HAAR WAVELET COLLOCATION METHOD FOR SOLVING NONLINEAR KURAMOTO–SIVASHINSKY EQUATION

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Abstract. A collocation method based on Haar wavelet is presented for solving numerical solution of fourth order nonlinear Kuramoto-Sivashinsky equation. Efficiency and accuracy of the present method has been established by comparing the numerical results with exact solutions.
Keywords: Haar wavelet, Kuramoto-Sivashinsky equation, Numerical observations.

1. Introduction

Wavelet methods are more appealing, unsophisticated and reliable to obtain the numerical solutions of partial differential equations. A variety of methods have been developed for solving nonlinear partial differential equations. Haar wavelet is simple, computationally fast and give more accurate numerical results. It is discontinuous and therefore not differentiable. So, it is impossible to find the numerical results of partial differential equation using Haar wavelet method directly. Due to integrability of this function, it is utilised as a powerful mathematical tool for solving nonlinear equations. In [3], Haar wavelet method has been used for numerical solution of generalized Burger-Huxley equation. Haar wavelet method for solving lumped and distributed-parameter systems has been presented in [4]. Haar wavelet method has been presented for solving Fisher's equation, Fitzhugh–Nagumo equation and evolution equation in [6, 7, 12]. Haar and Legendre wavelets collocation methods has been used for finding numerical solution of Schrondinger and wave equation in [10]. Numerical solutions of higher degree partial differential equations using Haar wavelet have been presented in [1, 13, 15, 17, 18]. Consider the general Kuramoto-Sivashinsky equation [14]:

$$u_t + uu_x + \mu u_{xx} + \nu u_{xxxx} = 0, \quad (1)$$

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with initial and boundary conditions

$$u(x,0) = f(x), u_x(0,t) = 0, u_{xx}(0,t) = 0, u(0,t) = g_1(t), u(1,t) = g_2(t), \tag{2}$$

where $f(x)$, $g_1(t)$ and $g_2(t)$ are known functions.

Kuramoto-Sivashinsky (KS) equation which is a canonical nonlinear evolution equation that arises in a variety of physical contexts. This equation was originally derived in the context of plasma instabilities, flame front propagation, and phase turbulence in reaction-diffusion system [16]. It occur incontext of long waves on the interface between two viscous fluids [9], unstable drift waves in plasmas, reaction-diffusion systems [11], and flame front instability.

The main aim of this research is to find an accurate and efficient numerical method for solving Kuramoto-Sivashinsky equation. In Section 2, theory of Haar wavelets has been presented. In Section 3, we describe function approximation. Description of Haar wavelet method for solving such equations has been given in Section 4. Error analysis of Haar wavelet method has been presented in Section 6. In Section 7, numerical observations have been solved using the present methods and compared with exact solutions.

2. Haar wavelet

Haar wavelet is discontinuous function and is defined as:

$$\mathcal{H}_i(x) = \begin{cases} 1, & \alpha \leq x < \beta, \\ -1, & \beta \leq x < \gamma, \\ 0, & elsewhere, \end{cases} \tag{3}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$, $m = 2^j$, $j = 0, 1, 2, ..., J$. J denotes the level of resolution. The integer $k = 0, 1, 2, ..., m-1$ is the translation parameter. The index i is calculated as: $i = m + k + 1$. The minimal value of $i = 2$. The maximal value of i is 2^{j+1}. The collocation points are calculated as:

$$x_l = \frac{l - 0.5}{2M}, \quad l = 1, 2, 3, ..., 2M. \tag{4}$$

The operational matrices $\mathcal{P}$, which are $2M \times 2M$, are calculated as below:

$$\mathcal{P}_{1,i}(x) = \int_0^x \mathcal{H}_i(x) dx, \tag{5}$$

and

$$\mathcal{P}_{n+1,i}(x) = \int_0^x \mathcal{P}_{n,i}(x) dx. \tag{6}$$

In general, operational matrices can be obtained directly from the following relation (see, for example [8])

$$\mathcal{P}_{n,i}(x) = \begin{cases} 0, & x < \alpha, \\ \frac{1}{n!}(x-\alpha)^n, & x \in [\alpha, \beta], \\ \frac{1}{n!}\{(x-\alpha)^n - 2(x-\beta)^n\}, & x \in [\beta, \gamma], \\ \frac{1}{n!}\{(x-\alpha)^n - 2(x-\beta)^n + (x-\gamma)^n\}, & x > \gamma. \end{cases} \tag{7}$$

