

DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVING HIGH ORDER FUZZY INITIAL VALUE PROBLEMS

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Abstract. In this paper, we develop and analyze the use of the Differential transformation method (DTM) to find the semi analytical solution for high order fuzzy initial

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value problems (FIVPs) involving ordinary differential equations. DTM allows for the solution of FIVPs to be calculated in the form of an infinite series by which the components will be simply computed. Also DTM will be constructed and formulated to obtain a semi-analytical solution of high order FIVPs using the basic properties and definitions of fuzzy set theory. Numerical example involving high order linear FIVPs was solved to illustrate the capability of DTM in this regard. The results obtained by DTM have been compared with the exact solution in the form of figures and tables.

Keywords: fuzzy numbers, fuzzy functions, fuzzy differential equations, differential transformation method.

1. Introduction

Fuzzy set theory is a powerful tool for the modeling of vagueness, and for processing uncertainty or subjective information on mathematical models. The use of fuzzy sets can be an effective tool for a better understanding of the studied phenomena. Many dynamical real life problems may be formulated as a mathematical model. These problems can be formulated either as a system of ordinary or partial differential equations. Fuzzy differential equations are a useful tool to model a dynamical system when information about its behavior. Fuzzy ordinary differential equations may arise in the mathematical modeling of real world problems in which there is some uncertainty or vagueness. Fuzzy Initial value problems (FIVPs) appear when the modeling of these problems was imperfect and its nature is under uncertainty. So the study and solution of FIVPs is extremely necessary in applications, particularly when it involves uncertain parameters or uncertain initial conditions. Fuzzy Initial value problems arise in several areas of mathematics and science including population models [1, 2, 3], mathematical physics [4] and medicine [5, 6]. Approximate-analytical methods such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and Variational Iteration Method (VIM) have been used to solve high order FIVPs involving ordinary differential equations. The HPM was utilized to solve high order linear FIVPs [7]. The ADM was employed to solve second order linear FIVP [8] and fourth order linear FIVP in [9]. [10] used VIM to solve linear systems of first order FIVPs. The VIM was implemented also directly to solve second order linear FIVPs [11]. Also HAM was developed and formulated to solve high order linear and nonlinear FIVPs [12].

The differential transformation method (DTM) is one of the Approximate-analytical methods in ordinary differential equations, partial differential equations and integral equations. Since proposed by [13], there are tremendous interests on the applications of the DTM to solve various scientific problems [14, 15, 16]. One of the problems that solvable by this method is the initial value problems (IVPs). Previous studies concluded that the DTM can be easily applied in linear and nonlinear differential equations. This can be observed in [17, 18, 19]. The main aim of this paper is to formulate and employ DTM to solve high order FIVPs directly without reducing into first order system. The

main thrust of this method is that the solution which is expressed as an infinite series converges quick to the exact solutions. To the best of our knowledge, this is the first attempt at solving a high order FIVPs using the DTM. The structure of this paper is organized as follows. We will start in section 2 with some preliminary concepts about fuzzy sets. In section 3 we define the defuzzification procedure of n 'th order FIVP. In section 4, we reviewed the concept of DTM and formulated it to obtain a reliable approximate solution to n th order FIVPs. In section 5, we employ DTM on test examples involving second and third order linear FIVPs and finally, in section 6, we give the conclusion of this study.

2. Preliminaries

Definition [20]. The r -level (or r -cut) set of a fuzzy set \tilde{A} , labeled as \tilde{A}_r , is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}} \geq r$ i.e.

$$\tilde{A}_r = \{x \in X \mid \mu_{\tilde{A}} > r, r \in [0, 1]\}.$$

Definition [21]. Fuzzy numbers are a subset of the real numbers set, and represent uncertain values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set

In this paper the class of all fuzzy subsets of \mathfrak{R} will be denoted by \tilde{E} and satisfy the following properties [21, 22]:

1. $\mu(t)$ is normal, i.e $\exists t_0$ with $\mathfrak{R} \mu(t_0) = 1$.
2. $\mu(t)$ is convex fuzzy set, i.e. $\mu(\lambda t + (1 - \lambda) s) \geq \min\{\mu(t), \mu(s)\}$, $\forall t, s \in \mathfrak{R}, \lambda \in \{0, 1\}$.
3. μ upper semi-continuous on \mathfrak{R} , and $\overline{\{t \in \mathfrak{R} : \mu(t) > 0\}}$ is compact.

