MORE PROPERTIES OF AN OPERATION ON SEMI-GENERALIZED OPEN SETS

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Abstract. The paper continues studying properties of an operation on τ_{sg}. The notions of $sg\gamma$-generalized closed sets and some of its properties are investigated. It also introduces sg-γ-$T_{\frac{1}{2}}$ space via $sg\gamma$-generalized closed set and sg-γ-closed set. Some basic characterization of sg-(γ, β)-irresolute functions with sg-β-closed graphs have been obtained. It studies the concept of sg-γ_0-closed space. Finally, it gives some properties of sg-γ^*-regular and sg-γ^*-normal spaces by using sg-open and sg-closed sets.

Keywords: sg-γ-open sets, $sg\gamma g$-closed sets, sg-γ-T_i spaces ($i \in \{0, \frac{1}{2}, 1, 2\}$), sg-(γ, β)-irresolute functions, sg-β-closed graphs, sg-γ_0-closed space, sg-γ^*-regular and sg-γ^*-normal spaces.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms, compactness etc. by utilizing general-

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ized open sets. Levine [9] introduced the concept of semi-open sets and semi-continuity in topological spaces. In 1987, Battacharyya and Lahiri [4] used semi-open sets to define the notion of semi-generalized closed sets.

Kasahara [10] introduced the notion of an α operation approaches on a class τ of sets and studied the concept of α-continuous functions with α-closed graphs and α-compact spaces. After this, Jankovic [8] introduced the concept of α-closure of a set in X via α-operation and investigated further characterizations of function with α-closed graph. Later, Ogata [13] defined and studied the concept of γ-open sets, and applied it to investigate operation-functions and operation-separation axioms.

Recently, several researchers developed many concepts of operation γ in a space (X, τ). Krishnan, Ganster and Balachandran [11] introduced and studied the concept of the operation γ on the class of all semi-open sets of (X, τ), and defined the notion of semi γ-open sets and investigated some of their properties. An, Cuong and Maki [1] defined and investigated an operation γ on the class of all preopen sets of (X, τ) and introduced the notion of pre-γ-open sets, and developed some of their properties. Tahiliani [14] defined an operation γ on the class of all β-open sets of (X, τ), and described the notion of β-γ-open sets. Carpintero, Rajesh and Rosas [5] studied the operation γ on the class of all b-open sets of (X, τ), and defined the notion of b-γ-open sets. Asaad [2] defined the notion of an operation γ on the class of all generalized open sets in (X, τ) and study some of its applications. Asaad and Ahmad [3] introduced the concept of an operation γ on the collection of all semi-generalized open sets (i.e. τ_{sg}) in (X, τ). By using this operation, they defined the concept of sg-γ-open sets and studied some of their properties. Also, they introduced and investigated sg-γ-T_i spaces for $i \in \{0, 1, 2\}$.

The aim of this study is to introduce the concept of $sg\gamma$-generalized closed sets by utilizing the operation γ on τ_{sg} and then investigate some of its properties. In addition, sg-γ-$T_{\frac{1}{2}}$ spaces are introduced and investigated. Some basic properties of sg-(γ, β)-irresolute functions with sg-β-closed graphs have been obtained. We study the concept of sg-γ_0-closed space and some of its properties. Finally, we give some spaces called sg-γ^*-regular and sg-γ^*-normal by using sg-open and sg-closed sets and study some of their properties.

2. Preliminaries

In this study, the spaces (X, τ) and (Y, σ) (or simply X and Y) represent non-empty spaces on which no separation axioms are assumed, unless otherwise mentioned, and they are simply written as X and Y, respectively, when no confusion arises. The closure and the interior of a set S of a space X are denoted by $Cl(S)$ and $Int(S)$, respectively. A subset S of a space X is said to be semi-open [9] if $S \subseteq Cl(Int(S))$. The complement of a semi-open set is said to be semi-closed [6]. We denote by $SO(X)$ the set of all semi-open sets in (X, τ). The semi-closure of S is defined as the intersection of all semi-closed sets containing S and it is denoted by $sCl(S)$ [6]. A subset S of a space (X, τ) is said to be semi-generalized closed (in short sg-closed) [4] if $sCl(S) \subseteq U$ whenever $S \subseteq U$ and U is a semi-open set in X. The complement of an sg-closed set of X is sg-open. The family of all sg-open subsets of a space (X, τ) is denoted by τ_{sg}. In general, every semi-closed set of a space X is sg-closed. A space (X, τ) is semi-$T_{\frac{1}{2}}$ [4] if every sg-closed subset of X is semi-closed.

An operation γ on $SO(X)$ on X is a mapping $\gamma\colon SO(X) \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in SO(X)$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U. A non-empty subset S of a space (X, τ) with an operation γ on $SO(X)$ is said to be semi γ-open [11] if for each $x \in S$, there exists a semi-open set U containing x such that $\gamma(U) \subseteq S$. The complement of a semi γ-open subset of a space X as semi γ-closed. The family of all semi γ-open sets of a space (X, τ) is denoted by $SO(X)_\gamma$. A point $x \in X$ is in the semi γ-closure [11] of a set $S \subseteq X$ if $\gamma(U) \cap S \neq \phi$ for each semi-open set U containing x. The set of all semi γ-closure points of S is called semi γ-closure of S and is denoted by $sCl_\gamma(S)$. A subset S of (X, τ) with an operation γ on $SO(X)$ is said to be semi γ-g.closed [11] if $sCl_\gamma(S) \subseteq U$ whenever $S \subseteq U$ and U is a semi γ-open set in (X, τ).

Definition 2.1 ([3])**.** An operation γ on τ_{sg} is a mapping $\gamma\colon \tau_{sg} \to P(X)$ such that $U \subseteq \gamma(U)$ for every $U \in \tau_{sg}$. From this, for any operation $\gamma\colon \tau_{sg} \to P(X)$, we have $\gamma(X) = X$. A non-empty set S of X is said to be sg-γ-open if for each $x \in S$, there exists an sg-open set U such that $x \in U$ and $\gamma(U) \subseteq S$. The complement of an sg-γ-open set of X is sg-γ-closed. Assume that the empty set ϕ is also sg-γ-open set for any operation $\gamma\colon \tau_{sg} \to P(X)$. The family of all sg-γ-open subsets of a space (X, τ) is denoted by $\tau_{sg\gamma}$.

The union of any collection of sg-γ-open sets in a topological space X is sg-γ-open. While, the intersection of any two sg-γ-open sets in (X, τ) is generally not an sg-γ-open set. The relation between the concept of sg-open set and sg-γ-open set are independent [3].

Definition 2.2. Let S be any subset of a space (X, τ). Then the class of all sg-γ-open sets in S is defined in a natural way as:
$\tau_{sg\gamma_S} = \{G \cap S : \text{for all } G \in \tau_{sg\gamma}\}$
That is H is sg-γ-open in S if and only if $H = G \cap S$, where $G \in \tau_{sg\gamma}$.

Definition 2.3 ([3])**.** The point $x \in X$ is in the sg-closure$_\gamma$ of a set S if $\gamma(U) \cap S \neq \phi$ for each sg-open set U containing x. The set of all sg-closure$_\gamma$ points of S is called sg-closure$_\gamma$ of S and is denoted by $sgCl_\gamma(S)$.