3. Function approximation

The function $y(t) \in L^2(0,1)$ can be approximated as:

$$y(t) = \sum_{i=0}^{\infty} \mathcal{C}_i \mathcal{H}_i(t), \tag{8}$$

where the coefficient $\mathcal{C}_i$ are determined as:

$$\mathcal{C}_i = 2^j \int_0^1 y(t)\mathcal{H}_i(t), \tag{9}$$

where $i = 2^j + k$, $j \geq 0$, $0 \leq k < 2^j$. The series expansion of $y(t)$ contains infinite terms. If $y(t)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(t)$ will be terminated at finite terms, that is:

$$y(t) = \sum_{i=0}^{m-1} \mathcal{C}_i \mathcal{H}_i(t) = \mathcal{C}_m^T \mathcal{H}_m(x), \tag{10}$$

where $\mathcal{C}_m^T = [\mathcal{C}_0, \mathcal{C}_1, ..., \mathcal{C}_{m-1}]$ and $\mathcal{H}_m = [\mathcal{H}_0, \mathcal{H}_1, ..., \mathcal{H}_{m-1}]^T$, where T is transpose.

4. Description of method for solving fourth order Kuramoto-Sivashinsky equation

Consider the approximation

$$\dot{u}''''(x,t) = \sum_{i=0}^{2M} \mathcal{C}_i \mathcal{H}_i(x). \tag{11}$$

Here $(\cdot)$ represents the differentiation with respect to t and $(')$ represents the differentiation with respect to x. Integrating (11) one time with respect to t, from t_s to t, we obtain

$$u''''(x,t) = u''''(x,t_s) + (t-t_s)\sum_{i=0}^{2M} \mathcal{C}_i \mathcal{H}_i(x). \tag{12}$$

Now, integrating (12) four times with respect to x, from 0 to x, we obtain

$$u'''(x,t) = u'''(x,t_s) + u'''(0,t) - u'''(0,t_s) + (t-t_s)\sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{1,i}(x), \tag{13}$$

$$\begin{aligned} u''(x,t) &= u''(x,t_s) + u''(0,t) - u''(0,t_s) + x\Big[u'''(0,t) - u'''(0,t_s)\Big] \\ &\quad + (t-t_s)\sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{2,i}(x), \end{aligned} \tag{14}$$

$$\begin{aligned} u'(x,t) &= u'(x,t_s) + u'(0,t) - u'(0,t_s) + x\Big[u''(0,t) - u''(0,t_s)\Big] \\ &\quad + (\frac{x^2}{2})\Big[u'''(0,t) - u'''(0,t_s)\Big] + (t-t_s)\sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{3,i}(x), \end{aligned} \tag{15}$$

and

$$\begin{aligned} u(x,t) &= u(x,t_s) + u(0,t) - u(0,t_s) + x\Big[u'(0,t) - u'(0,t_s)\Big] \\ &\quad + (\frac{x^2}{2})\Big[u''(0,t) - u''(0,t_s)\Big] \\ &\quad + (\frac{x^3}{6})\Big[u'''(0,t) - u'''(0,t_s)\Big] + (t-t_s)\sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{4,i}(x), \end{aligned} \tag{16}$$

Substituting $x = 1$ in (14), we obtain

$$\begin{aligned} \Big[u'''(0,t) - u'''(0,t_s)\Big] &= u''(1,t) - u''(1,t_s) - u''(0,t) \\ &\quad + u''(0,t_s) - (t-t_s)\sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{2,i}(1). \end{aligned} \tag{17}$$

