SPECIAL HOOP ALGEBRAS

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Abstract. Hoops are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [5, 6]. In this paper, we introduce the concepts of special hoop algebra and special filter in hoop algebras and study some properties of them. We establish relation between special hoops with other structures such as simple hoops, local hoops, locally finite hoops, perfect hoops, semi-De Morgan algebras and Boolean algebras. Then, by define the notion of special filter in bounded hoops, we study the relationship between special filters and implicative (positive implicative, maximal, obstinate) filters on bounded and special hoops. Finally, we investigated the properties of a quotient structure, when it is produced by a special filter.

Keywords: Hoop algebra, special hoop, simple hoops, local hoops, locally finite hoops, perfect hoops, semi-De Morgan algebras, Boolean algebra, special filter, implicative (positive implicative, maximal, obstinate) filter.

1. Introduction

Bosbach [5, 6] undertook the investigation of a class of residuated structures that were related to considerably more general than the Brouwerian semilattices and the algebras associated with Lukasiewicz's calculus mentioned above. The requirement he added was that the partial order be *natural*; in the commutative case (to which we will restrict ourselves in this paper) this means that $a \leq b$ if and only if there is an element c such that $a = b \odot c$. Brouwerian semilattices as well as the models of many-valued logic satisfy this requirement, but the models of linear logic do not in general. He showed that the resulting class of structures can be viewed as an equational class, and that the class is congruence distributive and congruence permutable. In a manuscript by J. R. Büchi and

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T. M. Owens [7], devoted to a study of Bosbachs algebras, written in the mid-seventies, the commutative members of this equational class were given the name hoops. The manuscript is a rich source of ideas, but of a preliminary nature and was never published. Some of the results obtained there can be found in two joint papers with Blok [2]; in particular the description of subdirectly irreducible hoops ([2], Theorem 2.9) will play a crucial role in this paper. In the last years, hoops theory was enriched with deep structure theorems(see [1, 8]). Many of these results have a strong impact with fuzzy logic. The algebraic structures corresponding to Hájek's propositional basic logic, BL-algebras, are particular cases of hoops. Kondo in [9], considered fundamental properties of some types of filters (implicative, positive implicative and fantastic filters) of hoops. R. A. Borzooei and M. Aaly Kologani investigate the relation between these filters in [3]. Also they introduce in [4], the concepts of local and perfect semihoops and state and prove some related results. Specially, they defined the concepts of locally finite semihoop and found a relation between local and perfect semihoops. N. Mohtashamnia and A. Borumand Saeid in 2012 introduced the notion of special type of BL-algebras [10].

The aim of this paper is the introduction of a new structure from hoop as a special and compare it with other structures hoop. After that we define special filters and consider relationships between special filters and some other filters. We show that in some examples any special filter is not (obstinate, fantastic, implicative, maximal, prime and perfect) filter and conversely is not true, too. Also, we study the relationship between special filter and congruence relations on hoop and obtain equivalent relation with special filter.

2. Preliminaries

In this section, we recollect some definitions and results which will be used not cite them every time they are used.

Definition 2.1 ([1]). A *hoop algebra* or *hoop* is an algebra $(A, \odot, \to, 1)$ of type $(2, 2, 0)$ such that for all $x, y, z \in A$:

$(HP1)$ $(A, \odot, 1)$ is a commutative monoid,

$(HP2)$ $x \to x = 1$,

$(HP3)$ $(x \odot y) \to z = x \to (y \to z)$,

$(HP4)$ $x \odot (x \to y) = y \odot (y \to x)$.

On hoop A, we define $x \leq y$ if and only if $x \to y = 1$. It is easy to see that $\leq$ is a partial order relation on A. A hoop A is *bounded* if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. We let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. Let A be a bounded hoop. We define a negation "$'$" on A by, $x' = x \to 0$, for all $x \in A$. If $x'' = x$, for all $x \in A$, then the bounded hoop A is said to have the *double negation property*, or (DNP), for short. The *order* of $1 \neq x \in A$, in symbols $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If no such n exists, then $ord(x) = \infty$. A hoop is called *locally finite* if for any $x \in A$, $x \neq 1$, has finite order. An element $x \in A$ is called *dense* if and only if $x' = 0$ and the set of all dense elements of A shown that by $D_e(A) = \{x \in A \mid x' = 0\}$. An element $a \in A$ is called *atom* if it is minimal among elements in bounded hoop $A \backslash \{0\}$.(See [8])

Proposition 2.2 ([5, 6]). *In any hoop $(A, \odot, \to, 1)$ the following properties hold, for all $x, y, z \in A$:*

(i) *$(A, \leq)$ is a meet-semilattice with $x \wedge y = x \odot (x \to y)$,*

(ii) *$x \odot y \leq x, y$ and $x^n \leq x$, for any $n \in \mathbb{N}$,*

(iii) *$1 \to x = x,\ x \to x = 1,\ x \to 1 = 1$,*

(iv) *$x \leq y \to x$,*

(v) *$x \odot (y \to z) \leq y \to x \odot z$.*

Proposition 2.3 ([5, 6]). *Let A be a bounded hoop. Then the following properties hold, for all $x, y, z \in A$:*