Definition 2.4 ([3])**.** Let S be any subset of a topological space (X, τ) and γ be an operation on τ_{sg}. The sg-γ-closure of S is defined as the intersection of all sg-γ-closed sets of X containing S and it is denoted by $sg_\gamma Cl(S)$. That is,

$$sg_\gamma Cl(S) = \bigcap\{F : S \subseteq F,\ X \backslash F \in \tau_{sg\gamma}\}.$$

Theorem 2.5 ([3]). *Let S be any subset of a topological space (X, τ) and γ be an operation on τ_{sg}. Then $x \in sg_\gamma Cl(S)$ if and only if $S \cap U \neq \phi$ for every sg-γ-open set U of X containing x.*

Lemma 2.6 ([3]). *The following statements are true for any subsets S and T of a topological space (X, τ) with an operation γ on τ_{sg}.*

1. *$sg_\gamma Cl(S)$ is sg-γ-closed set in X and $sgCl_\gamma(S)$ is sg-closed set in X.*
2. *$S \subseteq sgCl_\gamma(S) \subseteq sg_\gamma Cl(S)$.*
3. *(a) S is sg-γ-closed if and only if $sg_\gamma Cl(S) = S$ and,*
   (b) S is sg-γ-closed if and only if $sgCl_\gamma(S) = S$.
4. *If $S \subseteq T$, then $sg_\gamma Cl(S) \subseteq sg_\gamma Cl(T)$ and $sgCl_\gamma(S) \subseteq sgCl_\gamma(T)$.*
5. *(a) $sg_\gamma Cl(S \cap T) \subseteq sg_\gamma Cl(S) \cap sg_\gamma Cl(T)$ and,*
   (b) $sgCl_\gamma(S \cap T) \subseteq sgCl_\gamma(S) \cap sgCl_\gamma(T)$.
6. *(a) $sg_\gamma Cl(S) \cup sg_\gamma Cl(T) \subseteq sg_\gamma Cl(S \cup T)$ and,*
   (b) $sgCl_\gamma(S) \cup sgCl_\gamma(T) \subseteq sgCl_\gamma(S \cup T)$.
7. *$sg_\gamma Cl(sg_\gamma Cl(S)) = sg_\gamma Cl(S)$.*

Theorem 2.7 ([3]). *Let S be any subset of a topological space (X, τ) and γ be an operation on τ_{sg}. Then the following statements are equivalent:*

1. *S is sg-γ-open set.*
2. *$sgCl_\gamma(X \backslash S) = X \backslash S$.*
3. *$sg_\gamma Cl(X \backslash S) = X \backslash S$.*
4. *$X \backslash S$ is sg-γ-closed set.*

Theorem 2.8 ([4]). *A topological space (X, τ) is semi-$T_{\frac{1}{2}}$ if and only if $\tau_{sg} = SO(X)$.*

Lemma 2.9 ([3]). *If the space (X, τ) is semi-$T_{\frac{1}{2}}$, then $\tau_{sg\gamma} = SO(X)_\gamma$.*

3. $sg\gamma$-generalized closed sets and sg-γ-$T_{\frac{1}{2}}$ spaces

Definition 3.1. A subset S of a topological space (X, τ) with an operation γ on τ_{sg} is said to be $sg\gamma$-generalized closed (in short $sg\gamma g$-closed) if $sgCl_\gamma(S) \subseteq U$ whenever $S \subseteq U$ and U is an sg-γ-open set in X.

Lemma 3.2. *Let (X, τ) be a topological space and γ be an operation on τ_{sg}. A set S in (X, τ) is $sg\gamma g$-closed if and only if $S \cap sg_\gamma Cl(\{x\}) \neq \phi$ for every $x \in sgCl_\gamma(S)$.*

Proof. Suppose S is $sg\gamma g$-closed set in X and suppose (if possible) that there exists an element $x \in sgCl_\gamma(S)$ such that $S \cap sg_\gamma Cl(\{x\}) = \phi$. This follows that $S \subseteq X \backslash sg_\gamma Cl(\{x\})$. Since $sg_\gamma Cl(\{x\})$ is sg-γ-closed implies $X \backslash sg_\gamma Cl(\{x\})$ is sg-γ-open and S is $sg\gamma g$-closed set in X. Then, we have that $sgCl_\gamma(S) \subseteq X \backslash sg_\gamma Cl(\{x\})$. This means that $x \notin sgCl_\gamma(S)$. This is a contradiction. Hence $S \cap sg_\gamma Cl(\{x\}) \neq \phi$.

Conversely, let $U \in \tau_{sg\gamma}$ such that $S \subseteq U$. To show that $sgCl_\gamma(S) \subseteq U$. Let $x \in sgCl_\gamma(S)$. Then by hypothesis, $S \cap sg_\gamma Cl(\{x\}) \neq \phi$. So there exists an element $y \in S \cap sg_\gamma Cl(\{x\})$. Thus $y \in S \subseteq U$ and $y \in sg_\gamma Cl(\{x\})$. By Theorem 2.5, $\{x\} \cap U \neq \phi$. Hence $x \in U$ and so $sgCl_\gamma(S) \subseteq U$. Therefore, S is $sg\gamma g$-closed set in (X, τ). □

Theorem 3.3. *Let S be a subset of topological space (X, τ) and γ be an operation on τ_{sg}. If S is $sg\gamma g$-closed, then $sgCl_\gamma(S) \backslash S$ does not contain any non-empty sg-γ-closed set.*

Proof. Let F be a non-empty sg-γ-closed set in X such that $F \subseteq sgCl_\gamma(S) \backslash S$. Then $F \subseteq X \backslash S$ implies $S \subseteq X \backslash F$. Since $X \backslash F$ is sg-γ-open set and S is $sg\gamma g$-closed set, then $sgCl_\gamma(S) \subseteq X \backslash F$. That is $F \subseteq X \backslash sgCl_\gamma(S)$. Hence $F \subseteq X \backslash sgCl_\gamma(S) \cap sgCl_\gamma(S) \backslash S \subseteq X \backslash sgCl_\gamma(S) \cap sgCl_\gamma(S) = \phi$. This shows that $F = \phi$. This is contradiction. Therefore, $F \nsubseteq sgCl_\gamma(S) \backslash S$. □

Recall that an operation γ on τ_{sg} is said to be sg-open [3] if for each $x \in X$ and for every sg-open set U containing x, there exists an sg-γ-open set W containing x such that $W \subseteq \gamma(U)$.

Theorem 3.4 ([3])**.** *Let S be any subset of a topological space (X, τ). If γ is an sg-open operation on τ_{sg}, then $sgCl_\gamma(S) = sg_\gamma Cl(S)$, $sgCl_\gamma(sgCl_\gamma(S)) = sgCl_\gamma(S)$ and $sgCl_\gamma(S)$ is sg-γ-closed set in X.*

Theorem 3.5. *If $\gamma \colon \tau_{sg} \to P(X)$ is an sg-open operation, then the converse of the Theorem 3.3 is true.*

Proof. Let U be an sg-γ-open set in (X, τ) such that $S \subseteq U$. Since $\gamma \colon \tau_{sg} \to P(X)$ is an sg-open operation, then by Theorem 3.4, $sgCl_\gamma(S)$ is sg-γ-closed set in X. Thus, we have $sgCl_\gamma(S) \cap X \backslash U$ is an sg-γ-closed set in (X, τ). Since $X \backslash U \subseteq X \backslash S$, $sgCl_\gamma(S) \cap X \backslash U \subseteq sgCl_\gamma(S) \backslash S$. Using the assumption of the converse of the Theorem 3.3, $sgCl_\gamma(S) \subseteq U$. Therefore, S is $sg\gamma g$-closed set in (X, τ). □

Corollary 3.6. *Let S be an $sg\gamma g$-closed subset of topological space (X, τ) and let γ be an operation on τ_{sg}. Then S is sg-γ-closed if and only if $sgCl_\gamma(S) \backslash S$ is sg-γ-closed set.*

Proof. Let S be an sg-γ-closed set in (X, τ). Then by Lemma 2.6 (3b), $sgCl_\gamma(S) = S$ and hence $sgCl_\gamma(S) \backslash S = \phi$ which is sg-γ-closed set.

