_j\}_{j=1}^{i+1})\tag{77}$$

for all $k,l\in\mathbb{Z}_{\ge0}$. By the replacement $k\leftrightarrow l$ in the right-hand sides of the identities (28) and (77), we see that they have a symmetry with respect to k and

l. That symmetry gives the identity $\zeta(\{1\}^l, k+2) = \zeta(\{1\}^k, l+2)$ $(k, l \in \mathbb{Z}_{\geq 0})$ (Hoffman [15, Theorem 4.4]). Taking $\alpha = \beta = 1$ in (71), we get the expression for $\zeta^\star(\{1\}^l, k+2)$ $(k, l \in \mathbb{Z}_{\geq 0})$ in terms of MZVs in Ohno [25, Theorem 2 and its proof].

(ii) Taking $\beta = (\alpha+1)/2$ in (66) and (67), $\alpha = 1$ in (70), and $\beta = 1$ in (71), we get relations among the MHZVs $\zeta^{\leq}_{n,\pm}(\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{c_i\}_{i=1}^n; ((x+1)/2, x, 1))$.

(iii) Making the replacement $\alpha \leftrightarrow w$ in (72) and using the result of the replacement, the identity (69), the case $a = b = c = \beta - \alpha + 1$, $d = \beta$, $X = z - \alpha + 1$, $Y = 1$, $Z = w + 1 - \alpha$, $W = z$ of Lemma 2.1, a calculation and a way like those of proving (70), we can prove the following modification of (71) also, which is similar to (70):

$$\begin{aligned}&\zeta^{\leq}_{l+1,+}(\{1\}^{l+1} | \{0\}^l, k+1; (\alpha, \beta))\\ =&\sum\nolimits^{(k,l)} (1-\alpha)^{\sum_{j=1}^i e(k_j)} \psi_{i+1,+}(\{k_j\}_{j=1}^{i+1} | \{e(k_j)\}_{j=1}^i, 0 | \{l_j\}_{j=1}^{i+1}; (\beta, \beta-\alpha+1, 1))\end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, $\operatorname{Re}\alpha + 1 > \operatorname{Re}\beta > \max\{\operatorname{Re}\alpha/2, \operatorname{Re}\alpha - 1\}$, where $\psi_{n,+}(\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{c_i\}_{i=1}^n; (x, y, z))$ is the same as that in Theorem 2.15 *(ii)*.

Theorem 2.16. *The identity*

$$\begin{aligned}&\sum_{i=0}^{k} 2^{k-i} \binom{i+r}{i} \sum_{\substack{k_1+\cdots+k_s=k-i\\ k_j \in \mathbb{Z}_{\geq 0}}} \zeta^{\leq}_{s,+}(i+r+1, \{0\}^{s-1} | \{k_j+1\}_{j=1}^s |\\ &0, \{1\}^{s-1}; ((\alpha+1)/2, \alpha, 1)) = \sum_{i=0}^{r} \sum_{\substack{k_1+\cdots+k_{i+2}=k\\ r_1+\cdots+r_{i+1}=r+1\\ k_j \in \mathbb{Z}_{\geq 0}, r_j \in \mathbb{Z}_{\geq 1}}} 2^{i+1+k_{i+2}}\\ &\times \left\{\prod_{j=1}^{i} \binom{k_j+r_j-1}{k_j}\right\} \binom{k_{i+1}+r_{i+1}-2}{k_{i+1}} \binom{k_{i+2}+s-1}{k_{i+2}}\\ &\times \zeta^{<}_{i+1,-}(\{k_j+r_j\}_{j=1}^i, k_{i+1}+r_{i+1}-1 | \{0\}^i, k_{i+2}+s | \{0\}^i, s; ((\alpha+1)/2, \alpha, 1))\end{aligned} \tag{78}$$

holds for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Proof. Applying the operator $(-1)^r \partial^{(r)}(z)|_{z=(\alpha+1)/2}$ $(r \in \mathbb{Z}_{\geq 0}$; $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0)$ to both sides of the case $\beta = 1$, $w = \alpha$ of (25) and using the case