Using (17), from (14)-(16), we obtain

$$\begin{aligned} u''(x,t) &= u''(x,t_s) + u''(0,t) - u''(0,t_s) \\ &\quad + x\Big[u''(1,t) - u''(1,t_s) - u''(0,t) + u''(0,t_s)\Big] \\ &\quad + (t-t_s)\sum_{i=0}^{2M} \mathcal{C}_i \Big[\mathcal{P}_{2,i}(x) - x\mathcal{P}_{2,i}(1)\Big], \end{aligned} \tag{18}$$

$$\begin{aligned} u'(x,t) &= u'(x,t_s) + u'(0,t) - u'(0,t_s) + x\Big[u''(0,t) - u''(0,t_s)\Big] \\ &\quad + (\frac{x^2}{2})\Big[u''(1,t) - u''(1,t_s) - u''(0,t) + u''(0,t_s)\Big] \\ &\quad + (t-t_s)\sum_{i=0}^{2M} \mathcal{C}_i \Big[\mathcal{P}_{3,i}(x) - \frac{x^2}{2}\mathcal{P}_{2,i}(1)\Big], \end{aligned} \tag{19}$$

and

$$u(x,t) = u(x,t_s) + u(0,t) - u(0,t_s) + x\Big[u'(0,t) - u'(0,t_s)\Big]$$
$$+ (\frac{x^2}{2})\Big[u''(0,t) - u''(0,t_s)\Big] + (\frac{x^3}{6})\Big[u''(1,t) - u''(1,t_s) - u''(0,t) + u''(0,t_s)\Big] \tag{20}$$
$$+ (t - t_s)\sum_{i=0}^{2M} \mathcal{C}_i\Big[\mathcal{P}_{4,i}(x) - \frac{x^3}{6}\mathcal{P}_{2,i}(1)\Big],$$

Now, differentiating (20) with respect to t, we obtain

$$\dot{u}(x,t) = \dot{u}(0,t) + x\dot{u}'(0,t) + (\frac{x^2}{2})\dot{u}''(0,t) + (\frac{x^3}{6})\Big[\dot{u}''(1,t) - \dot{u}''(0,t)\Big]$$
$$+ \sum_{i=0}^{2M} \mathcal{C}_i\Big[\mathcal{P}_{4,i}(x) - \frac{x^3}{6}\mathcal{P}_{2,i}(1)\Big]. \tag{21}$$

From finite difference scheme, we obtain

$$\dot{u}''(0,t) = \Big[\frac{u''(0,t) - u''(0,t_s)}{t - t_s}\Big], \tag{22}$$

$$\dot{u}'(0,t) = \Big[\frac{u'(0,t) - u'(0,t_s)}{t - t_s}\Big], \tag{23}$$

and

$$\dot{u}(0,t) = \Big[\frac{u(0,t) - u(0,t_s)}{t - t_s}\Big]. \tag{24}$$

$$\dot{u}(x,t) = \Big[\frac{u(0,t) - u(0,t_s)}{t - t_s}\Big] + x\Big[\frac{u'(0,t) - u'(0,t_s)}{t - t_s}\Big]$$
$$+ (\frac{x^2}{2})\Big[\frac{u''(0,t) - u''(0,t_s)}{t - t_s}\Big]$$
$$+ (\frac{x^3}{6})\Big[(\frac{u''(1,t) - u''(1,t_s)}{t - t_s}) - (\frac{u''(0,t) - u''(0,t_s)}{t - t_s})\Big] \tag{25}$$
$$+ \sum_{i=0}^{2M} \mathcal{C}_i\Big[\mathcal{P}_{4,i}(x) - \frac{x^3}{6}\mathcal{P}_{2,i}(1)\Big].$$

Discretising (12), (18)-(21) by substituting $x \to x_l$ and $t \to t_{s+1}$, we obtain

$$u''''(x_l, t_{s+1}) = u''''(x_l, t_s) + (t_{s+1} - t_s)\sum_{i=0}^{2M} \mathcal{C}_i \mathcal{H}_i(x_l), \tag{26}$$

$$u''(x_l, t_{s+1}) = u''(x_l, t_s) + u''(0, t_{s+1}) - u''(0, t_s)$$
$$+ x_l\Big[u''(1, t_{s+1}) - u''(1, t_s) - u''(0, t_{s+1}) + u''(0, t_s)\Big] \tag{27}$$
$$+ (t_{s+1} - t_s)\sum_{i=0}^{2M} \mathcal{C}_i\Big[\mathcal{P}_{2,i}(x_l) - x_l\mathcal{P}_{2,i}(1)\Big],$$