(i) *$x \odot x' = 0,\ 0' = 1,\ 1' = 0, x''' = x', 0 \to x = 1$ and $0 \odot x = 0$,*

(ii) *$x' \leq x \to y$,*

(iii) *if $x \leq y$, then $y' \leq x'$,*

(iv) *$x \to y' = y \to x' = (x \odot y)'$,*

(v) *$x \leq x''$.*

Definition 2.4 ([8]). Let A be a bounded hoop and for any $x, y \in A$, we define $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. If $\vee$ is the join operation on A, then A is called a $\vee$-*hoop*.

Definition 2.5 ([8]). Let A be a hoop. A non-empty subset F of A is called a *filter* of A if,

$(F1)$ $x \in F$ and $x \leq y$, $y \in A$, then $y \in F$,

$(F2)$ $x \odot y \in F$, for any $x, y \in F$.

Clearly, $1 \in F$, for all filter of A. A filter F of A is called *proper filter* if $F \neq A$. It can be easily to see that, if A is a bounded hoop, then a filter is proper if and only if it is not containing 0.

Definition 2.6 ([3]). A proper filter F of a $\vee$-hoop A is called *prime filter* of A if $x \vee y \in F$ implies $x \in F$ or $y \in F$, for any $x, y \in A$. A *maximal filter* is a proper filter M of hoop A such that it is not included in any other proper filter.

Proposition 2.7 ([4]). *A proper filter M of bounded hoop A is a maximal filter of A if and only if $x \notin M$, then there exists $n \in \mathbb{N}$ such that $(x^n)' \in M$.*

Definition 2.8 ([4, 9, 11]). Let F be a subset of A such that $1 \in F$. Then for any $x, y, z \in A$:

(i) F is called a *positive implicative filter* of A, if $x \to (y \to z) \in F$ and $x \to y \in F$, then $x \to z \in F$.

(ii) F is called an *implicative filter* of A, if $x \to ((y \to z) \to y) \in F$ and $x \in F$, then $y \in F$.

(iii) F is called a *fantastic filter* of A, if $z \to (y \to x) \in F$ and $z \in F$, then $((x \to y) \to y) \to x \in F$.

(iv) F is called a *perfect filter* of A, if F is a filter such that, for any $x \in A$, $(x^n)' \in F$, for some $n \in \mathbb{N}$ if and only if $((x')^m)' \notin F$, for any $m \in \mathbb{N}$.

(v) F is called an *obstinate filter* of A, if F is a proper filter such that, for any $x, y \notin F$, $x \to y \in F$ and $y \to x \in F$.

(vi) F is called a *primary filter* of A, if for all $x, y \in A$, $(x \odot y)' \in F$ implies $(x^n)' \in F$ or $(y^n)' \in F$, for some $n \in \mathbb{N}$.

Definition 2.9 ([8])**.** Let A and B be two bounded hoops. A map $f : A \to B$ is called a *hoop homomorphism* if and only if for all $x, y \in A$, $f(0) = 0$, $f(1) = 1$, $f(x \odot y) = f(x) \odot f(y)$ and $f(x \to y) = f(x) \to f(y)$.

Proposition 2.10 ([8])**.** *Let A be a $\vee$-hoop. Then $(A, \vee, \wedge)$ is a distributive lattice.*

Proposition 2.11 ([3, 11])**.** (i) *If F is an implicative filter of A, then $x'' \to x \in F$, for any $x \in A$.*

(ii) *F is an implicative filter of A if and only if $(x' \to x) \to x \in F$, for any $x \in A$.*

(iii) *Any obstinate filter of A is a maximal filter of A.*

Definition 2.12 ([1, 4])**.** (i) A *simple hoop* is a hoop which has just two trivial filters.

(ii) A *cancellative hoop* is a hoop, where the monoid $(A, \odot, 1)$ is cancellative.

(iii) A *basic hoop* is a hoop and for any $x, y, z \in A$, $(x \to y) \to z \leq ((y \to x) \to z) \to z$.

(iv) A *finitely subdirectly irreducible hoop* is a hoop which any pair of non-trivial principal filters has a non-trivial intersection.

(v) A *local hoop* is a hoop, where $ord(x) < \infty$ or $ord(x') < \infty$, for all $x \in A$.

(vi) A *perfect hoop* is a hoop, where for any $x \in A$, if $ord(x) < \infty$, then $ord(x') = \infty$, and if $ord(x) = \infty$, then $ord(x') < \infty$.

Definition 2.13 ([12])**.** An algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ is called a *semi-De Morgan algebra* if $(L, \vee, \wedge, 0, 1)$ is a distributive lattice, $0' = 1$, $1' = 0$, and for any $x, y \in L$, $(x \vee y)' = x' \wedge y'$, $(x \wedge y)'' = x'' \wedge y''$ and $x''' = x'$.