$l=0$, $\beta=1$, $\gamma=(\alpha+1)/2$, $\delta=\alpha$ of (7), we get the identity

$$
\begin{aligned}
&\sum_{0\le m_1\le\cdots\le m_s<\infty}\frac{1}{(m_1+\alpha)(m_1+(\alpha+1)/2)^{r+1}}\left\{\prod_{i=2}^{s}\frac{1}{(m_i+\alpha)(m_i+1)}\right\}\\
&=\sum_{i=0}^{r}\sum_{\substack{r_1+\cdots+r_{i+1}=r+1\\ r_j\in\mathbb{Z}_{\ge1}}}2^{i+1}\sum_{0\le m_1<\cdots<m_{i+1}<\infty}\left\{\prod_{j=1}^{i}\frac{1}{(m_j+(\alpha+1)/2)^{r_j}}\right\}\\
&\quad\times\frac{1}{(m_{i+1}+(\alpha+1)/2)^{r_{i+1}-1}}\frac{(-1)^{m_{i+1}}}{(m_{i+1}+\alpha)^s(m_{i+1}+1)^s}
\end{aligned}\tag{79}
$$

for all $r\in\mathbb{Z}_{\ge0}$, $s\in\mathbb{Z}_{\ge1}$, $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>0$. Further, differentiating both sides of (79) k times with respect to α and multiplying both sides of the result by 2^k, we get (78) under the condition $\operatorname{Re}\alpha>0$. This condition can be changed into $\alpha\notin\mathbb{Z}_{\le0}$, because both sides of (78) are holomorphic in $\mathbb{C}\setminus\mathbb{Z}_{\le0}$ as functions of α. $\square$

Remark 8. Taking $\alpha=1$ in (78), we get relations between MZSVs and $\zeta_-(\{k_i\}_{i=1}^n)$. For example, taking $\alpha=1$, $r=0$ in (78) and using the identity $\zeta^<_{1,-}(a;1)=\sum_{m=0}^\infty(-1)^m(m+1)^{-a}=(1-2^{1-a})\zeta(a)$, we get the identity

$$
\begin{aligned}
&\sum_{i=0}^{k}2^{k-i}\sum_{\substack{k_1+\cdots+k_s=k-i\\ k_j\in\mathbb{Z}_{\ge0}}}\zeta^\star(k_1+i+2,\{k_j+2\}_{j=2}^s)\\
&=2^{1+k}\binom{k+s-1}{k}(1-2^{1-k-2s})\zeta(k+2s)
\end{aligned}
$$

for all $k\in\mathbb{Z}_{\ge0}$, $s\in\mathbb{Z}_{\ge1}$. Taking $k=1$ in this identity, we get the identity for $\zeta(2s+1)$ of Ihara, Kajikawa, Ohno and Okuda [22, Theorem 2].

Here we put

$$
\begin{aligned}
P(n;(\alpha,\beta)):=&\frac{1}{n!}\frac{\partial^n}{\partial\gamma^n}(\psi(\alpha+\beta-\gamma)-\psi(\gamma))\Big|_{\gamma=\beta}\\
=&\begin{cases}(\alpha-\beta)\zeta^<_{1,+}(1|1;(\alpha,\beta)), & \text{if } n=0,\\ -\zeta^<_{1,+}(n+1;\alpha)+(-1)^n\zeta^<_{1,+}(n+1;\beta), & \text{if } n\in\mathbb{Z}_{\ge1},\end{cases}
\end{aligned}
$$

where $\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_{\le0}$ and $\psi(z):=\Gamma'(z)/\Gamma(z)$ is the digamma function. The second equality for $P(n;(\alpha,\beta))$ follows from the following two properties of the digamma function: $\psi(z)-\psi(w)=(z-w)\sum_{m=0}^\infty(m+z)^{-1}(m+w)^{-1}$ ($z,w\in\mathbb{C}\setminus\mathbb{Z}_{\le0}$) and $\psi^{(n)}(z)=(-1)^{n+1}n!\zeta^<_{1,+}(n+1;z)$ ($n\in\mathbb{Z}_{\ge1}$), where $\psi^{(n)}(z)$ is the n-th derivative of $\psi(z)$. (For the property of the digamma function, see e.g. [27, Subsection 1.2].) We define the symbol $G(n;(\alpha,\beta))$ by

$$
G(n;(\alpha,\beta)):=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{i=1}^{n}\sum_{0=l_0<l_1<\cdots<l_{i-1}<l_i=n}\prod_{j=1}^{i}l_j^{-1}P(l_j-l_{j-1}-1;(\alpha,\beta))
$$