\tilde{E} is called the space of fuzzy numbers and \mathfrak{R} is a proper subset of \tilde{E} .

Define the r -level set $x \in \mathfrak{R}$, $[\mu]_r = \{x \mid \mu(x) \geq r\}$, $0 \leq r \leq 1$, where $[\mu]_0 = \{x \mid \mu(x) > 0\}$ is compact which is a closed bounded interval and denoted by $[\mu]_r = (\underline{\mu}(t), \bar{\mu}(t))$. In the parametric form, a fuzzy number is represented by an ordered pair of functions $(\underline{\mu}(t), \bar{\mu}(t))$, $r \in [0, 1]$ which satisfies [23]:

1. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over $[0, 1]$.
2. $\bar{\mu}(t)$ is a bounded left continuous non-increasing function over $[0, 1]$.
3. $\underline{\mu}(t) \leq \bar{\mu}(t)$, $r \in [0, 1]$.

A crisp number r is simply represented by $\underline{\mu}(r) = \bar{\mu}(r) = r$ for all $r \in [0, 1]$.

Definition [23]. If \tilde{E} be the set of all fuzzy numbers, we say that $\tilde{f}(t)$ is a fuzzy function if $\tilde{f}: \mathfrak{R} \rightarrow \tilde{E}$.

Definition [24]. A mapping $\tilde{f} : T \rightarrow \tilde{E}$ for some interval $T \subseteq \tilde{E}$ is called a fuzzy function process and we denote r -level set by:

$$[\tilde{f}(t)]_r = [\underline{f}(t; r), \bar{f}(t; r)], t \in T, r \in [0, 1].$$

The r -level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their r -level sets based on the resolution identity theorem.

Definition [25]. Each function $f : X \rightarrow Y$ induces another function $\tilde{f} : F(X) \rightarrow F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y) = \begin{cases} \text{Sup}_{x \in f^{-1}(y)} U(x), & \text{if } y \in \text{range}(f) \\ 0, & \text{if } y \notin \text{range}(f). \end{cases}$$

This is called the Zadeh extension principle.

Definition [27]. Consider $\tilde{x}, \tilde{y} \in \tilde{E}$. If there exists $\tilde{z} \in \tilde{E}$ such that $\tilde{x} = \tilde{y} + \tilde{z}$, then \tilde{z} is called the H-difference (Hukuhara difference) of x and y and is denoted by $\tilde{z} = \tilde{x} \ominus \tilde{y}$.

Definition [28]. If $\tilde{f} : I \rightarrow \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. We say that \tilde{f} Hukuhara Differentiable at y_0 , if there exists an element $[\tilde{f}']_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)$, $\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)$ and the limits are taken in the metric (\tilde{E}, D) .

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)}{h} = \lim_{h \rightarrow 0^+} \frac{\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)}{h}.$$

The fuzzy set $[\tilde{f}'(y_0)]_r$ is called the Hukuhara derivative of $[\tilde{f}]_r$ at y_0 .

These limits are taken in the space (\tilde{E}, D) if t_0 or T , then we consider the corresponding one-side derivation. Recall that $\tilde{x} \ominus \tilde{y} = \tilde{z} \in \tilde{E}$ are defined on r -level set, where $[\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r, \forall r \in [0, 1]$ By consideration of definition of the metric D all the r -level set $[\tilde{f}(0)]_r$ are Hukuhara differentiable at y_0 , with Hukuhara derivatives $[\tilde{f}'(y_0)]_r$, when $\tilde{f} : I \rightarrow \tilde{E}$ is Hukuhara differentiable at y_0 with Hukuhara derivative $[\tilde{f}'(y_0)]_r$ it lead to that \tilde{f} is Hukuhara differentiable for all $r \in [0, 1]$ which satisfies the above limits i.e. if f is differentiable at $t_0 \in [t_0 + \alpha, T]$ then all its r -levels $[\tilde{f}'(t)]_r$ are Hukuhara differentiable at t_0 .

Definition [27]. Define the mapping $\tilde{f}' : I \rightarrow \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. The fuzzy function \tilde{f}' be Hukuhara differentiable at y_0 , if there exists an element $[\tilde{f}^{(n)}]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small, exists

$$\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r), \tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)$$

and the limits are taken in the metric (\tilde{E}, D) .