Conversely, suppose $sgCl_\gamma(S)\backslash S$ is sg-γ-closed and S is $sg\gamma g$-closed. Then by Theorem 3.3, $sgCl_\gamma(S)\backslash S$ does not contain any non-empty sg-γ-closed set and since $sgCl_\gamma(S)\backslash S$ is sg-γ-closed subset of itself, then $sgCl_\gamma(S)\backslash S = \phi$ implies $sgCl_\gamma(S) \cap X\backslash S = \phi$. Hence $sgCl_\gamma(S) = S$. This follows from Lemma 2.6 (3b) that S is sg-γ-closed set in (X, τ). □

Theorem 3.7. *Let (X, τ) be a topological space and γ be an operation on τ_{sg}. If a subset S of X is $sg\gamma g$-closed and sg-γ-open, then S is sg-γ-closed.*

Proof. Since S is $sg\gamma g$-closed and sg-γ-open set in X, then $sgCl_\gamma(S) \subseteq S$ and hence by Lemma 2.6 (3b), S is sg-γ-closed. □

Theorem 3.8. *In any topological space (X, τ) with an operation γ on τ_{sg}. For an element $x \in X$, the set $X\backslash\{x\}$ is $sg\gamma g$-closed or sg-γ-open.*

Proof. Suppose that $X\backslash\{x\}$ is not sg-γ-open. Then X is the only sg-γ-open set containing $X\backslash\{x\}$. This implies that $sgCl_\gamma(X\backslash\{x\}) \subseteq X$. Thus $X\backslash\{x\}$ is an $sg\gamma g$-closed set in X. □

Corollary 3.9. *In any topological space (X, τ) with an operation γ on τ_{sg}. For an element $x \in X$, either the set $\{x\}$ is sg-γ-closed or the set $X\backslash\{x\}$ is $sg\gamma g$-closed.*

Proof. Suppose $\{x\}$ is not sg-γ-closed, then $X\backslash\{x\}$ is not sg-γ-open. Hence by Theorem 3.8, $X\backslash\{x\}$ is $sg\gamma g$-closed set in X. □

Definition 3.10. Let S be any subset of a topological space (X, τ) and γ be an operation on τ_{sg}. Then the $\tau_{sg\gamma}$-kernel of S is denoted by $\tau_{sg\gamma}$-ker(S) and is defined as follows:

$$\tau_{sg\gamma}\text{-ker}(S) = \cap\{U : S \subseteq U \text{ and } U \in \tau_{sg\gamma}\}$$

In other words, $\tau_{sg\gamma}$-ker(S) is the intersection of all sg-γ-open sets of (X, τ) containing S.

Theorem 3.11. *Let $S \subseteq (X, \tau)$ and γ be an operation on τ_{sg}. Then S is $sg\gamma g$-closed if and only if $sgCl_\gamma(S) \subseteq \tau_{sg\gamma}$-ker$(S)$.*

Proof. Suppose that S is $sg\gamma g$-closed. Then $sgCl_\gamma(S) \subseteq U$, whenever $S \subseteq U$ and U is sg-γ-open. Let $x \in sgCl_\gamma(S)$. Then by Lemma 3.2, $S \cap sg_\gamma Cl(\{x\}) \neq \phi$. So there exists a point z in X such that $z \in S \cap sg_\gamma Cl(\{x\})$ implies that $z \in S \subseteq U$ and $z \in sg_\gamma Cl(\{x\})$. By Theorem 2.5, $\{x\} \cap U \neq \phi$. Hence we show that $x \in \tau_{sg\gamma}$-ker(S). Therefore, $sgCl_\gamma(S) \subseteq \tau_{sg\gamma}$-ker$(S)$. Conversely, let $sgCl_\gamma(S) \subseteq \tau_{sg\gamma}$-ker$(S)$. Let U be any sg-γ-open set containing S. Let x be a point in X such that $x \in sgCl_\gamma(S)$. Then $x \in \tau_{sg\gamma}$-ker(S). Namely, we have $x \in U$, because $S \subseteq U$ and $U \in \tau_{sg\gamma}\}$. That is $sgCl_\gamma(S) \subseteq \tau_{sg\gamma}$-ker$(S) \subseteq U$. Therefore, S is $sg\gamma g$-closed set in X. □

Definition 3.12. A topological space (X, τ) is said to be:

1. sg-γ-T_0 [3] (resp., semi γ-T_0 [11]) if for any two distinct points x, y in X, there exists an sg-open (resp., a semi-open) set U such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.
2. sg-γ-T_1 [3] (resp., semi γ-T_1 [11]) if for any two distinct points x, y in X, there exist two sg-open (resp., semi-open) sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.
3. sg-γ-T_2 [3] (resp., semi γ-T_2 [11]) if for any two distinct points x, y in X, there exist two sg-open (resp., semi-open) sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \phi$.
4. semi γ-$T_{\frac{1}{2}}$ [11] if every semi γ-g.closed set in X is semi γ-closed.

Definition 3.13. A topological space (X, τ) with an operation γ on τ_{sg} is said to be sg-γ-$T_{\frac{1}{2}}$ if every $sg\gamma g$-closed set in X is sg-γ-closed set.

Theorem 3.14. *For any topological space (X, τ) with an operation γ on τ_{sg}. Then (X, τ) is sg-γ-$T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is sg-γ-closed or sg-γ-open.*

Proof. Let X be an sg-γ-$T_{\frac{1}{2}}$ space and let $\{x\}$ is not sg-γ-closed set in (X, τ). By Corollary 3.9, $X \backslash \{x\}$ is $sg\gamma g$-closed. Since (X, τ) is sg-γ-$T_{\frac{1}{2}}$, then $X \backslash \{x\}$ is sg-γ-closed set which means that $\{x\}$ is sg-γ-open set in X.

Conversely, let F be any $sg\gamma g$-closed set in the space (X, τ). We have to show that F is sg-γ-closed (that is $sgCl_\gamma(F) = F$ (by Lemma 2.6 (3b))). It is sufficient to show that $sgCl_\gamma(F) \subseteq F$. Let $x \in sgCl_\gamma(F)$. By hypothesis $\{x\}$ is sg-γ-closed or sg-γ-open for each $x \in X$. So we have two cases:

Case (1): If $\{x\}$ is sg-γ-closed set. Suppose $x \notin F$, then $x \in sgCl_\gamma(F) \backslash F$ contains a non-empty sg-γ-closed set $\{x\}$. A contradiction since F is $sg\gamma g$-closed set and according to the Theorem 3.3. Hence $x \in F$. This follows that $sgCl_\gamma(F) \subseteq F$ and hence $sgCl_\gamma(F) = F$. This means from by Lemma 2.6 (3b) that F is sg-γ-closed set in (X, τ). Thus (X, τ) is sg-γ-$T_{\frac{1}{2}}$ space.

Case (2): If $\{x\}$ is sg-γ-open set. Then by Theorem 2.5, $F \cap \{x\} \neq \phi$ which implies that $x \in F$. So $sgCl_\gamma(F) \subseteq F$. Thus by Lemma 2.6 (3b), F is sg-γ-closed. Therefore, (X, τ) is sg-γ-$T_{\frac{1}{2}}$ space. □

Theorem 3.15. *For any topological space (X, τ) and any operation γ on τ_{sg}, the following properties hold.*

1. *Every sg-γ-T_2 space is sg-γ-T_1, and every sg-γ-T_1 space is sg-γ-T_0 [3].*
2. *Every sg-γ-T_1 space is sg-γ-$T_{\frac{1}{2}}$.*
3. *Every sg-γ-$T_{\frac{1}{2}}$ space is sg-γ-T_0.*

Remark 3.16. The following diagram of implications follows directly from Theorem 3.15, Remark 3.5 in [11] and Remark 4.12 in [11], we obtain the following diagram of implications.

$$\begin{array}{ccccccc}
sg-\gamma-T_2 & \longrightarrow & sg-\gamma-T_1 & \longrightarrow & sg-\gamma-T_{\frac{1}{2}} & \longrightarrow & sg-\gamma-T_0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{semi}\gamma-T_2 & \longrightarrow & \mathrm{semi}\gamma-T_1 & \longrightarrow & \mathrm{semi}\gamma-T_{\frac{1}{2}} & \longrightarrow & \mathrm{semi}\gamma-T_0
\end{array}$$

Where $S \to T$ represents S implies T.

In the sequel, we shall show that none of the implications that concerning sg-γ-$T_{\frac{1}{2}}$ space in the above diagram is reversible.