$(n \in \mathbb{Z}_{\geq 1})$ and $G(0;(\alpha,\beta)) := \Gamma(\alpha+\beta)/(\Gamma(\alpha)\Gamma(\beta))$, where $\alpha,\beta \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ such that $\alpha+\beta \notin \mathbb{Z}_{\leq 0}$. We consider the series

$$\psi^{\sharp}_{1,-}(a|b;(x,y)) := \sum_{m=0}^{\infty} \frac{(x+y)_m}{m!} \frac{2m+x+y}{(m+x)^a(m+y)^b}(-1)^m,$$

where $a,b \in \mathbb{Z}$, $x,y \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ such that $a+b-\mathrm{Re}\,(x+y) > 1$. We can prove the following evaluation for the multiple series (12) in terms of the single series $\psi^{\sharp}_{1,-}(a|b;(x,y))$, $\zeta^{<}_{1,+}(a|b;(x,y))$:

Theorem 2.17. *The identity*

$$\begin{aligned}
&\sum_{\substack{r_1+\cdots+r_s=r\\ r_i\in\mathbb{Z}_{\geq 0}}} \phi^{s,\leq}_{A^{(s)},B^{(s)}}\left(\begin{matrix}\{r_i+1\}_{i=1}^{s}\\ \{r_i+1\}_{i=1}^{s}|\{r_i+1\}_{i=1}^{s}|\{1\}^s\end{matrix};(\alpha,\beta)\right)\\
&= \sum_{\substack{r_1+r_2+r_3=r\\ r_i\in\mathbb{Z}_{\geq 0}}} (-1)^{r_3}\binom{r_2+s}{r_2}\binom{r_3+s}{r_3} G(r_1;(\alpha,\beta))\\
&\quad \times \psi^{\sharp}_{1,-}(r_2+s+1|r_3+s+1;(\alpha,\beta))
\end{aligned} \tag{80}$$

holds for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha,\beta \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ such that $2s+1 > \mathrm{Re}\,(\alpha+\beta) > 0$, where $A^{(s)} = \{(-1)^{r_i}\}_{i=1}^{s}$, $B^{(s)} = \{1\}^s$. The left-hand side of (80) *can be written as a linear combination of $\zeta^{\leq}_{s,+}(\{a_i\}_{i=1}^{s}|\{b_i\}_{i=1}^{s}|\{1\}^s;(\alpha,\beta,(\alpha+\beta)/2))$, explicitly.*

Remark 9. We see that

$$\begin{aligned}
\psi^{\sharp}_{1,-}(a|b;(2-\beta,\beta)) =& 2\zeta^{<}_{1,-}(a|b-2;(2-\beta,\beta)) + 4(1-\beta)\zeta^{<}_{1,-}(a|b-1;(2-\beta,\beta))\\
&+ 2(1-\beta)^2\zeta^{<}_{1,-}(a|b;(2-\beta,\beta)).
\end{aligned}$$

By this fact, the left-hand side of the case $\alpha = 2-\beta$ of (80) can be evaluated by only the single series $\zeta^{<}_{1,\pm}(a|b;(2-\beta,\beta))$.

To prove Theorem 2.17, we prove the following lemma:

Lemma 2.18. *The identity*

$$\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{r!}\frac{\partial^r}{\partial\gamma^r}\left(\frac{1}{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}\right)\Big|_{\gamma=\beta}\\
&= \sum_{i=1}^{r} \sum_{0=l_0<l_1<\cdots<l_{i-1}<l_i=r} \prod_{j=1}^{i} l_j^{-1} P(l_j-l_{j-1}-1;(\alpha,\beta))
\end{aligned} \tag{81}$$

holds for all $r \in \mathbb{Z}_{\geq 1}$, $\alpha,\beta \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$.