3. Special hoop

In this section, we introduce the notion of special hoop and investigate some properties of them. Also, we compare relation between special hoop and other structures of hoops.

Definition 3.1. A bounded hoop A is called *special hoop* if for any $x, y \in A \backslash \{0\}$,

$$(x \to y)' = (y \to x)'$$

By the following example we show the relationship between special hoop and other structures of hoops.

Example 3.2. (i) Let $(A = \{0, a, b, c, 1\}, \leq)$, be a poset with $0 < c < a, b < 1$, but a, b are incomparable. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Then $(A, \odot, \rightarrow, 1, 0)$ is a special hoop. But it is not Wajsberg hoop, because $1 = (a \rightarrow 0) \rightarrow 0 \neq (0 \rightarrow a) \rightarrow a = a$. Also, it is not a finitely subdirectly irreducible hoop, because $[a) \cap [b) = \{1\}$.

(ii) Let $(A = \{0, a, b, 1\}, \leq)$ be a chain, that is $0 < a < b < 1$. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $(A, \odot, \rightarrow, 1, 0)$ is a bounded hoop which is a finitely subdirectly irreducible hoop. But it is not a special hoop, because $0 = (a \rightarrow b)' \neq (b \rightarrow a)' = a$.

(iii) Let $A = \{0, 1\}$ be a two-element chain with the following operations.

$\rightarrow$	0	1
0	1	1
1	0	1

$\odot$	0	1
0	0	0
1	0	1

Then A is a special hoop. But A is not a cancellative hoop, because $0 \odot 1 = 0 \odot 0$, and $1 \neq 0$.

(iv) Let $G = (G, +, -, \vee, \wedge, 0)$ be an arbitrary l-group and $N(G)$ be the negative cone of G, that is $N(G) = \{a \in G \mid a \leq 0\}$. Define the operations $\odot$ and $\rightarrow$ on $N(G)$ as follows:

$$a \odot b = a + b \quad \text{and} \quad a \rightarrow b = (b - a) \wedge 0$$

Then $(N(G), \odot, \rightarrow, 0)$ is a basic hoop and cancellative hoop (See [8]). But it is not a special hoop, because it is not bounded.

(v) Let $A = [0, 1]$ and operations $\odot$ and $\to$ on A are defind by $x \odot y = min\{x, y\}$ and

$$x \to y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } otherwise \end{cases}$$

Then $(A, \odot, \to, 1, 0)$ is a special hoop.

Proposition 3.3. *For a bounded hoop A, the following conditions are equivalent, for any $x, y \in A$ and $x \neq 0$:*
(i) *A is a special hoop,*
(ii) $x' = 0$,
(iii) $x'' = 1$,
(iv) $x' \to y = 1$,
(v) $y \odot x' = 0$,
(vi) $y \to (x' \to y') = 1$,
(vii) $(x^n)' = 0$, *for any $n \in \mathbb{N}$.*

Proof. $(i) \Rightarrow (ii)$ Let A be a special hoop. Then we have $(x \to y)' = (y \to x)'$, for any $x, y \in A\backslash\{0\}$. Consider $y = 1$, hence we have $0 = 1' = (x \to 1)' = (1 \to x)' = x'$.

$(ii) \Rightarrow (i)$ Let $x, y \in A\backslash\{0\}$. By Proposition 2.2(iv), $y \leq x \to y$ and $x \leq y \to x$ and so by Proposition 2.3(iii), $(x \to y)' \leq y' = 0$ and $(y \to x)' \leq x' = 0$. Hence $(x \to y)' = 0 = (y \to x)'$ and this means that A is a special hoop.

$(ii) \Leftrightarrow (iii)$ By Proposition 2.3(i).

$(ii) \Rightarrow (iv), (v), (vi)$ The proof are clear by Proposition 2.3.

$(iv) \Rightarrow (ii)$ Let $y = 0$. Then $x'' = x' \to 0 = x' \to y = 1$ and so by Proposition 2.3(i), $x' = x''' = 0$.

$(v) \Rightarrow (ii)$ Let $y = 1$. Then $x' = 1 \odot x' = y \odot x' = 0$.

$(vi) \Rightarrow (ii)$ Let $y = 1$. Then by Propositions 2.2(iii) and 2.3(i), $1 = y \to (x' \to y') = 1 \to (x' \to 0) = 1 \to x'' = x''$. So by Proposition 2.3(i), $x' = 0$.

$(ii) \Rightarrow (vii)$ Let $0 \neq x \in A$. If $x^2 = 0$, then

$$1 = 0 \to 0 = x^2 \to 0 = (x \odot x) \to 0 = x \to (x \to 0) = x \to x' = x \to 0 = x' = 0$$

which is impossible. Moreover, if $x^3 = 0$, then

$$1 = 0 \to 0 = x^3 \to 0 = (x^2 \odot x) \to 0 = x^2 \to (x \to 0) = x^2 \to x' = x^2 \to 0 = 0$$

which is impossible. Hence by the same way, for any $n \in \mathbb{N}$, $x^n \neq 0$. Therefore, by (ii), $(x^n)' = 0$.