Example 3.17. Consider the space (X, τ) as in Example 3.6 in [3]. Then the space (X, τ) is sg-γ-$T_{\frac{1}{2}}$, but (X, τ) is not semi γ-$T_{\frac{1}{2}}$.

Example 3.18. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\} = SO(X)$. Then $\tau_{sg} = P(X)$. Let $\gamma\colon \tau_{sg} \to P(X)$ be an operation on τ_{sg} defined as follows:
For every set $S \in \tau_{sg}$

$$\gamma(S) = \begin{cases} S, & \text{if } b \in S \\ Cl(S), & \text{if } b \notin S \end{cases}$$

Thus, $\tau_{sg\gamma} = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, and the sets $\{a\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ are $sg\gamma g$-closed. Then the space (X, τ) is sg-γ-T_0, but it is not sg-γ-$T_{\frac{1}{2}}$. Since $\{a, b\}$ is $sg\gamma g$-closed set in (X, τ), but $\{a, b\}$ is not sg-γ-closed set in (X, τ).

Example 3.19. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then $\tau_{sg} = \tau$. Let $\gamma\colon \tau_{sg} \to P(X)$ be an operation on τ_{sg} defined as follows:
For every set $S \in \tau_{sg}$

$$\gamma(S) = \begin{cases} S, & \text{if } a \in S \\ Cl(S), & \text{if } a \notin S \end{cases}$$

Thus, $\tau_{sg\gamma} = \tau$. Therefore, the space (X, τ) is sg-γ-$T_{\frac{1}{2}}$, but it is not sg-γ-T_1.

4. sg-(γ, β)-irresolute functions with sg-β-closed graphs

Let (X, τ) and (Y, σ) be two topological spaces and let $\gamma\colon \tau_{sg} \to P(X)$ and $\beta\colon \sigma_{sg} \to P(Y)$ be operations on τ_{sg} and σ_{sg} respectively. In this section, we introduce a new class of functions called sg-(γ, β)-irresolute. Some characterizations and properties of this function are investigated.

Definition 4.1. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be sg-(γ, β)-irresolute if for each $x \in X$ and each sg-open set V containing $f(x)$, there exists an sg-open set U containing x such that $f(\gamma(U)) \subseteq \beta(V)$.

Theorem 4.2. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be an sg-(γ, β)-irresolute function, then,*

1. *$f(sgCl_\gamma(S)) \subseteq sgCl_\beta(f(S))$, for every $S \subseteq (X,\tau)$.*

2. *$f^{-1}(F)$ is sg-γ-closed set in (X,τ), for every sg-β-closed set F of (Y,σ).*

Proof. (1) Let $y \in f(sgCl_\gamma(S))$ and V be any sg-open set containing y. Then by hypothesis, there exists $x \in X$ and sg-open set U containing x such that $f(x) = y$ and $f(\gamma(U)) \subseteq \beta(V)$. Since $x \in sgCl_\gamma(S)$, we have $\gamma(U) \cap S \neq \phi$. Hence $\phi \neq f(\gamma(U) \cap S) \subseteq f(\gamma(U)) \cap f(S) \subseteq \beta(V) \cap f(S)$. This implies that $y \in sgCl_\beta(f(S))$. Therefore, $f(sgCl_\gamma(S)) \subseteq sgCl_\beta(f(S))$.

(2) Let F be any sg-β-closed set of (Y,σ). By using (1), we have

$$f(sgCl_\gamma(f^{-1}(F))) \subseteq sgCl_\beta(F) = F.$$

Therefore, $sgCl_\gamma(f^{-1}(F))$=$f^{-1}(F)$. Hence $f^{-1}(F)$ is sg-γ-closed set in (X,τ). □

Recall that a topological space (X,τ) with an operation γ on τ_{sg} is said to be sg-γ-regular [3] if for each $x \in X$ and for each sg-open set U containing x, there exists an sg-open set W such that $x \in W$ and $\gamma(W) \subseteq U$. The space (X,τ) is an sg-γ-regular if and only if $\tau_{sg} \subseteq \tau_{sg\gamma}$ [3].

Theorem 4.3. *In Theorem* 4.2*, the properties of sg-(γ,β)-irresoluteness of f,* (1) *and* (2) *are equivalent to each other if either the space (Y,σ) is sg-β-regular or the operation β is sg-open.*

Proof. It follows from the proof of Theorem 4.2 that we know the following implications: "sg-(γ,β)-irresoluteness of f" $\Rightarrow$ (1) $\Rightarrow$ (2). Thus, when the space (Y,σ) is sg-β-regular, we prove the implication: (2) $\Rightarrow$ sg-(γ,β)-irresoluteness of f. Let $x \in X$ and let $V \in \sigma_{sg}$ such that $f(x) \in V$. Since (Y,σ) is an sg-β-regular space, then $V \in \sigma_{g\beta}$. By using (2) of Theorem 4.2, $f^{-1}(V) \in \tau_{sg\gamma}$ such that $x \in f^{-1}(V)$. So there exists an sg-open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(V)$. This implies that $f(\gamma(U)) \subseteq V \subseteq \beta(V)$. Therefore, f is sg-(γ,β)-irresolute.

Now, when β is an sg-open operation, we show the implication: (2) $\Rightarrow$ sg-(γ,β)-irresoluteness of f. Let $x \in X$ and let $V \in \sigma_{sg}$ such that $f(x) \in V$. Since β is an sg-open operation, then there exists $W \in \sigma_{g\beta}$ such that $f(x) \in W$ and $W \subseteq \beta(V)$. By using (2) of Theorem 4.2, $f^{-1}(W) \in \tau_{sg\gamma}$ such that $x \in f^{-1}(W)$. So there exists an sg-open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(W) \subseteq f^{-1}(\beta(V))$. This implies that $f(\gamma(U)) \subseteq \beta(V)$. Hence f is sg-(γ,β)-irresolute. □

Definition 4.4. A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be

1. sg-(γ,β)-closed if the image of each sg-γ-closed set of X is sg-β-closed in Y.

2. sg-β-closed if the image of each sg-closed set of X is sg-β-closed in Y.

Theorem 4.5. *Suppose that a function $f\colon (X,\tau) \to (Y,\sigma)$ is both sg-(γ,β)-irresolute and sg-β-closed, then:*

1. *For every $sg\gamma g$-closed set S of (X,τ), the image $f(S)$ is $sg\beta g$-closed in (Y,σ).*
2. *If (X,τ) is semi-$T_{\frac{1}{2}}$, then the inverse set $f^{-1}(T)$ is $sg\gamma g$-closed in (X,τ), for every $sg\beta g$-closed set T of (Y,σ).*

Proof. (1) Let G be any *sg*-β-open set in (Y,σ) such that $f(S) \subseteq G$. Since f is *sg*-(γ,β)-irresolute function, then by using Theorem 4.2 (2), $f^{-1}(G)$ is *sg*-γ-open set in (X,τ). Since S is $sg\gamma g$-closed and $S \subseteq f^{-1}(G)$, we have $sgCl_\gamma(S) \subseteq f^{-1}(G)$, and hence $f(sgCl_\gamma(S)) \subseteq G$. Thus, by Lemma 2.6 (1), $sgCl_\gamma(S)$ is *sg*-closed set and since f is *sg*-β-closed, then $f(sgCl_\gamma(S))$ is *sg*-β-closed set in Y. Therefore, $sgCl_\beta(f(S)) \subseteq sgCl_\beta(f(sgCl_\gamma(S))) = f(sgCl_\gamma(S)) \subseteq G$. This implies that $f(S)$ is $sg\beta g$-closed in (Y,σ).

