ON SOME PROPERTIES OF ROUGH APPROXIMATIONS OF SUBRINGS VIA COSETS

Madhavi Reddy*
Research Scholar, JNIA
Budhabhavan, Hyderabad-500085
Telangana State
India
gmadhavireddy123@gmail.com

P. Venkatraman
Centre for Mathematics, JNIA
Budhabhavan, Hyderabad-500085
Telangana State
India
venkat_raman603@yahoo.com

E. Kesahva Reddy
Department of Mathematics
JNTUA College of Engineering
Ananthapuramu (dt), A.P
India
keshava_e@rediffmail.com

Abstract. In 1982, Zdzislaw Pawlak introduced the theory of Rough sets to deal with the problems involving imperfect knowledge. This present research article studies some interesting properties of Rough approximations of subrings via an equivalence relation involving cosets of an ideal. In this present work, a ring structure is assigned to the universe set and a few results on the Rough approximations of subrings of the universe set are established.

Keywords: ring, subring, ideal, maximal ideal, prime ideal, rough set, upper approximation, lower approximation, information system, equivalence relation.

Introduction. The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became also a crucial issue for computer scientists, particularly in the area of Artificial Intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful approaches to tackle this problem are the Fuzzy set theory and the Rough set theory. Theories of Fuzzy sets and Rough sets are powerful mathematical tools for modeling various types of uncertainties. Fuzzy set theory was introduced by L. A. Zadeh in his classical paper [7] of 1965.

* Corresponding author
A polish applied mathematician and computer scientist Zdzislaw Pawlak introduced Rough set theory in his classical paper [4] of 1982. Rough set theory is a new mathematical approach to imperfect knowledge. This theory presents still another attempt to deal with uncertainty or vagueness.

The Rough set theory has attracted the attention of many researchers and practitioners who contributed essentially to its development and application. Rough sets have been proposed for a very wide variety of applications.

In particular, the Rough set approach seems to be important for Artificial Intelligence and cognitive sciences, especially for machine learning, knowledge discovery, data mining, pattern recognition and approximate reasoning.

In this present work, we construct Rough sets by considering cosets of an Ideal in a ring. We investigate a few results on lower and upper approximations of subrings.

1. Rings and ideals

In this section, some basic definitions of Rings and Ideals that are necessary for further study of this work are presented.

**Definition 1.1.** A non-empty set of elements $G$ is said to form a semi-group if in $G$ there is defined a binary operation, called the product and denoted by $o$, such that

a. $a, b \in G \Rightarrow ab \in G$ (Closure);

b. $ao(boc) = (aob)oc$, for all $a, b, c \in G$ (Associative law).

**Definition 1.2.** A non-empty set of elements $G$ is said to form a group if in $G$ there is defined a binary operation, called the product and denoted by $o$, such that

a. $a, b \in G \Rightarrow ab \in G$ (Closure);

b. $ao(boc) = (aob)oc$, for all $a, b, c \in G$ (Associative law);

c. there exists an element $e \in G$ such that $ae = ea = a$, for all $a \in G$. The element $e$ is called the identity element in $G$. (Existence of identity);

d. for every $a \in G$ there exists an element $a^{-1} \in G$ such that $aao^{-1} = a^{-1}oa = e$. The element $a^{-1}$ is called the inverse element of $a$ in $G$. (Existence of inverse).

We denote a non-empty set $G$ with respect to a binary operation $o$ by the symbol $(G, o)$.

**Definition 1.3.** A group $G$ with respect to a binary operation $o$ is said to be an abelian group if $aob = boa$, for all $a, b \in G$. 
Definition 1.4. A non-empty set \( U \) is said to be a ring if in there are defined two operations, denoted by + and \( \cdot \) respectively, such that

a. \((U, +)\) is an abelian group;

b. \((U, \cdot )\) is a semi-group;

c. \(x \cdot (y + z) = x \cdot y + x \cdot z\), for all \( x, y, z \in U \);

d. \((x + y) \cdot z = x \cdot y + x \cdot z\), for all \( x, y, z \in U \).

We denote a ring \( U \) with respect to the binary operations + and \( \cdot \) by the symbol \((U; +, \cdot )\). We denote the identity element in \((U; +)\) by the symbol 0 and we call this element, the zero element of \( U \).

