THE HECKE ALGEBRA $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ AND ITS RELATION TO THE CROSSED PRODUCT $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$

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Abstract. The algebra $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}})$ arose in number theory has been studied by Bost and Connes in [2]. In [1] a related Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is considered wherein it is shown to be a universal $*$-algebra generated by the elements $\{\mu_n : n \in \mathbb{N}^*\}$, $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ and an element $u = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$. The goal of this paper is to study the relationship between the Hecke algebra of Bost and Connes and the Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$. By showing the existence of a $*$-automorphism α of $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}})$, we construct a covariant representation (ι, U) of $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$ on $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$. This leads to our main result that $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is realized as the crossed product $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$.
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1. Introduction

A Hecke pair (G, S) consists of a discrete group G and a subgroup S of G such that every double coset consists of finitely many left cosets.

The Hecke algebra $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}})$ first arose in Bost and Connes' study [2] and they have proved that it is a universal $*$-algebra generated by elements $\{\mu_n : n \in \mathbb{N}^*\}$ and $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ subject to six relations.

In [1] we introduced a new Hecke pair $(P_{\mathbb{Q}}, P_{\mathbb{Z}})$, where

$$P_{\mathbb{Q}} = \left\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r \neq 0 \right\}.$$

Then we showed that this closely related Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is a universal $*$-algebra generated by elements $\{\mu_n : n \in \mathbb{N}^*\}$, $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ and an element

$$u = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right].$$

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We begin with a preliminaries section in which we define Hecke algebras, set up our notation and give information about the Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$. Then we review dynamical systems and their covariant representations and recall the basic properties. In section 3, we show the existence of a $*$-automorphism α of $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$. In section 4, we show the existence of a $*$-homomorphism ϕ from $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ into $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$. We then construct a covariant representation (ι, U) of $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$ on $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$. This enables us to show our main theorem that $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is realized as the crossed product $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$.

2. Preliminaries

In this section we give the background required for this paper, give the necessary information about the Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$, and setup our notations.

Definition 2.1. *Let G be a discrete group and S a subgroup of G. The pair (G, S) ia called a* Hecke pair *if each double coset StS can be written as a finite union of left cosets.*

The following Proposition can be found in [1, Proposition 2.2].

Proposition 2.2. *Let (G, S) be a Hecke pair. Then the set*

$$H(G, S) = \big\{ f : S \backslash G / S \longrightarrow \mathbb{C} : f \text{ has finite support} \big\}$$

is a $*$-algebra with

$$(f * g)(StS) = \sum_{rS \in G/S} f(SrS) g(Sr^{-1}tS) \tag{2.1}$$

and

$$f^{*}(StS) = \overline{f(St^{-1}S)}.$$

This $*$-algebra is called a Hecke algebra.

Bost and Connes defined

$$P_{\mathbb{Q}}^{+} = \Big\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r > 0 \Big\},$$

and

$$P_{\mathbb{Z}} = \Big\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \Big\}.$$

Then they showed that $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$ is the universal $*$-algebra generated by the elements $\{\mu_n : n \in \mathbb{N}^{*}\}$ and $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ subject to the following four relations according to the improvement of their theorem given by Laca and Raeburn in [4]:

(a) $\mu_n^*\mu_n = 1$ for all $n \in \mathbb{N}^*$.
(b) $\mu_{mn} = \mu_m\mu_n$ for all m, n.
(c) $e(r)^* = e(-r)$, $e(r_1 + r_2) = e(r_1)e(r_2)$ for all $r_1, r_2 \in \mathbb{Q}/\mathbb{Z}$.
(d) $\mu_n e(r)\mu_n^* = (1/n)\sum_{j=1}^{n} e(r/n + j/n)$ for all n and all r.

Where,
$\mu_n = n^{-1/2}\Big[\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Big]$, and $e(r) = \Big[\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\Big]$.
In [1] we defined the group

$$P_{\mathbb{Q}} = \Big\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r \neq 0 \Big\},$$

and the subgroup

$$P_{\mathbb{Z}} = \Big\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \Big\}.$$

Then we proved the following theorem.