(2) Let H be any *sg*-γ-open set of a semi-$T_{\frac{1}{2}}$ space (X,τ) such that $f^{-1}(T) \subseteq H$. Let $C = sgCl_\gamma(f^{-1}(T)) \cap (X\backslash H)$, then by Theorem 2.7, $C = sgCl_\gamma(f^{-1}(T)) \cap sgCl_\gamma(X\backslash H)$ and hence by Lemma 2.6 (1) and Theorem 2.8, C is *sg*-closed set in (X,τ). Since f is *sg*-β-closed function. Then $f(C)$ is *sg*-β-closed in (Y,σ). Since f is *sg*-(γ,β)-irresolute function, then by using Theorem 4.2 (1), we have $f(C) = f(sgCl_\gamma(f^{-1}(T))) \cap f(X\backslash H) \subseteq sgCl_\beta(T) \cap f(X\backslash H) \subseteq sgCl_\beta(T) \cap (Y\backslash T) = sgCl_\beta(T)\backslash T$. Since T is an $sg\beta g$-closed set of (Y,σ). Thus, this implies from Theorem 3.3 that $f(C) = \phi$, and hence $C = \phi$. So $sgCl_\gamma(f^{-1}(T)) \subseteq H$. Therefore, $f^{-1}(T)$ is $sg\gamma g$-closed in (X,τ). □

Theorem 4.6. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be injection, sg-(γ,β)-irresolute and sg-β-closed function. If (Y,σ) is sg-β-$T_{\frac{1}{2}}$, then (X,τ) is sg-γ-$T_{\frac{1}{2}}$.*

Proof. Let G be any $sg\gamma g$-closed set of (X,τ). Since f is *sg*-(γ,β)-irresolute and *sg*-β-closed function. Then by Theorem 4.5 (1), $f(G)$ is $sg\beta g$-closed in (Y,σ). Since (Y,σ) is *sg*-β-$T_{\frac{1}{2}}$, then $f(G)$ is *sg*-β-closed in Y. Again, since f is *sg*-(γ,β)-irresolute, then by Theorem 4.2 (2), $f^{-1}(f(G))$ is *sg*-γ-closed in X. Hence G is *sg*-γ-closed in X since f is injection. Therefore, (X,τ) is an *sg*-γ-$T_{\frac{1}{2}}$ space. □

Theorem 4.7. *Let a function $f\colon (X,\tau) \to (Y,\sigma)$ be surjection, sg-(γ,β)-irresolute and sg-β-closed. If (X,τ) is sg-γ-$T_{\frac{1}{2}}$, then (Y,σ) is sg-β-$T_{\frac{1}{2}}$.*

Proof. Let H be an $sg\beta g$-closed set of (Y,σ). Since f is *sg*-(γ,β)-irresolute and *sg*-β-closed function. Then by Theorem 4.5 (2), $f^{-1}(H)$ is $sg\gamma g$-closed in (X,τ). Since (X,τ) is *sg*-γ-$T_{\frac{1}{2}}$, then we have, $f^{-1}(H)$ is *sg*-γ-closed set in X. Again, since f is *sg*-β-closed function, then $f(f^{-1}(H))$ is *sg*-β-closed in Y. Therefore, H is *sg*-β-closed in Y since f is surjection. Hence (Y,σ) is *sg*-β-$T_{\frac{1}{2}}$ space. □

Theorem 4.8. *If $f\colon (X,\tau) \to (Y,\sigma)$ is injection sg-(γ,β)-irresolute function and the space (Y,σ) is sg-β-T_2, then the space (X,τ) is sg-γ-T_2.*

Proof. Let x_1 and x_2 be any distinct points of a space (X,τ). Since f is injection function and (Y,σ) is sg-β-T_2. Then there exist two sg-open sets U_1 and U_2 in Y such that $f(x_1) \in U_1$, $f(x_2) \in U_2$ and $\beta(U_1) \cap \beta(U_2) = \phi$. Since f is sg-(γ,β)-irresolute, there exist sg-open sets V_1 and V_2 in X such that $x_1 \in V_1$, $x_2 \in V_2$, $f(\gamma(V_1)) \subseteq \beta(U_1)$ and $f(\gamma(V_2)) \subseteq \beta(U_2)$. Therefore $\beta(U_1) \cap \beta(U_2) = \phi$. Hence (X,τ) is sg-γ-T_2. □

Theorem 4.9. *If $f\colon (X,\tau) \to (Y,\sigma)$ is injection sg-(γ,β)-irresolute function and the space (Y,σ) is sg-β-T_i, then the space (X,τ) is sg-γ-T_i for $i \in \{0,1\}$.*

Proof. The proof is similar to Theorem 4.8. □

Definition 4.10. A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be sg-(γ,β)-homeomorphism if f is bijection, sg-(γ,β)-irresolute and f^{-1} is sg-(β,γ)-irresolute.

Theorem 4.11. *Suppose that the function $f\colon (X,\tau) \to (Y,\sigma)$ is bijection sg-(γ,β)-irresolute and β be an sg-open operation. Then f is sg-(γ,β)-open (resp., sg-(γ,β)-closed) if and only if f^{-1} is sg-(β,γ)-irresolute.*

Proof. Obvious. □

Theorem 4.12. *Assume that a function $f\colon (X,\tau) \to (Y,\sigma)$ is sg-(γ,β)-homeomorphism. If (X,τ) is sg-γ-$T_{\frac{1}{2}}$, then (Y,σ) is sg-β-$T_{\frac{1}{2}}$.*

Proof. Let $\{y\}$ be any singleton set of (Y,σ). Then there exists an element x of X such that $y = f(x)$. So by hypothesis and Theorem 3.14, we have $\{x\}$ is sg-γ-closed or sg-γ-open set in X. By using Theorem 4.2, $\{y\}$ is sg-β-closed or sg-β-open set. Hence the space by Theorem 3.14, (Y,σ) is sg-β-$T_{\frac{1}{2}}$. □

For a function $f\colon (X,\tau) \to (Y,\sigma)$, the subset $\{(x,f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f and is denoted by $G(f)$ [7]. In this section, we further investigate general operator approaches of closed graphs of functions. Let $\lambda\colon (\tau \times \sigma)_{sg} \to P(X \times Y)$ be an operation on $(\tau \times \sigma)_{sg}$.

Definition 4.13. The graph $G(f)$ of $f\colon (X,\tau) \to (Y,\sigma)$ is called sg-β-closed if for each $(x,y) \in (X \times Y)\backslash G(f)$, there exist sg-open sets $U \subseteq X$ and $V \subseteq Y$ containing x and y, respectively, such that $(U \times \beta(V)) \cap G(f) = \phi$.

The proof of the following lemma follows directly from the above definition.

Lemma 4.14. *A function $f\colon (X,\tau) \to (Y,\sigma)$ has sg-β-closed graph if and only if for each $(x,y) \in (X \times Y)\backslash G(f)$, there exist $U \in \tau_{sg}$ containing x and $V \in \sigma_{sg}$ containing y such that $f(U) \cap \beta(V) = \phi$.*

Definition 4.15. An operation $\lambda\colon (\tau \times \sigma)_{sg} \to P(X \times Y)$ is said to be *sg*-associated with γ and β if $\lambda(U \times V) = \gamma(U) \times \beta(V)$ holds for each $U \in \tau_{sg}$ and $V \in \sigma_{sg}$.

Recall that an operation γ on τ_{sg} is said to be *sg*-regular [3] if for each $x \in X$ and for every pair of *sg*-open sets U_1 and U_2 such that both containing x, there exists an *sg*-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$.

Definition 4.16. The operation $\lambda\colon (\tau \times \sigma)_{sg} \to P(X \times Y)$ is said to be *sg*-regular with respect to γ and β if for each $(x, y) \in X \times Y$ and each *sg*-open set W containing (x, y), there exist *sg*-open sets U in X and V in Y such that $x \in U$, $y \in V$ and $\gamma(U) \times \beta(V) \subseteq \lambda(W)$.