$(vii) \Rightarrow (ii)$ The proof is clear. □

Proposition 3.4. *In any special hoop A, the following properties hold:*
(i) $x \to x' = x'$, *for all $x \in A$,*
(ii) $x \odot y \neq 0$, *for any $x, y \in A\backslash\{0\}$,*
(iii) $D_e(A) = \{x \in A \mid x' = 0\} = A\backslash\{0\}$,

(iv) $D_e(F) = F$, *for any filter* F *of* A,
(v) $x \to y'' = y \to x''$, *for all* $x, y \in A\backslash\{0\}$,
(vi) $x'' \to x = x$, *for all* $x \in A\backslash\{0\}$,
(vii) $(x' \to y) \to x = x$, *for all* $x, y \in A\backslash\{0\}$.

Proof. *(i)* By Proposition 3.3(ii), it is clear.

(ii) Let $x, y \in A\backslash\{0\}$, and $x \odot y = 0$, by the contrary. Since by Proposition 3.3(ii), $x' = 0$ and $y' = 0$, so by Proposition 2.3(i),(iv), $1 = 0' = (x \odot y)' = x \to y' = x' = 0$, which is a contradiction. Hence $x \odot y \neq 0$.

(iii) By Proposition 3.3(ii), the proof is clear.

(iv) Let F be a filter of A. Then by (iii), $D_e(F) = D_e(A) \cap F = A\backslash\{0\} \cap F = F$.

(v) Let $x, y \in A\backslash\{0\}$. Then by Propositions 3.3(iii) and 2.2(iii),

$$x \to y'' = x \to 1 = 1 = y \to 1 = y \to x''$$

(vi) By Propositions 3.3(iii) and 2.2(iii), $x'' \to x = 1 \to x = x$, for all $x \in A\backslash\{0\}$.

(vii) By Propositions 3.3(ii) and 2.2(iii), $(x' \to y) \to x = (0 \to y) \to x = 1 \to x = x$, for any $x, y \in A\backslash\{0\}$. □

In the following example we show that the converse of some properties of above proposition is not correct, in general.

Example 3.5. (i) Let $(A = \{0, a, b, c, 1\}, \leq)$ be a poset with $0 < a, b < c < 1$ but a, b are incomparable. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop. It is clear that $x \to x' = x'$, for all $x \in A$. But it is not a special hoop, since $a = (a \to b)' \neq (b \to a)' = b$.

(ii) Consider Example 3.2(i), we can see that the unique maximal filter of A is $\{1, b\} = D_e(A)$, but it is not a special hoop.

(iii) In Example 3.2(i), it is clear that for all filter of A, $D_e(F) = F$, but A is not a special hoop.

Theorem 3.6. *Let* A *be a special hoop. Then:*
(i) A *has only one atom,*
(ii) $ord(x) = \infty$, *for any* $x \in A\backslash\{0\}$.

Proof. (i) Let x, y be two atoms of A. Since by Proposition 2.2(ii), $x \odot y \leq x, y$ and x, y are atoms, we get $x \odot y = 0$. Hence by Propositions 2.2(iii), 2.3(iv) and 3.3(ii), $1 = 0 \to 0 = (x \odot y) \to 0 = x \to y' = x' = 0$, which is a contradiction.

(ii) Let $x \in A$ and there exists $n \in N$ such that $ord(x) = n$, by the contrary. Then $x^n = 0$ but $x, x^{n-1} \neq 0$, and so by Proposition 3.4(ii), $x^n = x \odot x^{n-1} \neq 0$, which is a contradiction. Therefore, for any $x \in A\backslash\{0\}$, $ord(x) = \infty$. □

Example 3.7. (i) Let A be as in Example 3.2(ii). Then A has only one atom, but it is not special hoop.

(ii) Let A be as in Example 3.5(i). Then $ord(x) = \infty$, for any $x \in A\backslash\{0\}$, but it is not a special hoop.

Definition 3.8. A hoop is called *meet zero divisor hoop* or mzd-hoop, if $x \wedge x' = 0$, for any $x \in A$.

Example 3.9. (i) Let A be as in Example 3.2(i). Then A is a mzd-hoop.

(ii) Let A be as in Example 3.2(ii). Then A is not a mzd-hoop. Because $a \wedge a' = a \neq 0$.

Proposition 3.10. (*i*) *If A is a special hoop, then A is a mzd-hoop,*

(*ii*) *If A is a linear mzd-hoop, then A is a special hoop.*

Proof. (i) Let A be a special hoop. Then by Proposition 3.3(ii), if $x \neq 0$, then $x' = 0$. So $x \wedge x' = 0$. Therefore, A is a mzd-hoop.