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)}{h} = \lim_{h \rightarrow 0^+} \frac{\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)}{h}$$

exists and equal to $\tilde{f}^{(n)}$ and for $n = 2$ we have second order Hukuhara derivative and so on.

Theorem [27]. Let $\tilde{f} : [t_0 + \alpha, T] \rightarrow \tilde{E}$ be Hukuhara differentiable denoted by

$$[\tilde{f}'(t)]_r = [\underline{f}'(t), \overline{f}'(t)]_r = [\underline{f}'(t; r), \overline{f}'(t; r)].$$

Then the boundary functions $\underline{f}'(t; r), \overline{f}'(t; r)$ are differentiable such that:

$$[\tilde{f}'(t)]_r = \left[(\underline{f}'(t; r))', (\overline{f}'(t; r))' \right], \forall r \in [0, 1].$$

Theorem [29]. Let $\tilde{f} : [t_0 + \alpha, T] \rightarrow \tilde{E}$ be Hukuhara differentiable denoted by

$$[\tilde{f}'(t)]_r = [\underline{f}'(t), \overline{f}'(t)]_r = [\underline{f}'(t; r), \overline{f}'(t; r)].$$

When the boundary functions $\underline{f}'(t; r), \overline{f}'(t; r)$ are differentiable we can write for n^{th} order fuzzy derivative.

$$[\tilde{f}^{(n)}(t)]_r = \left[(\underline{f}^{(n)}(t; r))', (\overline{f}^{(n)}(t; r))' \right], \forall r \in [0, 1].$$

3. Defuzzification of high order FIVPs

Consider the following general high order FIVP [22]:

$$(1) \quad \tilde{y}^{(n)}(t) = \tilde{f}\left(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)\right) + \tilde{w}(t), \quad t \in [t_0, T]$$

subject to the initial fuzzy conditions

$$(2) \quad \tilde{y}(t_0) = \tilde{y}_0, y'(t_0) = \tilde{y}'_0, \dots, y^{(n-1)}(t_0) = \tilde{y}_0^{(n-1)},$$

where \tilde{y} is a fuzzy function of the crisp variable t with \tilde{f} being a fuzzy function of the crisp variable t , the fuzzy variable \tilde{y} and the fuzzy Hukuhara-derivatives $\tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)$. Here $y^{(n)}$ is the fuzzy n^{th} order ($n \geq 2$) Hukuhara-derivative and $\tilde{y}(t_0), \tilde{y}'(t_0), \dots, \tilde{y}^{(n-1)}(t_0)$ are convex fuzzy numbers. We denote the fuzzy function y by $\tilde{y} = [\underline{y}, \overline{y}]$ for $t \in [t_0, T]$ and $r \in [0, 1]$. It means that the r -level set of $\tilde{y}(t)$ can be defined as:

$$[\tilde{y}(t)]_r = [\underline{y}(t; r), \overline{y}(t; r)],$$

$$\begin{aligned} [\tilde{y}'(t)]_r &= [\underline{y}'(t; r), \bar{y}'(t_0; r)], \dots [\tilde{y}^{(n-1)}(t)]_r \\ &= [\underline{y}^{(n-1)}(t; r), \bar{y}^{(n-1)}(t; r)], \\ [\tilde{y}(t_0)]_r &= [\underline{y}(t_0; r), \bar{y}(t_0; r)], [\tilde{y}'(t_0)]_r \\ &= [\underline{y}'(t_0; r), \bar{y}'(t_0; r)], \dots, [\tilde{y}^{(n-1)}(t_0)]_r \\ &= [\underline{y}^{(n-1)}(t_0; r), \bar{y}^{(n-1)}(t_0; r)], \end{aligned}$$

where is the fuzzy inhomogeneous term such that $[\tilde{w}(t)]_r = [\underline{w}(t; r), \bar{w}(t; r)]$.

Since $\tilde{y}^{(n)}(t) = \tilde{f}(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)) + \tilde{w}(t)$.