Theorem 4.17. *Let $\lambda\colon (\tau \times \tau)_{sg} \to P(X \times X)$ be an sg-associated operation with γ and γ. If $f\colon (X, \tau) \to (Y, \sigma)$ is an sg-(γ, β)-irresolute function and (Y, σ) is an sg-β-T_2 space, then the set $S = \{(x, y) \in X \times X : f(x) = f(y)\}$ is an sg-λ-closed set of $(X \times X, \tau \times \tau)$.*

Proof. We want to prove that $sgCl_\lambda(S) \subseteq S$. Let $(x, y) \in (X \times X)\backslash S$. Since (Y, σ) is *sg*-β-T_2. Then there exist two *sg*-open sets U and V in (Y, σ) such that $f(x) \in U$, $f(y) \in V$ and $\beta(U) \cap \beta(V) = \phi$. Moreover, for U and V there exist *sg*-open sets G and H in (X, τ) such that $x \in G$, $y \in H$ and $f(\gamma(G)) \subseteq \beta(U)$ and $f(\gamma(H)) \subseteq \beta(V)$ since f is *sg*-(γ, β)-irresolute. Therefore we have $(x, y) \in \gamma(G) \times \gamma(H) = \lambda(G \times H) \cap S = \phi$ because $G \times H \in (\tau \times \tau)_{sg}$. This shows that $(x, y) \notin sgCl_\lambda(S)$. □

Corollary 4.18. *Suppose $\lambda\colon (\tau \times \tau)_{sg} \to P(X \times X)$ is sg-associated operation with γ and γ, and it is sg-regular with γ and γ. A space (X, τ) is sg-γ-T_2 if and only if the diagonal set $\triangle = \{(x, x) : x \in X\}$ is sg-λ-closed of $(X \times X, \tau \times \tau)$.*

Theorem 4.19. *Let $\lambda\colon (\tau \times \sigma)_{sg} \to P(X \times Y)$ be an sg-associated operation with γ and β. If $f\colon (X, \tau) \to (Y, \sigma)$ is sg-(γ, β)-irresolute and (Y, σ) is sg-β-T_2, then the graph of f, $G(f) = \{(x, f(x)) \in X \times Y\}$ is an sg-λ-closed set of $(X \times Y, \tau \times \sigma)$.*

Proof. The proof is similar to Theorem 4.17. □

Definition 4.20. Let (X, τ) be a topological space and γ be an operation on τ_{sg}. A subset S of X is said to be *sg*-γ-compact if for every *sg*-open cover $\{U_i, i \in \mathbb{N}\}$ of S, there exists a finite subfamily $\{U_1, U_2, ..., U_n\}$ such that $S \subseteq \gamma(U_1) \cup \gamma(U_2) \cup ... \cup \gamma(U_n)$.

Theorem 4.21. *Suppose that γ is sg-regular and $\lambda\colon (\tau \times \sigma)_{sg} \to P(X \times Y)$ is sg-regular with respect to γ and β. Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function whose graph $G(f)$ is sg-λ-closed in $(X \times Y, \tau \times \sigma)$. If a subset S is sg-β-closed in (Y, σ), then $f^{-1}(S)$ is sg-γ-closed in (X, τ).*

Proof. Suppose that $f^{-1}(S)$ is not sg-γ-closed then there exist a point x such that $x \in sgCl_\gamma(f^{-1}(S))$ and $x \notin f^{-1}(S)$. Since $(x,s) \notin G(f)$ and each $s \in S$ and $sgCl_\lambda(G(f)) \subseteq G(f)$, there exists an sg-open set W of $(X \times Y, \tau \times \sigma)$ such that $(x,s) \in W$ and $\beta(W) \cap G(f) = \phi$. By sg-regularity of λ, for each $s \in S$ we can take two sg-open sets $U(s)$ and $V(s)$ in (Y,σ) such that $x \in U(s)$, $s \in V(s)$ and $\gamma(U(s)) \times \beta(V(s)) \subseteq \lambda(W)$. Then we have $f(\gamma(U(s))) \cap \beta(V(s)) = \phi$. Since $\{V(s) : s \in S\}$ is sg-open cover of S, then by sg-γ-compactness there exists a finite number $s_1, s_2, ..., s_n \in S$ such that $S \subseteq \beta(V(s_1)) \cup \beta(V(s_2)) \cup ... \cup \beta(V(s_n))$. By the sg-regularity of γ, there exist an sg-open set U such that $x \in U$, $\gamma(U) \subseteq \gamma(U(s_1)) \cap \gamma(U(s_2)) \cap ... \cap \gamma(U(s_n))$. Therefore, we have $\gamma(U) \cap f^{-1}(S) \subseteq U(s_i) \cap f^{-1}(\beta(V(s_i))) = \phi$. This shows that $x \notin sgCl_\gamma(f^{-1}(S))$. This is a contradiction. Therefore, $f^{-1}(S)$ is sg-γ-closed. □

Theorem 4.22. *Suppose that the following condition hold:*

1. $\gamma \colon \tau_{sg} \to P(X)$ *is sg-open*
2. $\beta \colon \sigma_{sg} \to P(Y)$ *is sg-regular, and*
3. $\lambda \colon (\tau \times \sigma)_{sg} \to P(X \times Y)$ *is associated with* γ *and* β*, and* λ *is sg-regular with respect to* γ *and* β*.*

Let $f \colon (X,\tau) \to (Y,\sigma)$ *be a function whose graph* $G(f)$ *is sg-*λ*-closed in* $(X \times Y, \tau \times \sigma)$*. If every cover of* S *by sg-*γ*-open sets of* (X,τ) *has finite sub cover, then* $f(S)$ *is sg-*β*-closed in* (Y,σ)*.*

Proof. Similar to Theorem 4.21. □

5. sg-γ_0-closed spaces

Definition 5.1. A space X is said to be sg-γ_0-closed if for every cover $\{U_i : i \in I\}$ of X by sg-γ-open sets of X, there exists a finite subset I_0 of I such that $X = \bigcup_{i \in I_0} sgCl_\gamma(U_i)$.

Theorem 5.2. *A space* X *is sg-*γ_0*-closed if and only if every class of sg-*γ*-open and sg-*γ*-closed sets with empty intersection has a finite subclass with empty intersection.*

Proof. Obvious. □

Lemma 5.3. *Let* $f \colon (X,\tau) \to (Y,\sigma)$ *be a function and* β *be an sg-open operation. Then* f *is sg-*(γ,β)*-irresolute if and only if* $f^{-1}(V)$ *is sg-*γ*-open set in* X*, for every sg-*β*-open set* V *of* Y*.*

Proof. Obvious. □

Theorem 5.4. *Let* $f \colon (X,\tau) \to (Y,\sigma)$ *be an sg-*(γ,β)*-irresolute function from an sg-*γ_0*-closed* X *onto a space* Y *and* β *be an sg-open operation. Then* Y *is sg-*β_0*-closed.*

Proof. Let $\{V_i : i \in I\}$ be a cover of Y by sg-β-open sets of Y. Since f is sg-(γ, β)-irresolute and β is an sg-open operation. Then by Lemma 5.3, the cover $\{U_i : U_i = f^{-1}(V_i) : i \in I\}$ of X is an sg-γ-open sets of X. Since X is sg-γ_0-closed, then there exists a finite subset I_0 of I such that $X = \bigcup_{i\in I_0} sgCl_\gamma(U_i)$. This gives that

$$\begin{aligned} Y = f(X) &= f(\bigcup_{i\in I_0} sgCl_\gamma(U_i)) \\ &= \bigcup_{i\in I_0} f(sgCl_\gamma(U_i)) \ (by\ Theorem\ 4.2) \\ &\subseteq \bigcup_{i\in I_0} sgCl_\beta(f(U_i)) = \bigcup_{i\in I_0} sgCl_\beta(V_i). \end{aligned}$$

Therefore, the space Y is sg-β_0-closed. □

Definition 5.5. A subset S of a topological space (X, τ) is said to be sg-γ_0-closed if for every cover $\{U_i : i \in I\}$ of S by sg-γ-open sets of X, there exists a finite subset I_0 of I such that $S \subseteq \bigcup_{i\in I_0} sgCl_\gamma(U_i)$.