(ii) Let A be a linear mzd-hoop. Then for any $x \in A$, $x \leq x'$ or $x' \leq x$. Since A is mzd-hoop, $x \wedge x' = 0$ and so $x = 0$ or $x' = 0$. Hence if $x \neq 0$, then $x' = 0$ and so by Proposition 3.3(ii), A is a special hoop. □

In the following example we show that the converse of Proposition 3.10(ii), is not correct, in general.

Example 3.11. Let A be as Example 3.5(i). Then A is mzd-hoop, but it is not a special hoop.

By the following example we study the relationship between a special hoop and a simple hoop.

Example 3.12. (i) Let $A = \{0, a, 1\}$ be a chain. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

$\odot$	0	a	1
0	0	0	0
a	0	0	a
1	0	a	1

Then $(A, \odot, \to, 1, 0)$ is a bounded simple hoop. But it is not a special hoop, because $a' = a \neq 0$.

(ii) Let A be as in Example 3.2(i). Then A is a special hoop, but it is not a simple hoop.

Theorem 3.13. *If A is a special $\vee$-hoop, then A is a semi-De Morgan algebra.*

Proof. Since A is a $\vee$-hoop, by Proposition 2.10, $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $(x \vee y)' = x' \wedge y'$. It is sufficient to show that $(x \wedge y)'' = x'' \wedge y''$. If $x = 0$ or $y = 0$, then it is true. If $x, y \neq 0$, then by Theorem 3.6(ii), x and y are not atoms, together. Hence $x \wedge y \neq 0$, and so by Proposition 3.3(iii), $(x \wedge y)'' = 1 = x'' \wedge y''$. □

In the following example we show that the converse of Theorem 3.13, is not correct, in general.

Example 3.14. Let A be as in Example 3.2(ii). Then A is a semi-De Morgan algebra, but it is not a special hoop.

Proposition 3.15. *Special hoop A is a locally finite hoop if and only if $A = \{0, 1\}$.*

Proof. Let A be a special hoop and there is $x \in A \backslash \{0, 1\}$ such that $x^n = 0$ for any $n \in \mathbb{N}$. Then by Proposition 2.3(i), $(x^n)' = 1$. On the other hands, by Proposition 3.3(vii), $(x^n)' = 0$, which is a contradiction. The converse is clear. □

Example 3.16. Hoop A in Example 3.12(i), is a locally finite hoop, but it is not a special hoop. Also, hoop A in Example 3.2(ii), is a special hoop, but it is not a locally finite hoop, because $c^2 = c \neq 0$.

Proposition 3.17. *Every special hoop is a local and perfect hoop.*

Proof. Let A be a special hoop. Then by Proposition 3.3(ii), we have $ord(x') < \infty$, for any $0 \neq x \in A$. Therefore, A is a local hoop. Moreover, if A is a special hoop, then by Proposition 3.3(ii), and Theorem 3.6(iii), we have $ord(x') < \infty$ and $ord(x) = \infty$. Therefore, A is a perfect hoop. □

We show that by the following example, every local or perfect hoop is not a special hoop, in general.

Example 3.18. (i) Let $A = \{0, a, b, 1\}$ be a chain that is $0 < a < b < 1$. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded local hoop. But it is not a special hoop, because $0 = (a \to b)' \neq (b \to a)' = a$.

(ii) Let $A = \{0, a, b, 1\}$ be a chain. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	b	b
1	0	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded perfect hoop. But it is not a special hoop, because $0 = (a \to b)' \neq (b \to a)' = b$.

Proposition 3.19. *Let A be a special hoop. Then A is a Boolean algebra if and only if $A = \{0, 1\}$.*

Proof. Let A be a Boolean algebra and $0 \neq x \in A$. Then by Proposition 3.3(ii), $x' = 0$. Hence $x = x \vee 0 = x \vee x' = 1$. Therefore, $A = \{0, 1\}$. The proof of converse is clear. □

4. Filters in special hoops

In this section, we study some properties of filters in a special hoop.

Proposition 4.1. *Let A be a special hoop. Then the following properties hold:*

(i) *A has only one maximal filter,*

(ii) *any filter of A is a perfect filter,*

(iii) *A has only one obstinate filter,*

(iv) *any filter of A is a primary filter,*

(v) *any implicative filter of A is a maximal filter.*

Proof. (i) Let F and G be two maximal filters of A such that $x \in G \backslash F$. Then by Proposition 2.7, there exists $n \in \mathbb{N}$ such that $(x^n)' \in F$. Since A is a special hoop, by Proposition 3.3(vii), $0 = (x^n)' \in F$, which is a contradiction. This maximal filter is $F = A \backslash \{0\}$.

(ii) Let $x = 0$. Then for any $n \in \mathbb{N}$, $(x^n)' = 1 \in F$, and $((x')^m)' = 0 \notin F$. If $0 \neq x \in A$, then by Proposition 3.3(vii), $0 = (x^n)' \notin F$ and $((x')^m)' = 1 \in F$, for any $n, m \in \mathbb{N}$.