$$
\begin{aligned}
u'(x_l, t_{s+1}) =& u'(x_l, t_s) + u'(0, t_{s+1}) - u'(0, t_s) + x_l\Big[u''(0, t_{s+1}) - u''(0, t_s)\Big] \\
&+ (\frac{{x_l}^2}{2})\Big[u''(1, t_{s+1}) - u''(1, t_s) - u''(0, t_{s+1}) + u''(0, t_s)\Big] \\
&+ (t_{s+1} - t_s)\sum_{i=0}^{2M} \mathcal{C}_i\Big[\mathcal{P}_{3,i}(x_l) - \frac{{x_l}^2}{2}\mathcal{P}_{2,i}(1)\Big],
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
u(x_l, t_{s+1}) =& u(x_l, t_s) + u(0, t_{s+1}) - u(0, t_s) + x_l\Big[u'(0, t_{s+1}) - u'(0, t_s)\Big] \\
&+ (\frac{{x_l}^2}{2})\Big[u''(0, t_{s+1}) - u''(0, t_s)\Big] \\
&+ (\frac{{x_l}^3}{6})\Big[u''(1, t_{s+1}) - u''(1, t_s) - u''(0, t_{s+1}) + u''(0, t_s)\Big] \\
&+ (t_{s+1} - t_s)\sum_{i=0}^{2M} \mathcal{C}_i\Big[\mathcal{P}_{4,i}(x_l) - \frac{{x_l}^3}{6}\mathcal{P}_{2,i}(1)\Big],
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
\dot{u}(x_l, t_{s+1}) =& \Big[\frac{u(0, t_{s+1}) - u(0, t_s)}{t_{s+1} - t_s}\Big] \\
&+ x_l\Big[\frac{u'(0, t_{s+1}) - u'(0, t_s)}{t_{s+1} - t_s}\Big] + (\frac{{x_l}^2}{2})\Big[\frac{u''(0, t_{s+1}) - u''(0, t_s)}{t_{s+1} - t_s}\Big] \\
&+ (\frac{{x_l}^3}{6})\Big[(\frac{u''(1, t_{s+1}) - u''(1, t_s)}{t_{s+1} - t_s}) - (\frac{u''(0, t_{s+1}) - u''(0, t_s)}{t_{s+1} - t_s})\Big] \\
&+ \sum_{i=0}^{2M} \mathcal{C}_i\Big[\mathcal{P}_{4,i}(x_l) - \frac{{x_l}^3}{6}\mathcal{P}_{2,i}(1)\Big].
\end{aligned} \tag{30}
$$

The nonlinear term in the partial differential equation (1) is linearized, using the following time discretized form

$$
u_t(x_l, t_{s+1}) + u(x_l, t_s)u_x(x_l, t_s) + \mu.u_{xx}(x_l, t_{s+1}) + \nu.u_{xxxx}(x_l, t_{s+1}) = 0. \tag{31}
$$

Substituting the values from (25) – (30) in (31), we obtain

$$
\begin{aligned}
&\sum_{i=0}^{2M} \Big[\mathcal{P}_{4,i}(x_l) - (\frac{{x_l}^3}{6})\mathcal{P}_{2,i}(1) + \nu.(t_{s+1} - t_s)\mathcal{H}_i(x_l) \\
&+ \mu.(t_{s+1} - t_s)\Big[\mathcal{P}_{2,i}(x_l) - x_l\mathcal{P}_{2,i}(1)\Big]\Big] \\
&= -\nu.u''''(x_l, t_s) - u(x_l, t_s)u'(x_l, t_s) \\
&- \mu.u''(x_l, t_s) - \mu.u''(0, t_{s+1}) + \mu.u''(0, t_s) \\
&- \mu.x_l\Big[u''(1, t_{s+1}) - u''(1, t_s) - u''(0, t_{s+1}) + u''(0, t_s)\Big]
\end{aligned}
$$