Definition 1.5. A non-empty subset \( S \) of a ring \((U; +, \cdot )\) is said to be a subring of \((U; +, \cdot )\) if and only if \( a, b \in S \) for every \( a \) and \( b \) in \( S \).

Definition 1.6. A non-empty subset \( I \) of a ring \((U; +, \cdot )\) is said to be an ideal of \( U \) if

a. \( x \in I \) and \( y \in I \Rightarrow x - y \in I \);

b. \( u \in U \) and \( x \in I \Rightarrow u \cdot x \in I \);

c. \( u \in U \) and \( x \in I \Rightarrow x \cdot u \in I \).

Definition 1.7. An ideal \( P \) of a ring \((U; +, \cdot )\) is said to be a prime ideal if \( x \cdot y \in P \) then either \( x \in P \) or \( y \in P \).

Definition 1.8. An ideal \( M \) of a ring \((U; +, \cdot )\) is said to be a maximal ideal if for any ideal \( I \) of \( U \), \( M \subseteq I \subseteq U \Rightarrow M = I \) or \( I = U \).

2. Rough sets

This section is devoted to the basic concepts of Rough set theory. In what follows \( \phi \) and \( U \) stand for the empty set and the universe set respectively.

Definition 2.1. A relation \( R \) on a non-empty set \( S \) is said to be an equivalence relation on \( S \) if

a. \( xRx \), for all \( x \in S \) (reflexivity);

b. \( xRy \Rightarrow yRx \) (symmetry);

c. \( xRy \) and \( yRz \Rightarrow xRz \) (transitivity).

We denote the equivalence class of an element \( x \in S \) with respect to the equivalence relation \( R \) by the symbol \( R[x] \) and \( R[x] = \{ y \in S : yRx \} \).
Definition 2.2. Let $X \subseteq U$. Let $R$ be an equivalence relation on $U$. Then we define the following

a. The lower approximation of $X$ with respect to $R$ is the set of all objects, which can be for certain classified as $X$ using $R$. That is the set $R_s(X) = \{x : R[x] \subseteq X\}$.

b. The upper approximation of $X$ with respect to $R$ is the set of all objects, which can be possibly classified as $X$ using $R$. That is the set $R^*(X) = \{x : R[x] \cap X \neq \emptyset\}$.

c. The boundary region of $X$ with respect to $R$ is the set of all objects, which can be classified neither as $X$ nor as not-$X$ using $R$. That is the set $B_R(X) = R^*(X) - R_s(X)$.

It is clear that $R_s(X) \subseteq X \subseteq R^*(X)$.

Definition 2.3. A set $X \subseteq U$ is said to be a Roughset with respect to an equivalence relation $R$ on $U$, if the boundary region $B_R(X) = R^*(X) - R_s(X)$ is non-empty.

3. Construction of rough sets

In the Literature of Rough set theory, information systems are considered. An information system is a pair $(U, A)$ where $A$ is a set of attributes. Each attribute $a \in A$ is a mapping $a : U \to V_a$ where $V_a$ is the range set of the attribute $a \in A$. Corresponding to each attribute $a \in A$, an equivalence relation $R_a$ is defined on $U$ such that $xR_a y \iff a(x) = a(y)$. Rough sets are constructed through this relation as usual.

In this section, we slightly deviate from the above traditional setting to construct Rough sets. We consider an equivalence relation on a ring to construct Rough sets and present a few results in this context.

In what follows $U$ stands for a ring and we take our universe set to be the ring $(U, +, \cdot)$.

Definition 3.1. Let $I$ be an ideal of a ring $U$ such that $I \neq \{0\}$ and $I \neq U$.

We define a relation $R$ on $U$ as follows.

For $x, y \in U$, $xRy \iff y - x \in I$.

Proposition 3.1. The relation $R$ on $U$ is an equivalence relation.

Remark 3.1. If $x \in U$ then we denote the equivalence class of $x$ under the equivalence relation $R$ by the symbol $R[x]$ and we have $R[x] = \{y \in U : yRx\} = X + I$ thus the equivalence class of $x \in U$ is the left coset $X + I$ of $I$ in $U$.

Proposition 3.2. For any subset $A$ of $U$, the following conditions are equivalent to one another.

(a) $R_s(A) = A$;
(b) \( R^*(A) = A \);

(c) \( R_*(A) = R^*(A) \).