(iii) Let F be an obstinate filter of A. Then for any $x \in A$, by Proposition 3.3(ii), $x' = 0 \notin F$, hence $x \in F$ and so $F = A \backslash \{0\}$.

(iv) Let F be a proper filter of A. If $x = 0$ or $y = 0$ and $(x \odot y)' \in F$, then $(x^n)' \in F$ or $(y^n)' \in F$, for $n \in \mathbb{N}$. If $x, y \neq 0$, then $(x \odot y)' = x \to y' = x \to 0 = 0 \notin F$. Therefore, F is a primary filter of A.

(v) Let F be an implicative filter of A. Then by Proposition 2.11(i), $x'' \to x \in F$, for any $x \in A$. So by Propositions 2.2(iii) and 3.3(iii), $1 \to x = x \in F$. Hence, $F = A \backslash \{0\}$ and so F is a maximal filter of A. □

Example 4.2. Let $A = \{0, a, b, c, 1\}$, with $0 < a < b, c < 1$, but c, b are incomparable. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

Then A is a special hoop and $F = \{1, b\} \neq A\backslash\{0\}$ is a positive implicative filter and fantastic filter of A, but it is not a maximal filter of A.

Corollary 4.3. *Let F be a filter of special hoop A. Then F is an implicative filter of A if and only if F is an obstinate filter of A if and only if F is a maximal filter of A if and only if $F = A\backslash\{0\}$.*

Theorem 4.4. *Let A be a special hoop and F be a maximal ((positive)implicative, obstinate, fantastic) filter of A. Then A/F:*
(i) is a mzd-hoop,
(ii) is a Boolean algebra,
(iii) is a special hoop,
(iv) is a local hoop,
(v) is a perfect hoop.

Proof. (i) Let $x/F \in A/F$. Then $x = 0$ or $x \neq 0$. If $x = 0$, then $x/F = 0/F$. If $x \neq 0$, then by Proposition 3.3(ii), $x'/F = 0/F$. Hence $x/F \wedge x'/F = 0/F$.

(ii) By Corollary 4.3, the proof is clear.

(iii) Let $0/F \neq x/F \in A/F$. Then by Proposition 3.3(ii), $x' = 0$, and so $x'/F = 0/F$.

(iv) By (iii), A/F is a special hoop. Then by Proposition 3.17, A/F is a local hoop.

(v) By (iii) and Proposition 3.17, the proof is clear. □

Proposition 4.5. *Let A/P be a special hoop. Then P is a primary filter of A.*

Proof. Assume that A/P is a special hoop and $(x \odot y)' = y \to x' \in P$, for some $x, y \in A$. Then $y/P \to x'/P = (y \to x')/P = 1/P$, and so $y/P \leq x'/P$. Assume that $(x^n)' \notin P$, for all $n \in \mathbb{N}$. Then $(x^n)'/P \neq 1/P$. Hence x^n/P and $x/P \neq 0/P$. Since A/P is a special hoop, $x'/P = 0/P$. Also, $y^m/P \leq (x')^m/P = 0/P$, for some $m \in \mathbb{N}$. Hence $(y^m)'/P = 1/P$, i.e. $(y^m)' \in P$. Therefore, P is a primary filter of A. □

In the following example we show that the converse of above Proposition is not correct, in general.

Example 4.6. Let A be as in Example 3.2(ii). Then $F = \{1, b\}$ is a primary filter of A, but A/F is not a special hoop. Because $a/F = a'/F \neq 0/F$.

In the following Theorem we show that if in Theorem 4.4, A is not a special hoop, then this Theorem is not correct, in general.

Theorem 4.7. *Let A be a bounded hoop and F be a subset of A. If F is an implicative (maximal, obstinate) filter of A. Then A/F is not a special hoop.*

Proof. Let F be an implicative filter of A. Then by Proposition 2.11(i), $x'' \to x \in F$. So $x''/F \to x/F = 1/F$ and by Proposition 2.3(v), $x/F \to x''/F = 1/F$. Hence $x/F = x''/F$ and so $D_e(A/F) = \{1/F\}$. So $x' \neq 0$, for all $1 \neq x \in A/F$. Then by Proposition 3.3(ii), A is not a special hoop.

If F is a maximal filter of A and $x/F \neq 0/F$, $x \notin F$, then by Proposition 2.7, $(x^n)' \in F$. So $(x^n)'/F = 1/F$. If A/F is a special hoop, then Proposition 3.3(vii), $(x^n)'/F = 0/F$, which is a contradiction.