$$- \Big[\frac{u(0,t_{s+1}) - u(0,t_s)}{t_{s+1} - t_s}\Big] - x_l\Big[\frac{u'(0,t_{s+1}) - u'(0,t_s)}{t_{s+1} - t_s}\Big] - (\frac{{x_l}^2}{2})\Big[\frac{u''(0,t_{s+1}) - u''(0,t_s)}{t_{s+1} - t_s}\Big] - (\frac{{x_l}^3}{6})\Big[(\frac{u''(1,t_{s+1}) - u''(1,t_s)}{t_{s+1} - t_s}) - (\frac{u''(0,t_{s+1}) - u''(0,t_s)}{t_{s+1} - t_s})\Big]. \tag{32}$$

After applying initial and boundary conditions in (32), we obtain the system of equations. The wavelet coefficients are obtained from this system of linear equations. The numerical solution of (1) is obtained by substituting the values of wavelet coefficients into (29).

5. Error analysis of Haar wavelet method

Let $u(x,t)$ be a differentiable function and assume that $u(x,t)$ have bounded first derivative on $[0,1]$, that is, there exist $K > 0$, such that

$$|u'(x,t)| \leq K, \qquad x \in [0,1]. \tag{33}$$

Consider the Haar wavelet approximation as below

$$u_{2M}(x,t) = \sum_{i=1}^{2M} \mathcal{C}_i \mathcal{H}_i(x). \tag{34}$$

L_2-error norm for Haar wavelet approximation [2] is given by

$$\parallel u(x,t) - u_{2M}(x,t) \parallel^2 \leq \frac{K^2}{3}\frac{1}{(2M)^2}. \tag{35}$$

After simplification, from (35), we obtain

$$\parallel u(x,t) - u_{2M}(x,t) \parallel \leq \frac{1}{(M)}. \tag{36}$$

As J is the maximal level of resolution and $M = 2^J$. From (36), we obtain

$$\parallel u(x,t) - u_{2M}(x,t) \parallel \leq \frac{1}{(2^J)}. \tag{37}$$

From (37), we conclude that error is inversely proportional to the level of resolution. It ensures the convergence of Haar wavelet approximation at higher level of resolution J.

6. Numerical observations

Here, we present some numerical observations to establish the efficiency and accuray of the present collocation method based on Haar wavelet.

Example 1. Consider nonlinear Kuramoto-Sivashinsky equation with $\mu = 1$ and $\nu = 1$. The exact solution of the problem is given in [14] and is

$$u(x,t)=\rho+\frac{15}{19}\sqrt{\frac{11}{19}}[-9\tanh(\sigma(x-\rho t-x_0))+11\tanh^3(\sigma(x-\rho t-x_0))]. \quad (38)$$

The initial and boundary conditions are obtained from exact solution. Table 1 shows the comparison of absolute errors at different values of t, ρ, x_0 with $\sigma = \frac{1}{2}\sqrt{\frac{11}{19}}$. Figure 1 shows the comparison absolute errors of Example 1 at $J = 3$.

xL/32	Absolute errors for $t = 0.1, \rho = 5, \sigma = -25$	Absolute errors for $t = 1, \rho = 5, \sigma = -35$	Absolute errors for $t = 1, \rho = 10, \sigma = -35$
1	7.8636E-010	6.8077E-010	7.1905E-008
3	4.0585E-009	6.2014E-010	6.5375E-008
5	7.0443E-009	8.2882E-010	6.0913E-008
7	9.6528E-009	1.4740E-009	5.9622E-008
9	1.1790E-008	2.7120E-009	6.2531E-008
11	1.3362E-008	4.6840E-009	7.0576E-008
13	1.4282E-008	7.5119E-009	8.4563E-008
15	1.4475E-008	1.1294E-008	1.0514E-007
17	1.3878E-008	1.6100E-008	1.3279E-007
19	1.2452E-008	2.1971E-008	1.6775E-007
21	1.0180E-008	2.8908E-008	2.1006E-007
23	7.0749E-009	3.6877E-008	2.5946E-007
25	3.1816E-009	4.5800E-008	3.1544E-007
27	1.4170E-009	5.5553E-008	3.7717E-007
29	6.5976E-009	6.5964E-008	4.4347E-007
31	1.2192E-008	7.6810E-008	5.1285E-007

Table 1: Comparison of absolute errors of Example 1 for $J = 3$ and different x_0, ρ, σ.