Let $\mathcal{Y}(t) = \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)$, such that

$$\begin{aligned} \tilde{\mathcal{Y}}(t; r) &= [\underline{\mathcal{Y}}(t; r), \bar{\mathcal{Y}}(t; r)] \\ &= [\underline{y}(t; r), \underline{y}'(t; r), \dots, \underline{y}^{(n-1)}(t; r), \bar{y}(t; r), \bar{y}'(t; r), \dots, \bar{y}^{(n-1)}(t; r)]. \end{aligned}$$

Also, we can write

$$[\tilde{f}(t, \tilde{\mathcal{Y}})]_r = [\underline{f}(t, \tilde{\mathcal{Y}}; r), \bar{f}(t, \tilde{\mathcal{Y}}; r)].$$

By using Zadeh extension principles as mentioned in [29, 30], we have $\tilde{f}(t, \tilde{\mathcal{Y}}(t; r)) = [\underline{f}(t, \tilde{\mathcal{Y}}(t; r)), \bar{f}(t, \tilde{\mathcal{Y}}(t; r))]$, such that

$$\begin{aligned} \underline{f}(t, \tilde{\mathcal{Y}}(t; r)) &= \mathcal{F}(t, \underline{\mathcal{Y}}(t; r), \bar{\mathcal{Y}}(t; r)) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)), \\ \bar{f}(t, \tilde{\mathcal{Y}}(t; r)) &= \mathcal{G}(t, \underline{\mathcal{Y}}(t; r), \bar{\mathcal{Y}}(t; r)) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)). \end{aligned}$$

Then Eq. (1) can as written as follows:

$$(3) \quad \underline{y}^{(n)}(t; r) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r),$$

$$(4) \quad \bar{y}^{(n)}(t; r) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \bar{w}(t; r),$$

where for all $r \in [0, 1]$, the membership functions $\mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r)$ and $\mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \bar{w}(t; r)$ can be defined as

$$\begin{aligned} \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r) &= \min\{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu | \mu \in [\tilde{\mathcal{Y}}(t; r)]_r\}, \\ \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \bar{w}(t; r) &= \max\{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu | \mu \in [\tilde{\mathcal{Y}}(t; r)]_r\}. \end{aligned}$$

4. Differential transformation method in fuzzy environment

The DTM is developed based on the Taylor series expansion [30]. This method constructs an analytical solution in the form of polynomial.

Definition. If $\tilde{f}: \mathfrak{R} \rightarrow \tilde{E}$, $\exists c \in [t_0, T]$ a Taylor polynomial of degree n is defined as follows:

$$(5) \quad \begin{cases} \underline{P}_n(t; r) = \sum_{k=0}^n \frac{1}{k!} \left(\underline{f}^k(c; r) \right) (t - c)^k, \\ \overline{P}_n(t; r) = \sum_{k=0}^n \frac{1}{k!} \left(\overline{f}^k(c; r) \right) (t - c)^k, \end{cases}$$

where $r \in [0, 1]$ and $t \in [t_0, T]$. Suppose that, the fuzzy function $\tilde{f}(t)$ is a continuously H-differentiable function on the interval $[t_0, T]$.

Definition. If $\tilde{f}: \mathfrak{R} \rightarrow \tilde{E}$, the differential transform of the fuzzy function $\tilde{f}(t)$ for the k th H-derivative is defined as follows:

$$(6) \quad \begin{cases} \mathcal{F}(k; r) = \frac{1}{k!} \left[\frac{d^k \underline{f}(t; r)}{d^k t} \right]_{t=t_0}, \\ \mathcal{G}(k; r) = \frac{1}{k!} \left[\frac{d^k \overline{f}(t; r)}{d^k t} \right]_{t=t_0}, \end{cases}$$

where $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are the original functions with $\mathcal{F}(k; r)$ and $\mathcal{G}(k; r)$ are the lower and upper fuzzy transformed functions for all $r \in [0, 1]$.

Definition. If $\tilde{f}: \mathfrak{R} \rightarrow \tilde{E}$, the inverse differential transform $\mathcal{F}(k; r)$ and $\mathcal{G}(k; r)$ are define as follows:

$$(7) \quad \begin{cases} \underline{f}(t; r) = \sum_{k=0}^n \mathcal{F}(k; r) (t - t_0)^k, \\ \overline{f}(t; r) = \sum_{k=0}^n \mathcal{G}(k; r) (t - t_0)^k. \end{cases}$$

Substitution of Eq. (6) into Eq. (7) yields:

$$(8) \quad \begin{cases} \underline{f}(t; r) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k \underline{f}(t; r)}{d^k t} \right]_{t=t_0} (t - t_0)^k, \\ \overline{f}(t; r) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k \overline{f}(t; r)}{d^k t} \right]_{t=t_0} (t - t_0)^k. \end{cases}$$

Note that, this is the Taylor series of the fuzzy functions $\tilde{f}(t; r)$ at $t = t_0$. The basic operations of differential transformation for the crisp problems were given in [30, 15, 16]. Now according to Section 2 and 3, the following basic operations of differential transformation can be deduced from equations Eqs. (6) and (7):

Table 1: Various differential transform operators in fuzzy environment.