If F is an obstinate filter of A and $x/F \neq 0/F$, $x \notin F$, then $x' \in F$. So $x'/F = 1/F$, which is a contradiction by Proposition 3.3(ii). Then A/F is not a special hoop. □

Proposition 4.8. *Let A be a hoop with DNP. Then A/F is a special hoop if and only if F is a maximal filter of A.*

Proof. Let A/F be a special hoop. Then by Proposition 3.3(ii), for any $0/F \neq x/F \in A/F$, $x'/F = 0/F$. By Proposition 2.3(i), $x''/F = 1/F$ and so $x'' \in F$, for all $0 \neq x \in A$. By assumption, $x \in F$, for all $0 \neq x \in A$. Hence $F = A \backslash \{0\}$. The converse is similar, too. □

5. Special filter

In this section, we introduce the concept of special filter in a bounded hoop and investigate some properties of them. Also, we study relation between special filter and some other filters in hoops.

Note. In this section, we consider A, as a bounded hoop.

Definition 5.1. A proper filter F of A is called *special filter* of A if and only if $(x \to y)' = (y \to x)'$, for all $x, y \in F$.

Example 5.2. (i) $\{1\}$ is a special filter in A.

(ii) Let $A = \{0, a, b, c, 1\}$ be as in Example 4.2. Then $F = \{1, c\}$ is a special filter of A.

Proposition 5.3. *F is a proper special filter of A if and only if $D_e(F) = \{x \in F \mid x' = 0\} = F$.*

Proof. It is clear that $D_e(F) \subseteq F$. If $x \in F$, then $(x \to 1)' = (1 \to x)'$ and so $x' = 0$. Hence $x \in D_e(F)$.

Conversely, if $F = D_e(F)$, then $x' = y' = 0$, for all $x, y \in F$. In the other hand, by Propositions 2.2(iv) and 2.3(iii), $x \le y \to x$, then $(y \to x)' \le x' = 0$. Hence $(x \to y)' = (y \to x)' = 0$, for all $x, y \in F$. Therefore, F is a special filter of A. □

Proposition 5.4. *For all proper filter F of A, $D_e(F) = F$ if and only if A is a special hoop.*

Proof. If $D_e(F) = F$, for all filter F of A, then by Proposition 5.3, A is a special hoop.

Conversely, if A is a special hoop, then $x' = 0$, for all $0 \neq x \in A$. Therefore, $D_e(F) = F$, for all filter F of A. □

Corollary 5.5. *Any filter of special hoop A is a special filter of A.*

Proposition 5.6. *Let F and G be two special filters of A. Then:*
(*i*) *$[F \cup G)$ is special filter of A,*
(*ii*) *$F \cap H$ is special filter of A, for any $H \in \mathcal{F}(A)$.*

Proof. (i) Let $x \in [F \cup G)$. Then for $f \in F$ and $g \in G$, $x \ge f \odot g$. Since $f \in F$, $g \in G$, and F and G are special filters, then $f' = g' = 0$. Also, by Proposition 2.3(iii), (iv), $x' \le (f \odot g)' = f \to g' = f' = 0$. By Proposition 5.3, $[F \bigcup G)$ is a special filter of A.

(ii) By Proposition 5.3, $D_e(F) = F$. If $x \in F \cap H$, then $x \in F$ and $x' = 0$. Hence $D_e(F \cap H) = F \cap H$. □

In the following example we show that the converse of Proposition 5.6, is not correct, in general.

Example 5.7. Let A be a hoop in Example 3.5(i). Then for $F = \{1, c, b\}$ and $G = \{1, c, a\}$, we have $F \cap G = \{1, c\}$ is a special filter of A. But F and G are not a special filters of A. Because $a' = b \neq 0$ and $b' = a \neq 0$.

Proposition 5.8. (*i*) *Let F be a special filter and G be an obstinate filter of A. Then $F \subseteq G$.*
(*ii*) *If F is a special filter and G is an implicative filter of A, then $F \subseteq G$.*

Proof. (i) Let $F \nsubseteq G$. Then there exists $x \in F \backslash G$. Hence by Proposition 5.3, since G is an obstinate filter, $0 = x' \in G$, which is a contradiction.

(ii) Let $F \nsubseteq G$. Then there exists $x \in F \backslash G$. By Propositions 2.11(i), 2.2(iii) and 3.3(iii), $x'' \to x = 1 \to x = x \in G$, which is a contradiction. □

Proposition 5.9. (*i*) *Let F be a special filter of A and $x \in A$ be a dense. Then $F(x)$ is a special filter of A.*
(*ii*) *$[x)$ is a special filter of A, if and only if x is a dense of A.*
(*iii*) *Let x be a dense and $x \le y$, for $y \in A$. Then $[y)$ is a special filter of A.*

Proof. (i) Let $y \in F(x)$. Then $f \odot x^n \leq y$, for $f \in F$ and $n \in \mathbb{N}$. Since $f \in F$ and F is a special filter, then $f' = 0$. By Proposition 2.3(iii) and (iv), $y' \leq (f \odot x^n)' = x^n \to f' = (x^n)' = x^{n-1} \to x' = ... = x' = 0$. Hence, by Proposition 5.3, $F(x)$ is a special filter of A.