Proof. The left-hand side of (81) can be calculated as follows:

$$
\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{r!}\frac{\partial^r}{\partial\gamma^r}\left(\frac{1}{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}\right)\Big|_{\gamma=\beta}\\
&=\frac{\Gamma(\alpha)\Gamma(\beta)}{r!}\frac{\partial^{r-1}}{\partial\gamma^{r-1}}\left(\frac{1}{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}\{\psi(\alpha+\beta-\gamma)-\psi(\gamma)\}\right)\Big|_{\gamma=\beta}\\
&=\frac{1}{r}\sum_{i=0}^{r-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{i!}\frac{\partial^i}{\partial\gamma^i}\left(\frac{1}{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}\right)\Big|_{\gamma=\beta}P(r-1-i;(\alpha,\beta))
\end{aligned}
\tag{82}
$$

for $r\in\mathbb{Z}_{\geq1}$, $\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_{\leq0}$. By using (82) and the induction on $r\in\mathbb{Z}_{\geq1}$, we prove (81). Taking $r=1$ in (82), we get the case $r=1$ of (81). Now we suppose that the identity (81) holds for any $r\in\mathbb{Z}_{\geq1}$. Then, using (82) and the inductive hypothesis, we get the following:

$$
\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{(r+1)!}\frac{\partial^{r+1}}{\partial\gamma^{r+1}}\left(\frac{1}{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}\right)\Big|_{\gamma=\beta}\\
&=\frac{1}{r+1}\sum_{i=0}^{r}\frac{\Gamma(\alpha)\Gamma(\beta)}{i!}\frac{\partial^i}{\partial\gamma^i}\left(\frac{1}{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}\right)\Big|_{\gamma=\beta}P(r-i;(\alpha,\beta))\\
&=(r+1)^{-1}P(r;(\alpha,\beta))\\
&+\sum_{i=1}^{r}\left(\sum_{j=1}^{i}\sum_{0=l_0<l_1<\cdots<l_{j-1}<l_j=i}\prod_{k=1}^{j}l_k^{-1}P(l_k-l_{k-1}-1;(\alpha,\beta))\right)\\
&\times(r+1)^{-1}P(r-i;(\alpha,\beta))\\
&=(r+1)^{-1}P(r;(\alpha,\beta))\\
&+\sum_{j=1}^{r}\sum_{i=j}^{r}\sum_{0=l_0<l_1<\cdots<l_{j-1}<l_j=i}\left(\prod_{k=1}^{j}l_k^{-1}P(l_k-l_{k-1}-1;(\alpha,\beta))\right)\\
&\times(r+1)^{-1}P(r-i;(\alpha,\beta))\\
&=\sum_{j=0}^{r}\sum_{0=l_0<l_1<\cdots<l_j<l_{j+1}=r+1}\prod_{k=1}^{j+1}l_k^{-1}P(l_k-l_{k-1}-1;(\alpha,\beta)).
\end{aligned}
$$

This completes the proof of (81). □

Now we prove Theorem 2.17.

Proof of Theorem 2.17. Taking $a=\alpha+\beta$, $b_i=\alpha+\beta-\gamma$, $c_i=\gamma$ ($i=1,\dots,s+1$), where $s\in\mathbb{Z}_{\geq1}$, $\alpha,\beta,\gamma\in\mathbb{C}$ such that $2s+1>\mathrm{Re}\,(\alpha+\beta)>0$, $\alpha+\beta-\gamma,\gamma\notin\mathbb{Z}_{\leq0}$, in Theorem A (i), we get the identity

$$
\begin{aligned}
&\sum_{0\leq m_1\leq\cdots\leq m_s<\infty}\prod_{i=1}^{s}\frac{1}{(m_i+\alpha+\beta-\gamma)(m_i+\gamma)}\\
&=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_m}{m!}\frac{2m+\alpha+\beta}{(m+\alpha+\beta-\gamma)^{s+1}(m+\gamma)^{s+1}}(-1)^m.
\end{aligned}
\tag{83}
$$

By a direct calculation, we get the identities

$$
\begin{aligned}
&\frac{1}{r!}\frac{\partial^r}{\partial\gamma^r}\left(\prod_{i=1}^{s}\frac{1}{(m_i+\alpha+\beta-\gamma)(m_i+\gamma)}\right)\Big|_{\gamma=\beta}\\
=&\frac{1}{r!}\frac{\partial^r}{\partial\gamma^r}\left(\prod_{i=1}^{s}\left(\frac{1}{(m_i+\alpha+\beta-\gamma)(2m_i+\alpha+\beta)}\right.\right.\\
&\left.\left.+\frac{1}{(m_i+\gamma)(2m_i+\alpha+\beta)}\right)\right)\Big|_{\gamma=\beta}\\
=&\sum_{\substack{r_1+\cdots+r_s=r\\ r_i\in\mathbb{Z}_{\geq 0}}}\prod_{i=1}^{s}\left(\frac{1}{(m_i+\alpha)^{r_i+1}(2m_i+\alpha+\beta)}\right.\\
&\left.+\frac{(-1)^{r_i}}{(m_i+\beta)^{r_i+1}(2m_i+\alpha+\beta)}\right)
\end{aligned}
\tag{84}
$$