Theorem 2.3 (Theorem 7.4 of [1])**.** *$H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is the universal $*$-algebra generated by elements $\{\mu_n : n \in \mathbb{N}^*\}$,$\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$, and $u = \big[\big(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\big)\big]$ subject to the relations*
(a) $\mu_n^*\mu_n = 1$ *for all* $n \in \mathbb{N}^*$.
(b) $\mu_{mn} = \mu_m\mu_n$ *for all* m, n.
(c) $e(r)^* = e(-r)$, $e(r_1 + r_2) = e(r_1)e(r_2)$ *for all* $r_1, r_2 \in \mathbb{Q}/\mathbb{Z}$.
(d) $\mu_n e(r)\mu_n^* = (1/n)\sum_{j=1}^{n} e(r/n + j/n)$ *for all n and all r.*
(e) $u^* = u$, $u^2 = 1$.
(f) $u\mu_n = \mu_n u$ *for all* $n \in \mathbb{N}^*$.
(g) $e(r)u = ue(-r)$ *for all* $r \in \mathbb{Q}/\mathbb{Z}$.

Remark 2.4. An action of a group G on a $*$-algebra A is a homomorphism $\beta : G \longrightarrow \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of $*$-automorphisms of A. The pair (A, G) is referred to as a dynamical system. We usually write $\beta_s(a)$ for $\beta(s)(a)$.

Background 2.5. Let A be a unital $*$-algebra, G a group and β an action of the group G on A. A covariant representation of the dynamical system (A, G, β) on a unital $*$-algebra B is a pair (π, U) consisting of a unital $*$-algebra homomorphism $\pi : A \longrightarrow B$ and a unitary homomorphism $U : G \longrightarrow u(B)$, such that

$$\pi\big(\beta_t(a)\big) = U_t\pi(a)U_t^* \text{ for all } a \in A,\, t \in G.$$

Let (A, G, β) be a dynamical system; we shall assume that A has an identity 1_A. Define the crossed product $A \times_\beta G$ to be $k(G, A)$, which is the vector space of finitely supported functions $f : G \longrightarrow A$, with operations given by

$(\lambda f + \gamma h)(s) = \lambda f(s) + \gamma h(s)$. By [4, Lemma 42] $k(G, A)$ is a $*$-algebra with multiplication and involution given by

$$f *_\beta h(t) := \sum_{s \in G} f(s)\beta_s\big(h(s^{-1}t)\big)$$

and

$$f^*(s) := \beta_s\big(f(s^{-1})\big).$$

Also if (π, U) is a covariant representation of (A, G, β) on a unital $*$-algebra B, then there is a unital $*$-representation $\pi \times U$ of $k(G, A)$ on B such that

$$\pi \times U(f) = \sum_{s \in G} \pi(f(s))U_s \text{ for } f \in k(G, A). \tag{2.2}$$

To go from representations of $k(G, A)$ to covariant representations of the system, we define $i_G : G \longrightarrow k(G, A)$ by $i_G(s) := \delta_s 1_A$, i.e.

$$i_G(s)(t) = \begin{cases} 1_A, & \text{if } s = t \\ 0, & \text{otherwise,} \end{cases}$$

and $i_A : A \longrightarrow k(G, A)$ by $i_A(a) := \delta_e a$. By [4, proposition 44] i_G is a homomorphism of G into the group $u(k(G, A))$ of unitary elements in the $*$-algebra $k(G, A)$, and i_A is a $*$-homomorphism of A into $k(A, G)$.

3. The automorphism of $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$

Lemma 3.1. *There is a $*$-automorphism α of $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$ such that*

(i) $\alpha(\mu_n) = \mu_n$ *for all* $n \in \mathbb{N}^*$.

(ii) $\alpha(e(r)) = e(-r)$ *for all* $r \in \mathbb{Q}/\mathbb{Z}$.

(iii) $\alpha^2 = id$.

Proof. Define $\{\tilde{\mu}_n = \mu_n : n \in \mathbb{N}^*\}$ and $\{\tilde{e}(r) = e(-r) : r \in \mathbb{Q}/\mathbb{Z}\}$, then these elements are in $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$. If these elements satisfy the relations (a)-(d) of Theorem 5.1 in [1], then the proof follows.

Parts (a) and (b) are trivially true because $\tilde{\mu}_n = \mu_n$.