Theorem 5.6. *Suppose that $f\colon (X, \tau) \to (Y, \sigma)$ is injection sg-(γ, β)-irresolute function and the operation β is sg-open. If S is an sg-γ_0-closed subspace of a space X. Then $f(S)$ is sg-β_0-closed subspace of a space Y.*

Proof. Let $\{V_i : i \in I\}$ be a cover of $f(S)$ by sg-β-open sets of Y. Then $V_i = f(S) \cap U_i$, where U_i is an sg-β-open sets in Y. Since f is sg-(γ, β)-irresolute and β is an sg-open operation. Then by Lemma 5.3, $f^{-1}(U_i)$ is sg-γ-open sets in X. Therefore, we have $f^{-1}(V_i) = S \cap f^{-1}(U_i)$ and hence the cover $\{f^{-1}(V_i) : i \in I\}$ of S is an sg-γ-open sets of X. Since S is sg-γ_0-closed, then there exists a finite subset I_0 of I such that $S \subseteq \bigcup_{i\in I_0} sgCl_\gamma(f^{-1}(V_i))$. This gives that

$$\begin{aligned} f(S) &\subseteq f(\bigcup_{i\in I_0} sgCl_\gamma(f^{-1}(V_i))) \\ &= \bigcup_{i\in I_0} f(sgCl_\gamma(f^{-1}(V_i))) \subseteq \bigcup_{i\in I_0} sgCl_\beta(f(f^{-1}(V_i))) \ (\text{by Theorem } 4.2) \\ &= \bigcup_{i\in I_0} sgCl_\beta(V_i). \end{aligned}$$

Hence, $f(S)$ is sg-β_0-closed subspace of a space Y. □

Theorem 5.7. *If γ is an sg-regular operation. Then each sg-γ-closed subset of an sg-γ_0-closed space X is sg-γ_0-closed.*

Proof. Let F be an sg-γ-closed subset of an sg-γ_0-closed space X, and let $\{V_i : i \in I\}$ be a cover of F by sg-γ-open sets of X. Then for sg-γ-open set V_i in F, we have $V_i = F \cap U_i$, where U_i is an sg-γ-open sets in X. Since X is sg-γ_0-closed space, then there exists a finite subset I_0 of I such that $X = \bigcup_{i\in I_0} sgCl_\gamma(U_i)$. Since γ is an sg-regular operation. Therefore, we have

$$\begin{aligned} F &= F \cap X \\ &= sgCl_\gamma(F) \cap \bigcup_{i\in I_0} sgCl_\gamma(U_i) \; (by\; Lemma\; 2.6) \\ &= \bigcup_{i\in I_0} (sgCl_\gamma(F) \cap sgCl_\gamma(U_i)) \\ &\subseteq \bigcup_{i\in I_0} (sgCl_\gamma(F \cap U_i)) \\ &\subseteq \bigcup_{i\in I_0} (sgCl_\gamma(V_i)). \end{aligned}$$

Thus, the set F is sg-γ_0-closed. $\square$

Definition 5.8. An operation γ on τ_{sg} is said to be sg-γ-open if $\gamma(U)$ is sg-γ-open for each sg-γ-open set U.

Theorem 5.9. *Let γ be both sg-regular and sg-γ-open operation, and let X be an sg-γ-T_2 space. Suppose that F is an sg-γ_0-closed subset of X and $x \in X\backslash F$. Then there are sg-open sets U_x and V_x in X such that $x \in \gamma(U_x)$, $F \subseteq \gamma(V_x)$ and $\gamma(U_x) \cap \gamma(V_x) = \phi$.*

Proof. Suppose that F is an sg-γ_0-closed subset of X and $x \in X\backslash F$. For every $y \in F$, $y \neq x$. Since the space X is sg-γ-T_2, there exist sg-open sets U_x and V_y containing x and y respectively such that $\gamma(U_x) \cap \gamma(V_y) = \phi$. Now, let $\{\gamma(V_y) \cap F : y \in F\}$ be a cover of F by sg-γ-open sets of X. Therefore, this cover has a finite subset $\{\gamma(V_{y_1}) \cap F, \gamma(V_{y_2}) \cap F, ..., \gamma(V_{y_n}) \cap F\}$ such that $F = \bigcup_{i=1}^n sgCl_\gamma(\gamma(V_{y_i}) \cap F)$. Let $\gamma(U_{y_1})$, $\gamma(U_{y_2})$, ..., $\gamma(U_{y_n})$ be the corresponding sg-γ-open sets containing x. Take

$$\gamma(U_x) = \bigcap_{i=1}^n sgCl_\gamma(\gamma(U_{y_i})) \text{ and } \gamma(V_x) = \bigcup_{i=1}^n sgCl_\gamma(\gamma(V_{y_i}))$$

Then $x \in \gamma(U_x)$ and $F \subseteq \gamma(V_x)$, where $\gamma(U_x)$ and $\gamma(V_x)$ are sg-γ-closed since γ is sg-regular. Also,

$$\begin{aligned} \gamma(U_x) \cap \gamma(V_x) &= \bigcap_{i=1}^n sgCl_\gamma(\gamma(U_{y_i})) \cap \bigcup_{i=1}^n sgCl_\gamma(\gamma(V_{y_i})) \\ &= \bigcap_{i=1}^n \bigcup_{i=1}^n sgCl_\gamma(\gamma(U_{y_i})) \cap sgCl_\gamma(\gamma(V_{y_i})) \\ &= \bigcap_{i=1}^n \bigcup_{i=1}^n sgCl_\gamma[\gamma(U_{y_i}) \cap \gamma(V_{y_i})] \; (\gamma\; is\; sg{-}regular) \\ &= \bigcap_{i=1}^n \bigcup_{i=1}^n sgCl_\gamma(\phi) = \phi. \end{aligned}$$

This completes the proof. □

Theorem 5.10. *Let γ be both sg-regular and sg-γ-open operation, and let X be an sg-γ-T_2 space. Then each sg-γ_0-closed subset of X is sg-γ-closed.*

Proof. The proof is similar to Theorem 5.9. □

Theorem 5.11. *Let the operations γ be sg-regular, and β be sg-regular, sg-β-open and sg-open. If the function $f\colon (X,\tau) \to (Y,\sigma)$ is injection sg-(γ,β)-irresolute from an sg-γ_0-closed X into an sg-β-T_2 space Y. Then it is sg-(γ,β)-closed.*

Proof. Let X be an sg-γ_0-closed space and S be an sg-γ-closed subset of X. Since γ is sg-regular, then by Theorem 5.7, S is sg-γ_0-closed subset of X. Since β is sg-open and f is injection sg-(γ,β)-irresolute, then by Theorem 5.6, $f(S)$ is sg-β_0-closed subset of Y. Since Y is sg-β-T_2, and β is sg-regular and sg-β-open. Thus, by Theorem 5.10, $f(S)$ is sg-β-closed in Y. Therefore, f is sg-(γ,β)-closed. □

The proof of the following theorem is immediate:

Theorem 5.12. *Let the operations γ be sg-regular, and β be sg-regular, sg-β-open and sg-open. If the function $f\colon (X,\tau) \to (Y,\sigma)$ is bijection sg-(γ,β)-irresolute from an sg-γ_0-closed space X onto an sg-β-T_2 space Y. Then it is sg-(γ,β)-homeomorphism.*

Theorem 5.13. *Let γ be sg-regular and sg-open operation. If S and T are sg-γ_0-closed subsets of X such that $X = sgCl_\gamma(S) \cup sgCl_\gamma(T)$, then X is sg-γ_0-closed.*

Proof. The proof is easy and hence it is omitted. □

6. sg-γ^*-regular and sg-γ^*-normal spaces

Definition 6.1. A space (X,τ) is said to be sg-γ^*-regular if for each sg-closed set F of X not containing $x \in X$, there exist sg-open sets G and H such that $x \in G$, $F \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$.

Example 6.2. Consider the space $X = \{a,b,c\}$ and $\tau = \{\phi, X, \{a\}, \{b,c\}\}$. Then $\tau_{sg} = P(X)$. Define an operation $\gamma\colon \tau_{sg} \to P(X)$ by $\gamma(S) = S$ for all $S \in \tau_{sg}$. Hence it is easy to show that X is sg-γ^*-regular space.