Example 2. Consider nonlinear Kuramoto-Sivashinsky equation with $\mu = -1$ and $\nu = 1$. The exact solution of the problem is given in [14] and is

$$u(x,t) = \rho + \frac{15}{19\sqrt{19}}[-3\tanh(\sigma(x-\rho t-x_0)) + \tanh^3(\sigma(x-\rho t-x_0))]. \quad (39)$$

The initial and boundary conditions are obtained from exact solution. Table 2 shows the comparison of absolute errors at different values of t, ρ, x_0 with $\sigma = \frac{1}{2}\sqrt{\frac{1}{19}}$. Figure 2 shows the comparison absolute errors of Example 2 at $J = 3$.

xL/32	Absolute errors for $t = 0.1, \rho = 5, \sigma = -25$	Absolute errors for $t = 1, \rho = 5, \sigma = -35$	Absolute errors for $t = 1, \rho = 10, \sigma = -35$
1	3.9296E-008	3.1345E-007	6.1496E-006
3	1.9555E-007	3.1226E-006	1.3651E-005
5	3.3756E-007	1.3809E-005	4.3476E-005
7	4.5837E-007	3.7423E-005	1.0993E-004
9	5.5089E-007	7.8717E-005	2.2649E-004
11	6.0821E-007	1.4202E-004	4.0540E-004
13	6.2385E-007	2.3112E-004	6.5740E-004
15	5.9213E-007	3.4914E-004	9.9132E-004
17	5.0845E-007	4.9840E-004	1.4138E-003
19	3.6959E-007	6.8030E-004	1.9286E-003
21	1.7403E-007	8.9517E-004	2.5370E-003
23	7.7708E-008	1.1422E-003	3.2363E-003
25	3.8278E-007	1.4191E-003	4.0205E-003
27	7.3566E-007	1.7223E-003	4.8791E-003
29	1.1278E-006	2.0466E-003	5.7973E-003
31	1.5474E-006	2.3848E-003	6.7551E-003

Table 2: Comparison of absolute errors of Example 2 for $J = 3$ and different x_0, ρ, σ.

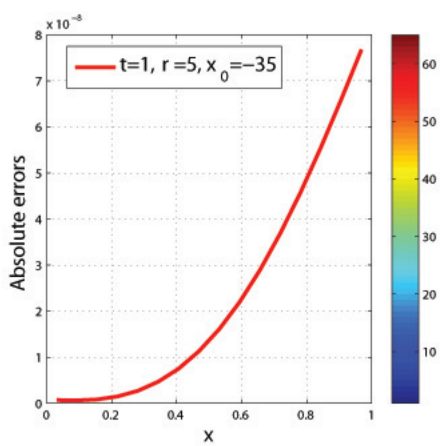

Figure 1: Absolute errors of Example 1 at $J = 3$.

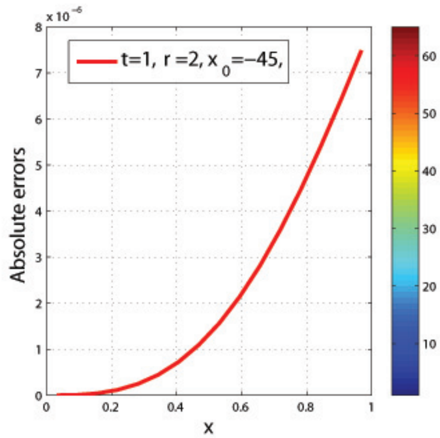

Figure 2: Absolute errors of Example 2 at $J = 3$.

7. Conclusion

From above, it is concluded that Haar wavelet method is a powerful mathematical tool for solving Kuramoto-Sivashinsky equation. The numerical solutions are much closer to the exact solutions. Also, it is concluded that Haar wavelet method is simplier, efficient and take low computational time for solving such equations.

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