for $r\in\mathbb{Z}_{\geq 0}$. Thus, applying the operator $\partial^{(r)}(\gamma)|_{\gamma=\beta}$ ($r\in\mathbb{Z}_{\geq 0}$; $\beta\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ such that $2s+1>\mathrm{Re}\,(\alpha+\beta)>0$, $\alpha\notin\mathbb{Z}_{\leq 0}$) to the left-hand side of (83) and using (84), we get the left-hand side of (80). The right-hand side of (80) can be proved by applying $\partial^{(r)}(\gamma)|_{\gamma=\beta}$ to the right-hand side of (83) and using Lemma 2.18. This completes the proof of (80). Applying (20) to the most right-hand side of (84), we can write the left-hand side of (80) as a linear combination of $\zeta^{\leq}_{s,+}(\{a_i\}_{i=1}^s|\{b_i\}_{i=1}^s|\{1\}^s;(\alpha,\beta,(\alpha+\beta)/2))$, explicitly. □

From (80), we can get the following identity:

Corollary 2.19. *The identity*

$$
\begin{aligned}
&\sum_{\substack{r_1+\cdots+r_s=r\\ r_i\in\mathbb{Z}_{\geq 0}(i=1,\dots,s):\\ \text{even}}}\zeta^{\leq}_{s,+}(\{r_i+2\}_{i=1}^s;\alpha)\\
=&2\sum_{i=0}^{[\frac{r}{2}]}\binom{i+s}{i}G(r-2i;(\alpha,\alpha))\psi^{\sharp}_{1,-}(2i+2s+1;\alpha)
\end{aligned}
\tag{85}
$$

holds for all $r\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, $\alpha\in\mathbb{C}$ with $s+1/2>\mathrm{Re}\,\alpha>0$, where $[x]$ denotes the greatest integer not greater than x and

$$
\psi^{\sharp}_{1,-}(a+b-1;\alpha):=\sum_{m=0}^{\infty}\frac{(2\alpha)_m}{m!}\frac{(-1)^m}{(m+\alpha)^{a+b-1}}=2^{-1}\psi^{\sharp}_{1,-}(a|b;(\alpha,\alpha)).
$$

Proof. Taking $\alpha=\beta$ in (80) and using the identity

$$
\prod_{i=1}^{s}(1+(-1)^{r_i})=\begin{cases}2^s, & \text{if } r_i \text{ is even for all } i,\\ 0, & \text{if } r_i \text{ is odd for some } i,\end{cases}
$$

we get the left-hand side of (85). The right-hand side of the case $\alpha = \beta$ of (80) can be rewritten as

$$
\begin{aligned}
& 2 \sum_{\substack{r_1+r_2+r_3=r \\ r_i \in \mathbb{Z}_{\geq 0}}} (-1)^{r_3} \binom{r_2+s}{r_2} \binom{r_3+s}{r_3} \\
& \quad \times G(r_1; (\alpha,\alpha)) \psi^{\sharp}_{1,-}(r_2+r_3+2s+1; \alpha) \\
=& 2 \sum_{r_1=0}^{r} G(r-r_1; (\alpha,\alpha)) \psi^{\sharp}_{1,-}(r_1+2s+1; \alpha) \\
& \quad \times \sum_{\substack{r_2+r_3=r_1 \\ r_2, r_3 \in \mathbb{Z}_{\geq 0}}} (-1)^{r_3} \binom{r_2+s}{r_2} \binom{r_3+s}{r_3}.
\end{aligned} \tag{86}
$$

Applying the identity

$$
\sum_{\substack{r_2+r_3=r_1 \\ r_2, r_3 \in \mathbb{Z}_{\geq 0}}} (-1)^{r_3} \binom{r_2+s}{r_2} \binom{r_3+s}{r_3} = \begin{cases} \dbinom{r_1/2+s}{r_1/2}, & \text{if } r_1 \text{ is even,} \\ 0, & \text{if } r_1 \text{ is odd} \end{cases}
$$