Example 6.3. Let $X = \{a,b,c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}\{b,c\}, X\} = SO(X)$. Then $\tau_{sg} = \tau$. Let $\gamma\colon \tau_{sg} \to P(X)$ be an operation on τ_{sg} defined as follows: For every set $S \in \tau_{sg}$

$$\gamma(S) = \begin{cases} S, & \text{if } b \in S \\ Cl(S), & \text{if } b \notin S \end{cases}$$

Clearly, the space X is not sg-γ^*-regular.

Theorem 6.4. *If γ is sg-regular, then every subspace of sg-γ^*-regular space X is sg-γ^*-regular.*

Proof. Let S be a subspace of an sg-γ^*-regular space X. We show that S is sg-γ^*-regular. Suppose F is sg-closed set in S and $a \in S$ such that $a \notin F$. Then $F = E \cap S$, where E is sg-closed in X. Then $a \notin E$. Since X is sg-γ^*-regular space, then there exist sg-open sets G, H in X such that $a \in G$, $E \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$. Then $G \cap S$ and $H \cap S$ are sg-open sets in S containing a and F respectively, also since γ is sg-regular, so we have

$$\begin{aligned}\gamma(G \cap S) \cap \gamma(H \cap S) &\subseteq (\gamma(G) \cap \gamma(S)) \cap (\gamma(H) \cap \gamma(S)) \\ &= (\gamma(G) \cap \gamma(H)) \cap \gamma(S) = \phi \cap \gamma(S) = \phi.\end{aligned}$$

Therefore, S is sg-γ^*-regular. □

Definition 6.5. A space (X, τ) is said to be sg-γ^*-normal if for each disjoint sg-closed sets E, F of X, there exist sg-open sets G and H such that $E \subseteq G$, $F \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$.

Example 6.6. It is obvious from Example 6.3 that the space (X, τ) is sg-γ^*-normal.

Example 6.7. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$. Then $SO(X) = \{\phi, \{a\}, \{a, b\}\{a, c\}, X\} = \tau_{sg}$. Let $\gamma\colon \tau_{sg} \to P(X)$ be an operation on τ_{sg} defined as follows:
For every set $S \in \tau_{sg}$

$$\gamma(S) = \begin{cases} S, & \text{if } a \in S \\ Cl(S), & \text{if } a \notin S \end{cases}$$

Obviously, the space X is not sg-γ^*-normal.

Theorem 6.8. *If γ is sg-regular, then every sg-closed subspace of sg-γ^*-normal space X is sg-γ^*-normal.*

Proof. Let S be an sg-closed subspace of an sg-γ^*-normal space X. We show that S is sg-γ^*-normal space. Suppose E_1 and E_2 are disjoint sg-closed sets of S. Then there exist sg-γ-closed sets F_1 and F_2 in X such that $E_1 = F_1 \cap S$ and $E_2 = F_2 \cap S$. Since S is sg-closed in X, thus E_1 and E_2 are sg-closed sets in X. Since X is sg-γ^*-normal space, then there exist sg-open sets G, H in X such that $E_1 \subseteq G$, $E_2 \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$. Then $G \cap S$ and $H \cap S$ are sg-open sets in S such that $E_1 \subseteq G \cap S$ and $E_2 \subseteq H \cap S$, also since γ is sg-regular, therefore we have

$$\begin{aligned}\gamma(G \cap S) \cap \gamma(H \cap S) &\subseteq (\gamma(G) \cap \gamma(S)) \cap (\gamma(H) \cap \gamma(S)) \\ &= (\gamma(G) \cap \gamma(H)) \cap \gamma(S) = \phi \cap \gamma(S) = \phi.\end{aligned}$$

This means that, S is sg-γ^*-normal. □

Theorem 6.9. *Let γ be both sg-regular and sg-γ-open operation. Then each sg-γ_0-closed and sg-γ-T_2 space is sg-γ^*-normal.*

Proof. Let X be an sg-γ_0-closed and sg-γ-T_2 space and let E and F be disjoint sg-closed sets of X. Then E and F are sg-γ-closed sets because γ is sg-regular. Since X is sg-γ_0-closed space, then by Theorem 5.7, E and F are sg-γ_0-closed. Since X is sg-γ-T_2, then by Theorem 5.9, for sg-γ_0-closed F and $x \in X\backslash F$. Then there are sg-open sets U_x and V_x in X such that $x \in \gamma(U_x)$, $F \subseteq \gamma(V_x)$ and $\gamma(U_x) \cap \gamma(V_x) = \phi$. Let $\{\gamma(U_x) : x \in E\}$ be a cover of E by sg-γ-open and sg-γ-closed sets of X (since by Theorem 5.10, E is sg-γ-closed). Therefore, there are finite number of elements x_1, x_2, ..., x_n such that $E \subseteq \bigcup_{i=1}^n sgCl_\gamma(\gamma(U_{x_i})) = \bigcup_{i=1}^n \gamma(U_{x_i})$. Hence

$$\gamma(\textstyle\bigcup_{i=1}^n \gamma(U_{x_i}) \cap \bigcap_{i=1}^n \gamma(V_{x_i})) = \gamma(\phi) = \phi.$$

Which implies that X is sg-γ^*-normal. □

Definition 6.10. An operation γ on τ_{sg} is said to be strongly sg-regular if for each $x \in X$ and for every pair of sg-open sets U_1 and U_2 such that both containing x, there exists an sg-open set W containing x such that $\gamma(W) = \gamma(U_1) \cap \gamma(U_2)$.

Theorem 6.11. *Let γ be both strongly sg-regular and sg-γ-open operation. If for each sg-closed set F in X and each sg-open set G containing F, there exists an sg-open set H containing F such that $sgCl_\gamma(\gamma(H)) \subseteq \gamma(G)$, then X is sg-γ^*-normal.*

Proof. Let E and F be two sg-closed sets in X such that $E \cap F = \phi$. Then $F \subseteq X\backslash E$, where $X\backslash E$ is sg-open set in X. By hypothesis, there exists an sg-open set H containing F such that $sgCl_\gamma(\gamma(H)) \subseteq \gamma(X\backslash E)$. This gives that $\gamma(E) \subseteq \gamma(X\backslash sgCl_\gamma(H))$ and $H \cap X\backslash sgCl_\gamma(\gamma(H)) = \phi$. Thus, $F \subseteq H$, $E \subseteq X\backslash sgCl_\gamma(\gamma(H))$ and $\gamma(H) \cap \gamma(X\backslash sgCl_\gamma(\gamma(H))) = \phi$. Consequently, X is sg-γ^*-normal space. □

Definition 6.12 ([12])**.** A topological space (X, τ) is said to be semi generalized-T_1 (in short sg-T_1) if for any two distinct points x, y in X, there exist two sg-open sets, one containing x but not y, and the other containing y but not x.

Theorem 6.13 ([12])**.** *The space (X, τ) is sg-T_1 if and only if for every point $x \in X$, $\{x\}$ is an sg-closed set.*

Now we prove the following theorem.

Theorem 6.14. *Every sg-γ^*-normal and sg-T_1 space (X, τ) is sg-γ^*-regular.*

Proof. Let F be an sg-closed set in X and $x \in X$ does not belong to F. Since X is sg-T_1. Then by Theorem 6.13, $\{x\}$ is sg-closed. So $\{x\}$ and F are two disjoint sg-closed sets of X. Since X is sg-γ^*-normal, then there exist sg-open sets G and H such that $x \in \{x\} \subseteq G$, $F \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$. Hence X is sg-γ^*-regular. □

7. Conclusion

This paper continues studied properties of an operation on τ_{sg}. The notions of $sg\gamma$-generalized closed sets and some of its properties have been investigated. It has been introduced sg-γ-$T_{\frac{1}{2}}$ space via $sg\gamma$-generalized closed set and sg-γ-closed set. Some basic characterization of sg-(γ, β)-irresolute functions with sg-β-closed graphs have been obtained. It has been studied the concept of sg-γ_0-closed space. Finally, some properties of sg-γ^*-regular and sg-γ^*-normal spaces by using sg-open and sg-closed sets have been given.

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Accepted: 30.10.2017