For (c), let us start with

$$\tilde{e}(r)^* = e(-r)^* = e(r) = \tilde{e}(-r)$$

and for the second part of (c) we have that

$$\tilde{e}(r_1 + r_2) = e(-(r_1 + r_2)) = e(-r_1)e(-r_2) = \tilde{e}(r_1)\tilde{e}(r_2).$$

For (d),

$$\begin{aligned}
\tilde{\mu}_n \tilde{e}(r) \tilde{\mu}_n^* &= \mu_n e(-r) \mu_n^* \\
&= \frac{1}{n} \sum_{j=1}^{n} e(-r/n + j/n) \\
&= \frac{1}{n} \sum_{j=1}^{n} e(-(r/n - j/n)) \\
&= \frac{1}{n} \sum_{j=1}^{n} \tilde{e}(r/n - j/n) \\
&= \frac{1}{n} \sum_{j=1}^{n} \tilde{e}(r/n + j/n) \text{ since we are summing over } 1 \leq j \leq n.
\end{aligned}$$

Thus by [1, Theorem 5.1], there exists a $*$-homomorphism $\alpha : H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}) \longrightarrow H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$ satisfying the relations (i) and (ii).

To show part (iii), notice that α is a $*$-homomorphism and $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$ is generated by elements μ_n and $e(r)$. So to show that $\alpha^2 = id$ it is enough to check the equality for μ_n and $e(r)$.

$$\alpha^2(\mu_n) = \alpha\big(\alpha(\mu_n)\big) = \alpha(\mu_n) = \mu_n$$

and for $e(r)$

$$\alpha^2(e(r)) = \alpha\big(\alpha(e(r))\big) = \alpha(e(-r)) = e(r).$$

□

Proposition 3.2. *Let G be a group, A a unital $*$-algebra and $\beta : G \longrightarrow Aut(A)$ be an action of the group G on A. Then $k(G, A)$ is the universal $*$-algebra generated by elements $\{i_A(a) : a \in A\}$ and $\{i_G(s) : s \in G\}$ such that*

(a) *i_A is a unital $*$-homomorphism.*

(b) *i_G is a homomorphism from G into the group $u(k(G, A))$.*

(c) *$i_A\big(\beta_s(a)\big) = i_G(s) i_A(a) i_G(s)^*$ for all $a \in A$ and $s \in G$.*

Proof. That $k(G, A)$ is a $*$-algebra generated by the elements $i_A(a)$ and $i_G(s)$ such that the relations (a)-(c) are satisfied follows directly from [4, Lemma 42 and Lemma 43]. So we just need to check that i_A is unital and that $k(G, A)$ is a universal $*$-algebra.

That i_A is unital is pretty clear since

$$i_A(1_A) = \delta_e 1_A = 1_{k(G,A)}.$$

To show that $k(G, A)$ is a universal $*$-algebra, suppose that $\{\hat{i}_A(a) : a \in A\}$ and $\{\hat{i}_G(s) : s \in G\}$ are elements in a $*$-algebra B which also satisfies (a)-(c). We need to find a $*$-homomorphism $\phi : k(G, A) \longrightarrow B$ such that $\phi(i_A(a)) = \hat{i}_A(a)$ and $\phi(i_G(s)) = \hat{i}_G(s)$. □

Claim. *The map $\phi : k(G, A) \longrightarrow B$ defined by $\phi(f) = \sum_{s \in G} \hat{i}_A(f(s))\hat{i}_G(s)$ is a $*$-homomorphism.*

Proof. Notice that ϕ is a linear combination of the linear maps $f \mapsto \hat{i}_A(f(s))$, hence ϕ is linear. Let $a \in A$. Then

$$\begin{aligned}\phi(i_A(a)) &= \sum_{s\in G} \hat{i}_A(i_A(a)(s))\hat{i}_G(s)\\ &= \sum_{s\in G} \hat{i}_A(\delta_e a(s))\hat{i}_G(s)\\ &= \hat{i}_A(a)\hat{i}_G(e)\\ &= \hat{i}_A(a)1_B\\ &= \hat{i}_A(a).\end{aligned}$$