($r_1 \in \mathbb{Z}_{\geq 0}$) to the right-hand side of (86), we get the right-hand side of (85). We remark that the above binomial sum identity follows from $(1-x)^{-s-1}(1+x)^{-s-1} = (1-x^2)^{-s-1}$. $\square$

3. Remarks

We give some remarks related to the contents of Section 2:

(R1) By the contents of Subsection 2.2, it seems to be interesting to study multiple series of the form

$$
\begin{aligned}
& \sum_{\substack{0 \leq = \lesssim_0 m_1 \lesssim_1 \cdots \lesssim_{P_1-1} m_{P_1} \\ \lesssim_{P_1} m_{P_1+1} \lesssim_{P_1+1} \cdots \lesssim_{P_2-1} m_{P_2} \\ \vdots \\ \lesssim_{P_{n-1}} m_{P_{n-1}+1} \lesssim_{P_{n-1}+1} \cdots \lesssim_{P_n-1} m_{P_n} \lesssim_{P_n} = < \infty}} \prod_{i=0}^{n-1} \frac{(\alpha)_{m_{P_i+1}}}{m_{P_i+1}!} \frac{(\beta)_{m_{P_i+1}}}{(\gamma)_{m_{P_i+1}}} \frac{m_{P_{i+1}}!}{(\alpha)_{m_{P_{i+1}}}} \frac{(\gamma)_{m_{P_{i+1}}}}{(\beta)_{m_{P_{i+1}}}} \\
& \times \left\{ \prod_{j=P_i+1}^{P_{i+1}} \frac{1}{(m_j+\alpha)^{a_j} (m_j+\beta)^{b_j} (m_j+\gamma)^{c_j} (m_j+\delta)^{d_j}} \right\},
\end{aligned} \tag{87}
$$

where $n \in \mathbb{Z}_{\geq 1}$; $p_1 \in \mathbb{Z}_{\geq 1}$, $p_2, \ldots, p_n \in \mathbb{Z}_{\geq 0}$; $P_0 := 0$, $P_i := p_1 + \cdots + p_i$ ($i = 1, \ldots, n$); $a_j, b_j, c_j, d_j \in \mathbb{Z}$ such that $a_j + b_j + c_j + d_j \geq 1$ ($j = 1, \ldots, P_n - 1$), $a_{P_n} + b_{P_n} + c_{P_n} + d_{P_n} \geq 2$; $\alpha, \beta, \gamma, \delta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ such that $\mathrm{Re}\,(\alpha + \beta - \gamma) > 0$. By Theorems 2.2, 2.5 and Corollaries 2.3, 2.4, 2.6, we think that the case $a_j = b_j = 0$ ($j = P_i + 1$; $i = 0, \ldots, n-1$), $a_j = b_j = 1$ ($j = P_i + 2, \ldots, P_{i+1}$; $i = 0, \ldots, n-1$),

$d_j = 0$ $(j = 1, \ldots, P_n)$ of (87) is particularly interesting. We hope that the multiple series (87) is useful for the study of relations among MHZVs. (The case $P_i = i$ $(i = 1, \ldots, n)$ of (87) is a MHZV.) We remark that some identities for (87) can be derived from Theorem A. It seems to be also interesting to consider whether the multiple series (87) with $\alpha = \beta = \gamma = \delta$, $\lesssim_i = \leq$ $(i = 1, \ldots, P_n - 1)$ satisfy the same relation as the two-one formula for MZSVs. This problem was deduced from my consideration for the case $\alpha = \beta = \gamma$ of (8) written in [21, **(R2)**].

(R2) We consider the multiple series

$$\begin{aligned} &Z_n^{\lesssim}(\{k_i\}_{i=1}^n; (\alpha, \beta)) \\ &:= \sum_{0 \leq m_1 \lesssim \cdots \lesssim m_n < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_n!}{(\alpha)_{m_n+1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \beta)^{k_i}} \right\} \frac{1}{(m_n + \beta)^{k_n - 1}}, \end{aligned}$$