(ii) If x is a dense of A, then for any $a \in [x)$, $x^n \leq a$. By Propositions 2.3(iii), and 5.3, $a' \leq (x^n)' = 0$, and so $[x)$ is a special filter of A. The converse is clear.

(iii) If x is a dense, then by (*ii*), $[x)$ is a special filter of A. On the other hands, $[y) \subseteq [x)$. Let $z \in [y)$. Then $z \in [x)$ and $z^{'} = 0$. Therefore, by Proposition 5.3, $[y)$ is a special filter of A. □

Corollary 5.10. *If A is a special hoop, then $[x)$ is a special filter of A, for any $x \in A$.*

Proof. By Theorem 3.3(ii), x is a dense of A. Hence by Proposition 5.9(ii), the proof is clear. □

We determine the relationship between the special filter and the other types of filters in hoop.

Example 5.11. Let $A = \{0, a, b, 1\}$ be as in Example 3.2(i). Then $F = \{b, 1\}$ is a special filter of A.

(i) F is not an obstinate filter of A, because $a, 0 \notin F$ and $a \to 0 \notin F$.

(ii) F is not a positive implicative filter of A, because $a \to (a \to 0) = a \to a = 1 \in F$ and $a \to a = 1 \in F$. But $a \to 0 = a \notin F$.

(iii) F is not an implicative filter A, because $1 \to ((a \to 0) \to a) = 1 \in F$ and $1 \in F$. But $a \notin F$.

(iv) F is not a perfect filter of A, because $(a^2)' = 1 \in F$ and $((a')^2)' = 1 \in F$.

Example 5.12. Let $A = \{0, a, b, c, 1\}$ be as in Example 3.5(i). Then $G = \{1, c\}$ is a special filter of A.

(i) G is not a maximal filter, because $G \subseteq \{1, c, a\}$.

(ii) G is not a primary filter, because $(a \odot b)' = 1 \in G$, but $(a^n)' = a' = b \notin G$ and $(b^n)' = b' = a \notin G$.

Example 5.13. Let $(A = \{0, a, b, c, d, 1\}, \leq)$ be a poset. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop. It is clear that $F = \{1, c, d\}$ is (obstinate, prime, (positive) implicative, maximal, perfect, primary and fantastic)filter of A. But it is not a special filter, because $0 = (c \to d)' \neq (d \to c)' = a$

Proposition 5.14. *Let F is a special filter of A, and $x' \to x = x$, for any $x \in A \backslash F$. Then F is an implicative filter of A.*

Proof. Let F be a special filter of A. If $x \in F$, then by Propositions 2.2(iii) and 5.3, $(x' \to x) \to x = 1 \to x = x \in F$. If $x \notin F$, then by assumption, $(x' \to x) \to x = x \to x = 1 \in F$. By Proposition 2.11(ii), F is an implicative filter of A. □

Proposition 5.15. *If A/F is a special hoop, then for any $0 \neq x,\ y \in A$, $x' \to (x'' \odot y) \in F$.*

Proof. Let $0/F \neq x/F \in A/F$. Then by Proposition 3.3(iii), $x''/F = 1/F$. So $x'' \in F$. Now, by Proposition 2.3(ii), $x'' \leq x' \to y$. By $(F1)$, $x' \to y \in F$. Now, by Proposition 2.2(v) and $(F2)$, $x'' \odot (x' \to y) \leq x' \to (x'' \odot y)$. Therefore, $x' \to (x'' \odot y) \in F$. □

Proposition 5.16. *If F is a special filter and obstinate filter of A, then A/F is a locally finite hoop.*

Proof. By Proposition 2.11(iii), F is a maximal filter of A. Let $x/F \neq 1/F$ be an arbitrary element of A/F. Since $x \notin F$, by Proposition 2.7, there exists $n \in \mathbb{N}$ such that $(x^n)' \in F$. Then $(x^n)'/F = 1/F$ and $(x^n)''/F = 0/F$. Hence $x^n/F = 0/F$. Therefore, A/F is a locally finite hoop. □

By the following example we show that F is a special filter of A, but A/F is not a special hoop.

Example 5.17. Let A be as in Example 3.2(i). Then it is clear that $F = \{1, b\}$, is a special filter of A, but A/F is not a special hoop. Because $a/F \neq 0/F$ and $a'/F = a/F \neq 0/F$.

6. Conclusion and future research

Hoops are a particular class of algebraic structures which were introduced in an unpublished manuscript by Büchi and Owens in the mid-1970s. In fact, hoops are partially ordered commutative residuated integral monoids satisfying a further divisibility condition.

In this note, we introduced the notion special hoop and we show that every special hoop is a local and perfect hoop. But the converse is not true. Then we studied special filter and relationships between special filter and some other filters.

Some important issues for future work are:

(i) define fuzzy special filter on hoop and special hoop,

(ii) investigate congruence relations of special filter in hoop and some of the application of them.

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