Let $t \in G$. Then

$$\begin{aligned}\phi(i_G(t)) &= \sum_{s\in G} \hat{i}_A(i_G(t)(s))\hat{i}_G(s)\\ &= \sum_{s\in G} \hat{i}_A(\delta_t 1_A(s))\hat{i}_G(s)\\ &= \hat{i}_A(1_A)\hat{i}_G(t)\\ &= 1_B\hat{i}_G(t)\\ &= \hat{i}_G(t).\end{aligned}$$

Next, we compute

$$\begin{aligned}\phi(f^*) &= \sum_{s\in G} \hat{i}_A(f^*(s))\hat{i}_G(s)\\ &= \sum_{s\in G} \hat{i}_A(\beta_s(f(s^{-1})^*))\hat{i}_G(s)\\ &= \sum_{s\in G} \hat{i}_A(\beta_s(f(s^{-1}))^*)\hat{i}_G(s)\\ &= \sum_{s\in G} \hat{i}_A(\beta_s(f(s^{-1})))^*\hat{i}_G(s)\end{aligned}$$

From relation (c) we conclude that

$$\hat{i}_G(s)^*\hat{i}_A(\beta_s(a)) = \hat{i}_A(a)\hat{i}_G(s)^*,$$

and this is equivalent to

$$\hat{i}_A(\beta_s(a))^*\hat{i}_G(s) = \hat{i}_G(s)\hat{i}_A(a)^*.$$

By noting that

$$\hat{i}_G(s)^* = \hat{i}_G(s^{-1}),$$

we have

$$\begin{aligned}
\phi(f^*) &= \sum_{s\in G} \hat{i}_G(s)\hat{i}_A(f(s^{-1}))^* \\
&= \sum_{s\in G} \hat{i}_G(s^{-1})^*\hat{i}_A(f(s^{-1}))^* \\
&= \sum_{p\in G} \hat{i}_G(p)^*\hat{i}_A(f(p))^* \\
&= \phi(f)^*.
\end{aligned}$$

Finally, let $f, h \in k(G, A)$. Then

$$\begin{aligned}
\phi(f *_\beta h) &= \sum_{s\in G} \hat{i}_A(f *_\beta h(s))\hat{i}_G(s) \\
&= \sum_{s\in G} \hat{i}_A\Big(\sum_{t\in G} f(t)\beta_t(h(t^{-1}s))\Big)\hat{i}_G(s) \\
&= \sum_{s\in G} \Big(\sum_{t\in G} \hat{i}_A(f(t))\hat{i}_A(\beta_t(h(t^{-1}s)))\Big) \\
&= \sum_{s\in G} \Big(\sum_{t\in G} \hat{i}_A(f(t))\hat{i}_G(t)\hat{i}_A(h(t^{-1}s))\hat{i}_G(t)^*\hat{i}_G(s)\Big) \\
&= \sum_{s\in G} \Big(\sum_{t\in G} \hat{i}_A(f(t))\hat{i}_G(t)\hat{i}_A(h(t^{-1}s))\hat{i}_G(t^{-1}s)\Big).
\end{aligned}$$

By writing $d = t^{-1}s$ and noting that $\sum_{s\in G} = \sum_{d\in G}$ (this is true since all sums are finite) we have

$$\phi(f *_\beta h) = \sum_{t\in G} \hat{i}_A(f(t))\hat{i}_G(t) \sum_{d\in G} \hat{i}_A(h(d))\hat{i}_G(d) = \phi(f)\phi(h).$$

Thus ϕ is multiplicative. □

4. The relation between $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ and $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$

Proposition 4.1. *Consider the group $G=\{1,-1\}$ and the algebra $A=H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$. Then there is a $*$-homomorphism $\phi : H(P_{\mathbb{Q}}, P_{\mathbb{Z}}) \to H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_\beta \{1,-1\}$ such that*

(i) $\phi(e(r)) = i_A(e(r))$ *for all* $r \in \mathbb{Q}/\mathbb{Z}$.
(ii) $\phi(\mu_n) = i_A(\mu_n)$ *for all* $n \in \mathbb{N}^*$.
(iii) $\phi(u) = i_G(-1)$.