where $n \in \mathbb{Z}_{\geq 1}$, $k_1, \ldots, k_{n-1} \in \mathbb{Z}_{\geq 1}$, $k_n \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\beta \notin \mathbb{Z}_{\leq 0}$. The multiple series $Z_n^{<}$ satisfy a sum formula, which generalizes the sum formula for MZVs (see (91) below). A sum formula for $Z_n^{\leq}$ can be proved by using the sum formula for $Z_n^{<}$ and Hoffman's argument in [15, p. 283] (see (93) below). The proof of the sum formula for $Z_n^{<}$ is based on a multiple integral representation of $Z_n^{<}$ (see [16], [18, Lemma 2.2]). We remark that these sum formulas can also be proved by using Theorem A (ii) and in a way similar to that of proving Corollary 2.4 (i) and (iii). Indeed, taking $z = w = \gamma$, $s = 2$ in (3), we get the identity

$$\begin{aligned} &\sum_{0 \leq m_1 \leq m_2 < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_2!}{(\alpha)_{m_2+1}} \frac{(\beta)_{m_1}}{(\beta)_{m_2+1}} \frac{(\gamma)_{m_2}}{(\gamma)_{m_1+1}} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)(m+\beta)(m+\alpha+\beta-\gamma)} + \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)(m+\beta)(m+\gamma)} \end{aligned} \tag{88}$$

for $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$. The identity (88) can be rewritten as

$$\begin{aligned} &\sum_{0 \leq m_1 < m_2 < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_2!}{(\alpha)_{m_2+1}} \frac{(\beta)_{m_1}}{(\beta)_{m_2+1}} \frac{(\gamma)_{m_2}}{(\gamma)_{m_1+1}} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)(m+\beta)(m+\alpha+\beta-\gamma)}. \end{aligned} \tag{89}$$

Differentiating both sides of (89) n times with respect to γ at $\gamma = \beta$ and using the case $m = m_2 - m_1 - 1$, $w = \gamma + m_1 + 1$ of (65), we get the identity

$$\begin{aligned} &\sum_{0 \leq m_1 < l_1 < \cdots < l_n < m_2 < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_2!}{(\alpha)_{m_2+1}} \frac{1}{(m_1+\beta)(m_2+\beta)} \left\{ \prod_{i=1}^{n} \frac{1}{l_i + \beta} \right\} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{n+2}(m+\beta)} \end{aligned} \tag{90}$$

for $n \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$. Further, differentiating both sides of (90) k (≥ 0) times with respect to β, we get the sum formula for $Z_n^<$ ([16], [18, Remark 2.4]):

$$\sum_{\substack{k_1+\cdots+k_n=k+n \\ k_i\in\mathbb{Z}_{\geq 1}(i=1,\dots,n-1), \\ k_n\in\mathbb{Z}_{\geq 2}}} Z_n^<(\{k_i\}_{i=1}^n;(\alpha,\beta)) = \zeta_{1,+}^<(n|k;(\alpha,\beta)) \tag{91}$$

for all $k, n \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re}\alpha > 0$, $\beta \notin \mathbb{Z}_{\leq 0}$. Similarly, differentiating both sides of (88) n times with respect to β at $\beta = \gamma$ and using the case $m = m_2 - m_1$, $z = \beta + m_1$ of (4), we get the identity

$$\begin{aligned} &\sum_{0\leq m_1\leq l_1\leq\cdots\leq l_n\leq m_2<\infty} \frac{(\alpha)_{m_1}}{m_1!}\frac{m_2!}{(\alpha)_{m_2+1}}\frac{1}{(m_1+\gamma)(m_2+\gamma)}\left\{\prod_{i=1}^n\frac{1}{l_i+\gamma}\right\} \\ =&\sum_{i=0}^{n+1}\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^{n-i+2}(m+\gamma)^{i+1}} \end{aligned} \tag{92}$$

for $n \in \mathbb{Z}_{\geq 0}$, $\alpha, \gamma \in \mathbb{C}$ with $\operatorname{Re}\alpha, \operatorname{Re}\gamma > 0$. Further, differentiating both sides of (92) k (≥ 0) times with respect to γ, we get the sum formula for $Z_n^{\leq}$ also:

$$\sum_{\substack{k_1+\cdots+k_n=k+n \\ k_i\in\mathbb{Z}_{\geq 1}(i=1,\dots,n-1), \\ k_n\in\mathbb{Z}_{\geq 2}}} Z_n^{\leq}(\{k_i\}_{i=1}^n;(\alpha,\gamma)) = \sum_{i=0}^{n-1}\binom{k-1+i}{i}\zeta_{1,+}^<(n-i|k+i;(\alpha,\gamma)) \tag{93}$$