Function form		DTM - exact	
If $\underline{f}(t; r) = \underline{a}(t; r) \pm \underline{b}(t; r)$	then	$\mathcal{F}(k; r) = \underline{A}(k; r) + \underline{B}(k; r)$	
If $\overline{f}(t; r) = \overline{a}(t; r) \pm \overline{b}(t; r)$	then	$\mathcal{G}(k; r) = \overline{A}(k; r) + \overline{B}(k; r)$	
If $\underline{f}(t; r) = s\underline{a}(t; r)$	then	$\mathcal{F}(k; r) = s\underline{A}(k; r)$, s constant	
If $\overline{f}(t; r) = s\overline{a}(t; r)$	then	$\mathcal{G}(k; r) = s\overline{A}(k; r)$, s constant	
If $\underline{f}(t; r) = \frac{d^n \underline{a}(t; r)}{d^n t}$	then	$\mathcal{F}(k; r) = \frac{(k+n)!}{k!} \underline{A}(k+n; r)$	
If $\overline{f}(t; r) = \frac{d^n \overline{a}(t; r)}{d^n t}$	then	$\mathcal{G}(k; r) = \frac{(k+n)!}{k!} \overline{A}(k+n; r)$	
If $\underline{f}(t; r) = \underline{a}(t; r) \underline{b}(t; r)$	then	$\mathcal{F}(k; r) = \sum_{l=0}^k \underline{A}(l; r) \underline{B}(k-l; r)$	
If $\overline{f}(t; r) = \overline{a}(t; r) \overline{b}(t; r)$	then	$\mathcal{G}(k; r) = \sum_{l=0}^k \overline{A}(l; r) \overline{B}(k-l; r)$	
If $\underline{f}(t; r) = t^n$	then	$\mathcal{F}(k; r) = \underline{\delta}(k-n; r)$,	
		$\underline{\delta}(k-n; r) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$	
If $\overline{f}(t; r) = t^n$	then	$\mathcal{G}(k; r) = \overline{\delta}(k-n; r)$,	
		$\overline{\delta}(k-n; r) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$	
If $\underline{f}(t; r) = \exp(\theta t)$	then	$\mathcal{F}(k; r) = \frac{\theta^k}{k!}$	
If $\overline{f}(t; r) = \exp(\theta t)$	then	$\mathcal{G}(k; r) = \frac{\theta^k}{k!}$	
If $\underline{f}(t; r) = \sin(st + c)$	then	$\mathcal{F}(k; r) = \frac{s^k}{k!} \sin(\pi \frac{k}{2!} + c)$	
If $\overline{f}(t; r) = \cos(st + c)$	then	$\mathcal{F}(k; r) = \frac{s^k}{k!} \cos(\pi \frac{k}{2!} + c)$	
If $\underline{f}(t; r) = \sin(st + c)$	then	$\mathcal{G}(k; r) = \frac{s^k}{k!} \sin(\pi \frac{k}{2!} + c)$	
If $\overline{f}(t; r) = \cos(st + c)$	then	$\mathcal{G}(k; r) = \frac{s^k}{k!} \cos(\pi \frac{k}{2!} + c)$	

5. Numerical examples

5.1 Example

Consider the circuit model problem [22] shown in Fig. 2, where $L = 1h$, $R = 2\Omega$, $C = 0.25f$ and $E(t) = 50\cos(t)$. Let $Q(t)$ be the charge on the capacitor at time $t > 0$.