Proof. Define a map $\beta : G \longrightarrow \mathrm{Aut}(A)$ by

$$\beta_1 = id \text{ and } \beta_{-1} = \alpha \text{ (the $*$-homomorphism of Lemma 3.1).}$$

To show that β is an action we only need to check that $\beta_{(1)(-1)} = \beta_1\beta_{-1}$ and $\beta_{-1}^2 = \beta_1$. Now $\beta_{(1)(-1)} = \beta_{-1} = \alpha = id\alpha = \beta_1\beta_{-1}$, and by Lemma 3.1 we have $\beta_{-1}^2 = \alpha^2 = id = \beta_1$. Thus β is an action of the group G on the group of automorphisms of A. Suppose that $\{\hat{\mu}_n = i_A(\mu_n) : n \in \mathbb{N}^*\}$, $\{\hat{e}(r) = i_A(e(r)) : r \in \mathbb{Q}/\mathbb{Z}\}$ and $\hat{u} = i_G(-1)$. If these elements satisfy the relations (a)-(g) of [1, Theorem 7.4] we are done.

For (a)

$$\hat{\mu}_n^* \hat{\mu}_n = i_A(\mu_n)^* i_A(\mu_n) = i_A(\mu_n^* \mu_n) = 1_{k(G,A)}.$$

For (b)

$$\hat{\mu}_{mn} = i_A(\mu_{mn}) = i_A(\mu_m \mu_n) = i_A(\mu_m) i_A(\mu_n) = \hat{\mu}_m \hat{\mu}_n.$$

For (c), let us start with

$$\hat{e}(r)^* = i_A(e(r))^* = i_A(e(r)^*) = i_A(e(-r)) = \hat{e}(-r).$$

and for the second part

$$\hat{e}(r_1 + r_2) = i_A(e(r_1 + r_2)) = i_A(e(r_1)e(r_2)) = i_A(e(r_1) i_A(e(r_2)) = \hat{e}(r_1)\hat{e}(r_2).$$

For (d)

$$\begin{aligned}
\hat{\mu}_n \hat{e}(r) \hat{\mu}_n^* &= i_A(\mu_n) i_A(e(r)) i_A(\mu_n)^* \\
&= i_A(\mu_n e(r) \mu_n^*) \\
&= i_A\Big(\frac{1}{n} \sum_{j=1}^{n} e(r/n + j/n)\Big) \\
&= \frac{1}{n} \sum_{j=1}^{n} i_A\big(e(r/n + j/n)\big) \\
&= \frac{1}{n} \sum_{j=1}^{n} \hat{e}(r/n + j/n).
\end{aligned}$$

For (e), let us start with $\hat{u}^* = i_G(-1)^* = i_G(-1) = \hat{u}$ and for the second part of (e) $\hat{u}^2 = i_G(-1) i_G(-1) = i_G(1) = 1_{k(G,A)}$.

For (f), by relation (c) of Proposition 3.2

$$\begin{aligned}
\hat{u}\hat{\mu}_n &= i_G(-1) i_A(\mu_n) \\
&= i_A\big(\beta_{-1}(\mu_n)\big) i_G(-1) \\
&= i_A\big(\alpha(\mu_n)\big) i_G(-1) \\
&= i_A(\mu_n) i_G(-1) \\
&= \hat{\mu}_n \hat{u}.
\end{aligned}$$

For (g), by relation (c) of Proposition 3.2

$$\begin{aligned}
\hat{u}\hat{e}(-r) &= i_G(-1) i_A(e(-r)) \\
&= i_A\big(\beta_{-1}(e(-r))\big) i_G(-1) \\
&= i_A\big(\alpha(e(-r))\big) i_G(-1) \\
&= i_A(e(r)) i_G(-1) \\
&= \hat{e}(r)\hat{u}.
\end{aligned}$$

Thus, [1, Theorem 7.4] says that there exists a $*$-homomorphism $\phi: H(P_{\mathbb{Q}}, P_{\mathbb{Z}}) \longrightarrow A \times_{\beta} G$. □

Lemma 4.2. *The pair (ι, U) is a covariant representation of $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$ on $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ where*

(i) *$\iota : H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}}) \longrightarrow H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is the unital $*$-homomorphism of [1, Lemma 2.8].*

(ii) *$U : \{1, -1\} \longrightarrow u(H(P_{\mathbb{Q}}, P_{\mathbb{Z}}))$ is defined by $U_1 = \mu_1 = 1_{H(P_{\mathbb{Q}}, P_{\mathbb{Z}})}$ and $U_{-1} = u$.*

(iii) *β is the group action defined in Proposition 4.1.*

Proof. The map U is a homomorphism by relation (e) of [1, Theorem 7.4]. We still need to show that $\iota\big(\beta_1(a)\big) = U_1 \iota(a) U_1^*$ and $\iota\big(\beta_{-1}(a)\big) = U_{-1} \iota(a) U_{-1}^*$ for all $a \in H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}})$.