for all $k, n \in \mathbb{Z}_{\geq 1}$, $\alpha, \gamma \in \mathbb{C}$ such that $\operatorname{Re}\alpha > 0$, $\gamma \notin \mathbb{Z}_{\leq 0}$. Taking $\alpha = \beta = \gamma = 1$ in (91) and (93), we get the sum formulas for MZVs and MZSVs, respectively (Granville [13], Hoffman [15], Zagier); therefore the sum formulas for MZ(S)Vs can also be derived by using the basic hypergeometric identity of Andrews [1, Theorem 4].

(R3) In [16], I proved the identity

$$\begin{aligned} &\sum_{\substack{k_1+\cdots+k_n=k \\ k_i\in\mathbb{Z}_{\geq 1}(i=1,\dots,n-1), \\ k_n\in\mathbb{Z}_{\geq 2}}} \zeta_{n,+}^<(\{k_i\}_{i=1}^n;\alpha) \\ =&\frac{1}{(k-n-1)!}\sum_{m=0}^{\infty}\frac{1}{(m+1)^n}\frac{\partial^{k-n-1}}{\partial x^{k-n-1}}\left(\frac{(1-x)_m}{(\alpha-x)_{m+1}}\right)\Big|_{x=0} \end{aligned} \tag{94}$$

for all $k, n \in \mathbb{Z}$ such that $0 < n < k$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. I remark that the identity (94) gives a relation between the MHZVs $\zeta_{n,+}^<(\{a_i\}_{i=1}^n;\alpha)$ and the multiple series (23) with $z = 1$. (See also [16, Examples].) Indeed, taking $a = c = X = 1 - x$, $b = d = Y = \alpha$, $Z = \alpha - x$, $W = 1$ in Lemma 2.1, we get the identity

$$\begin{aligned}&\frac{(1-x)_m}{(\alpha-x)_{m+1}}\frac{(\alpha)_m}{(m+1)!}\\ =&\frac{1}{(\alpha-x+m)(1+m)}\sum_{i=0}^{m}\sum_{0\leq m_1<\cdots<m_i<m}\prod_{j=1}^{i}\frac{x(1-\alpha)}{(\alpha-x+m_j)(1+m_j)}.\end{aligned}$$

Using this identity, we get the identity

$$\begin{aligned}&\frac{1}{r!}\frac{\partial^r}{\partial x^r}\left(\frac{(1-x)_m}{(\alpha-x)_{m+1}}\frac{(\alpha)_m}{(m+1)!}\right)\Big|_{x=0}\\ =&\sum_{i=0}^{m}\sum_{\substack{k_1+\cdots+k_{i+1}=r+1\\ k_j\in\mathbb{Z}_{\geq1}}}(1-\alpha)^i\sum_{0\leq m_1<\cdots<m_i<m_{i+1}=m}\prod_{j=1}^{i+1}\frac{1}{(\alpha+m_j)^{k_j}(1+m_j)}\end{aligned}\tag{95}$$

for $m,r\in\mathbb{Z}_{\geq0}$, $\alpha\in\mathbb{C}\setminus\mathbb{Z}_{\leq0}$. By using (95), the right-hand side of (94) can be rewritten as

$$\sum_{i=0}^{k-n-1}\sum_{\substack{k_1+\cdots+k_{i+1}=k-n\\ k_j\in\mathbb{Z}_{\geq1}}}(1-\alpha)^iH^{<}_{i+1,+}(\{k_j\}_{j=1}^{i+1}|\{1\}^i,n;\alpha)$$

($\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>0$), where $H^{<}_{n,+}(\{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n;\alpha)$ denotes the case $z=1$, $\lesssim=<$ of the multiple series (23).

I also remark that the identity (94) can also be proved by applying the product of operators $(-1)^{k-n-1}\partial^{(k-n-1)}(w)|_{w=z}\partial^{(n-1)}(\alpha)|_{\alpha=w}$ to the left-hand sides of the identities (69) and (74); the right-hand sides are completely the same. Using (94) and Hoffman's argument in [15, p. 283], I can also get a sum formula for $\zeta^{\leq}_{n,+}(\{a_i\}_{i=1}^n;\alpha)$ similar to (94).

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