Thus the FIVP of this model is given as follows

$$(9) \quad \begin{aligned} \tilde{Q}''(t; r) + 2\tilde{Q}'(t; r) + 4\tilde{Q}(t; r) &= 50\cos t, \quad t > 0, \\ \tilde{Q}(0; r) &= [4 + r, 6 - r], \tilde{Q}'(0; r) = [r, 2 - r], \quad r \in [0, 1]. \end{aligned}$$

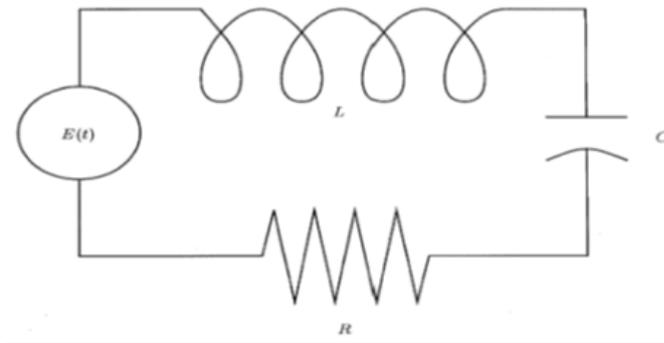


Figure 1: Electric circuit model

The exact analytical solution of Eq. (9) was given by [31]. Taking the differential transform of Eq. (9) by Table 1, we obtain for the lower bound:

$$(10) \quad \mathcal{F}(k+2; r) = \frac{50 \frac{1}{k!} \sin\left(\pi \frac{k}{2!}\right) - 2\{(k+1)\mathcal{F}(k+1; r)\} - 4\mathcal{F}(k; r)}{(k+1)(k+2)}.$$

The differential transform of the initial values of Eq. (10) for all $r \in [0, 1]$ are as follows

$$(11) \quad \mathcal{F}(0; r) = 4 + r, \mathcal{F}(1; r) = r.$$

The differential transform of the upper bound of Eq. (9) is as follows:

$$(12) \quad \mathcal{G}(k+2; r) = \frac{50 \frac{1}{k!} \sin\left(\pi \frac{k}{2!}\right) - 2\{(k+1)\mathcal{G}(k+1; r)\} - 4\mathcal{G}(k; r)}{(k+1)(k+2)}.$$

Now, the differential transform of the initial values of Eq. (10) for all $r \in [0, 1]$ are as follows

$$(13) \quad \mathcal{G}(0; r) = 6 - r, \mathcal{G}(1; r) = 2 - r.$$

Moreover, substituting Eq. (11) into Eq. (10) and Eq. (13) into Eq. (12) and by recursive method we can calculate another value of $\mathcal{F}(k; r)$ and $\mathcal{G}(k; r)$. Finally, by using Eq. 8 the final solution can be obtained. The 15-order DTM approximation solution of Eq. (9) can be obtained by using Mathematica software code as showing in the following table and figures:

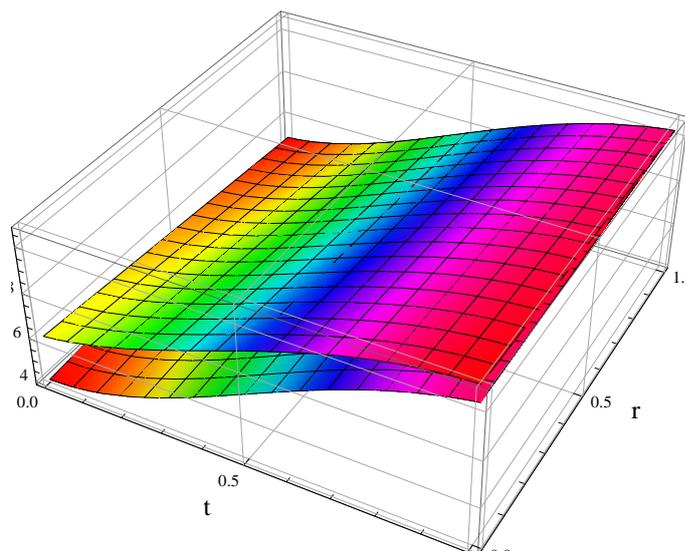


Figure 2: 15-order DTM solution of Eq. (9) for all $r \in [0,1]$ and $t \in [0,1]$.