On one hand, the first relation is true since both sides are $\iota(a)$, and on the other, it is enough to check the second relation when $a = \mu_n$ and $a = e(r)$.

If $a = \mu_n$, then we have relation (f) in [1, Theorem 7.4] and if $a = e(r)$ we have relation (g) in [1, Theorem 7.4]. Both β_{-1} and $a \mapsto uau^*$ are $*$-homomorphisms, so this implies the second relation is true for all $a \in H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}})$. □

Now we give our main theorem which allows us to realize the Bost and Connes Hecke Algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ as the crossed product $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$.

Theorem 4.3. *The map ϕ in Proposition 4.1 is a $*$-isomorphism of $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ onto $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$ with inverse $\iota \times U$ where*

(i) *β is the group action defined in Proposition 4.1.*

(ii) *The pair (ι, U) is the covariant representation in Lemma 4.2.*

Proof. Let $A = H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}})$ and $G = \{1, -1\}$. Lemma 4.2 and [4, Lemma 42] yield the map $\iota \times U : H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\} \longrightarrow H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ defined by

$$\iota \times U(f) = \sum_{s \in \{1,-1\}} \iota\big(f(s)\big) U_s$$

is a $*$-homomorphism. So if we show that $(\iota \times U) \circ \phi = id$ and $\phi \circ (\iota \times U) = id$ the proof of this theorem will follow. Since the elements $\mu_n, e(r)$ and u generate the $*$-algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ and $(\iota \times U) \circ \phi$ is a $*$-homomorphism. To show that $(\iota \times U) \circ \phi = id$, it is enough to check that it is true for $\mu_n, e(r)$ and u.

For μ_n we have

$$\begin{aligned}(\iota \times U) \circ \phi(\mu_n) &= (\iota \times U)\big(\phi(\mu_n)\big)\\ &= (\iota \times U)\big(i_A(\mu_n)\big)\\ &= \iota\big(\mu_n\big)\\ &= \mu_n.\end{aligned}$$

The same argument works for $e(r)$. For u we have

$$\begin{aligned}(\iota \times U) \circ \phi(u) &= (\iota \times U)\big(\phi(u)\big) \\ &= (\iota \times U)\big(i_G(-1)\big) \\ &= U_{-1} \\ &= u.\end{aligned}$$

For the second part of the proof, notice that the elements $i_A(\mu_n)$, $i_A(e(r))$ and $i_G(-1)$ generate the $*$-algebra $A \times_\beta G$, and $\phi \circ (\iota \times U)$ is a $*$-homomorphism. So to show that $\phi \circ (\iota \times U) = id$, it is enough to check the equality for $i_A(\mu_n)$, $i_A(e(r))$ and $i_G(-1)$.

For $i_A(\mu_n)$

$$\begin{aligned}\phi \circ (\iota \times U)\big(i_A(\mu_n)\big) &= \phi\big(\iota \times U(i_A(\mu_n))\big) \\ &= \phi\big(\iota(\mu_n)\big) \text{ [4, proposition 44]} \\ &= \phi\big(\mu_n\big) \\ &= i_A(\mu_n).\end{aligned}$$

The same argument works for $i_A(e(r))$. For $i_G(-1)$ we have

$$\begin{aligned}\phi \circ (\iota \times U)\big(i_G(-1)\big) &= \phi\big(\iota \times U(i_G(-1))\big) \\ &= \phi\big(U_{-1}\big) \text{ [4, proposition 44]} \\ &= \phi(u) \\ &= i_G(-1).\end{aligned}$$

Consequently, the two maps are inverses of each other, and hence the two $*$-algebras are isomorphic. □

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