Table 2: 15-order DTM solution and accuracy of Eq. (9) at $t = 1$.

x	$\underline{Q}(1; r)$	$ DTM - exact $	$\overline{Q}(1; r)$	$ DTM - exact $
0	9.959398092279500	8.172037×10^{-9}	10.679826450125848	6.060271×10^{-9}
0.2	10.031440928064136	7.960863×10^{-9}	10.607783614341212	6.271454×10^{-9}
0.4	10.103483763848772	7.749688×10^{-9}	10.535740772077789	6.482629×10^{-9}
0.6	10.175526599633404	7.538512×10^{-9}	10.463697942771944	6.693806×10^{-9}
0.8	10.247569435418038	7.327333×10^{-9}	10.391655106987310	6.904985×10^{-9}
1	10.319612271202672	7.116156×10^{-9}	10.319612271202672	7.116156×10^{-9}

From Table 1 and Fig. 3 one can note that the 15-order DTM approximate solution of Eq. (9) at $t = 1$ and for all $0 \leq r \leq 1$ satisfy the fuzzy numbers properties in Section 2 by taking the triangular fuzzy number shape.

5.2 Example 2

Consider the following third order fuzzy initial value problem [31]:

$$\begin{aligned}
 \tilde{y}'''(t; r) &= -\tilde{y}''(t; r) - 3\tilde{y}'(t; r) + 5\tilde{y}(t; r), \quad t \in [0, 1] \\
 \tilde{y}(0; r) &= [0.75 - 0.25r, 1.25 - 0.25r], \\
 \tilde{y}'(0; r) &= [1.5 + 0.5r, 2.5 - 0.5r], \quad r \in [0, 1] \\
 \tilde{y}''(0; r) &= [3.75 + 0.25r, 4.25 - 0.25r].
 \end{aligned}
 \tag{14}$$

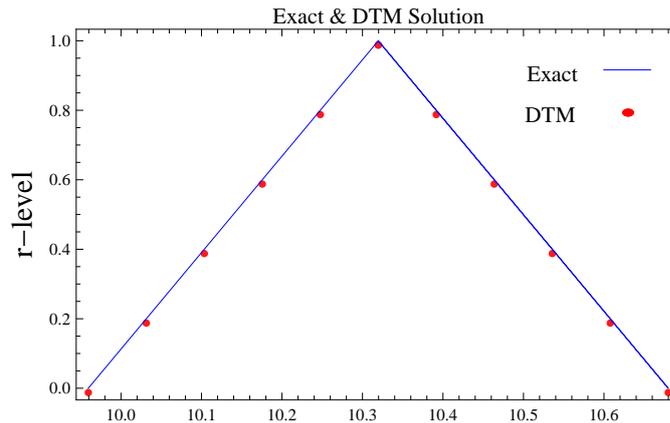


Figure 3: Exact and 15-order DTM solutions of Eq. (9) at t=1.

The exact analytical solution of Eq. (14) was given [32]. Taking the differential transform of Eq. (14) by Table 1, we obtain for the lower bound:

$$(15) \quad \begin{aligned} & \mathcal{F}(k+3; r) \\ &= \frac{-\{(k+1)(k+2)\mathcal{F}(k+2; r)\} - 3\{(k+1)\mathcal{F}(k+1; r)\} + 5\mathcal{F}(k; r)}{(k+1)(k+2)(k+3)}. \end{aligned}$$

The differential transform of the initial values of Eq. (10) for all $r \in [0, 1]$ are as follows

$$(16) \quad \mathcal{F}(0; r) = 0.75 + 0.25r, \mathcal{F}(1; r) = 1.5 + 0.5r, \mathcal{F}(2; r) = 0.5(3.75 + 0.25r).$$

The differential transform of the upper bound of Eq. (14) is as follows:

$$(17) \quad \begin{aligned} & \mathcal{G}(k+3; r) \\ &= \frac{-\{(k+1)(k+2)\mathcal{G}(k+2; r)\} - 3\{(k+1)\mathcal{G}(k+1; r)\} + 5\mathcal{G}(k; r)}{(k+1)(k+2)(k+3)}. \end{aligned}$$

Now, the differential transform of the initial values of Eq. 10 for all $r \in [0, 1]$ are as follows

$$(18) \quad \mathcal{G}(0; r) = 1.25 - 0.25r, \mathcal{G}(1; r) = 2.5 - 0.5r, \mathcal{G}(2; r) = 0.5(4.25 - 0.25r).$$

Moreover, substituting Eq. (16) into Eq. (15) and Eq. (18) into Eq. (17) and by recursive method we can calculate another value of $\mathcal{F}(k; r)$ and $\mathcal{G}(k; r)$. Finally, by using Eq. (8) the final solution can be obtained. The 15-order DTM approximation solution of Eq. (14) can be obtained by using Mathematica software code as showing in the following table and figures:

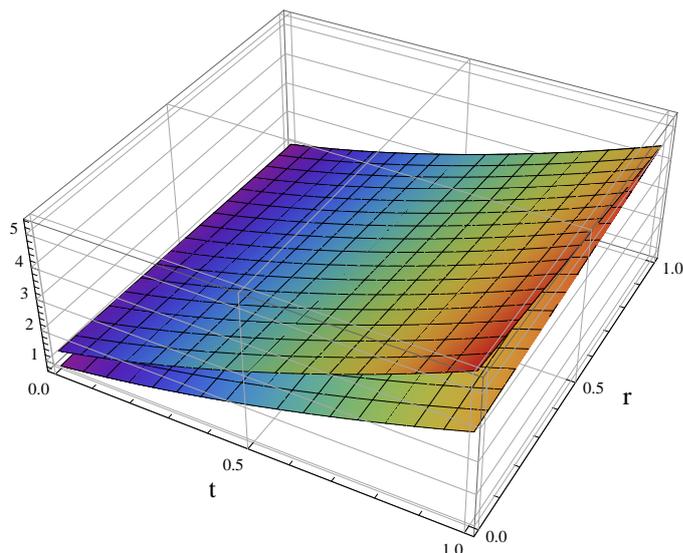


Figure 4: Exact and 15-order DTM solutions of Eq. (14) at $t=1$.

Table 3: Exact and 15-order DTM solutions of Eq. (14) at $t=1$.

x	$y(1;r)$	$ DTM - exact $	$\bar{y}(1;r)$	$ DTM - exact $
0	3.591138112568900	6.257013×10^{-9}	5.351014715840749	4.877771×10^{-9}
0.2	3.767125772896086	6.119089×10^{-9}	5.175027055513564	5.015696×10^{-9}
0.4	3.943113433223270	5.981164×10^{-9}	4.999039395186378	5.153618×10^{-9}
0.6	4.119101093550455	5.843240×10^{-9}	4.823051734859193	5.291543×10^{-9}
0.8	4.295088753877639	5.705315×10^{-9}	4.647064074532009	5.429467×10^{-9}
1	4.471076414204824	5.567392×10^{-9}	4.471076414204824	5.567392×10^{-9}

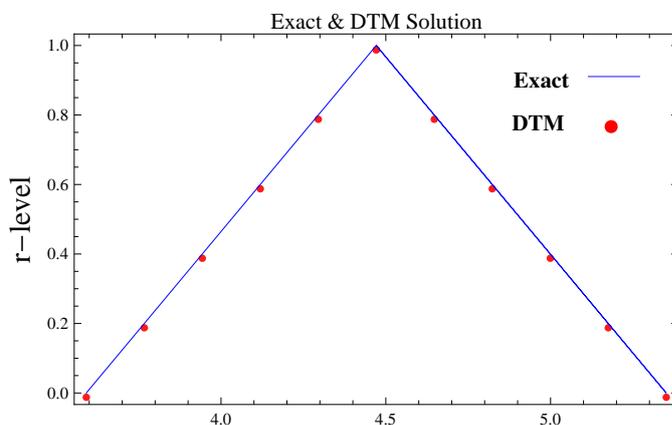


Figure 5: Exact and 15-order DTM solutions of Eq. (14) at $t = 1$.

As in Example 5.1, from Table 2 and Fig. 5 one can note that the 15-order DTM approximate solution of Eq. (14) at $t = 1$ and for all $0 \leq r \leq 1$ satisfy the fuzzy numbers properties in Section 2 by taking the triangular fuzzy number shape.

6. Conclusions

In this paper, we studied and applied the Differential Transform Method in finding solution of high order fuzzy initial value problems involving linear ordinary differential equations. To the best of our knowledge, this is the first attempt for solving the high order FIVPs with DTM. The method has been formulated to obtain an approximate solution of general high order FIVPs. Two numerical examples including second and third order linear fuzzy initial value problems showed the capability and the efficiency of DTM. Moreover, this technique converges to the exact solution and requires less computational work directly without reduced to first order system. The numerical result that obtained by DTM satisfy the fuzzy numbers properties by taking the convex fuzzy numbers shape.

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