**Proof.** For any subset \( A \) of \( U \), we always have \( R_*(A) \subseteq A \subseteq R^*(A) \).

Suppose \( R_*(A) = A \) and let \( x \in R^*(A) \). Then \( R[x] \cap A \neq \phi \)
\[ \Rightarrow R_*(A) \cap R^*(A) \neq \phi \]
\[ \Rightarrow \text{there exists a point } y \text{ in } U \text{ such that } y \in R[x] \cap R_*(A) \]
\[ \Rightarrow yRx \text{ and } y \in R_*(A) \]
\[ \Rightarrow R[x] = R[y] \text{ and } R[y] \subseteq A \]
\[ \Rightarrow R[x] \subseteq A \]
\[ \Rightarrow x \in A \]
This shows that \( R^*(A) \subseteq A \). Hence \( R^*(A) = A \).

Now consider \( R^*(A) = A \). Let \( x \in A \). Then \( x \in R^*(A) \).
\[ \Rightarrow R[x] \cap A \neq \phi \]

Let \( z \in R[x] \). Then \( xRz \Rightarrow R[z] = R[x] \)
\[ \Rightarrow R[z] \cap A \neq \phi \]
\[ \Rightarrow z \in R^*(A) \Rightarrow z \in A. \]
This proves that \( R[x] \subseteq A \) and hence \( x \in R_*(A) \).
Hence \( R_*(A) = A \). This completes the proof of (b) \( \Rightarrow \) (a).

Obviously (c) is equivalent to both (a) and (b).

4. Rough approximations of subrings

In this section, we present a few properties of lower and upper rough approximations of subrings of \( U \).

**Proposition 4.1.** \( \{0\} \) is a Rough set.

**Proof.** Since \( R_*(\{0\}) \subseteq \{0\} \subseteq R^*(\{0\}) \), either \( R_*(\{0\}) = \phi \) or \( R_*(\{0\}) = \{0\} \)
Assume that \( R_*(\{0\}) = \{0\} \).
Then \( 0 \in R_*(\{0\}) \Rightarrow I \subseteq \{0\} \Rightarrow I = \{0\} \).
This is a contradiction. Hence \( R_*(\{0\}) = \phi \).
Let \( x \in R^*(\{0\}) \). Then \( x + I \cap \{0\} \neq \phi \).
\[ \Rightarrow 0 \in x + I. \]
\[ \Rightarrow -x \in I. \]
\[ \Rightarrow x \in I. \]
This shows that
\[ R^*(\{0\}) \subseteq I \]

Let \( x \in I \). Then \( x + I = I \)
\[ \Rightarrow x + I \cap \{0\} = I \cap \{0\} = \{0\} \neq \phi. \]
\[ \Rightarrow x \in R^*(\{0\}). \]
This shows that

\[(4.2)\quad I \subseteq R^*(\{0\})\]

From (4) and (4), \(R^*(\{0\}) = I\).

Then \(B_R(\{0\}) = I\).

Hence \(\{0\}\) is a Rough set.

**Remark 4.1.** By the above Proposition-4.1, it follows that the lower approximation of a subring of \(U\) is not necessarily a subring of \(U\). Even though \(\{0\}\) is a subring of \(U\), its lower approximation is empty and hence \(R_*(\{0\})\) is not a subring of \(U\).

**Proposition 4.2.** \(I\) is not a Rough set.

**Proof.** Clearly \(R_*(I) \subseteq I \subseteq R^*(I)\). Let \(x \in I\).

Then \(x + I = I \Rightarrow x \in R_*(I)\).

Then \(I = R_*(I)\).

By Proposition-3.2, \(R_*(I) = I = R^*(I)\).

Hence \(I\) is not a Rough set.

**Proposition 4.3.** Let \(K\) be a subring of \(U\). Then the following are equivalent.

(a) \(I \subseteq K\);

(b) \(R_*(K) = K = R^*(K)\);

(c) \(R_*(K)\) is a subring of \(U\).

**Proof.** Let \(K\) be a subring of \(U\). Suppose that \(I \subset K\).

Clearly \(R_*(K) \subseteq K \subseteq R^*(K)\).

Let \(x \in K\). Since \(I \subseteq K\), \(x + I \subseteq x + K = K\)

\(\Rightarrow x \in R_*(K)\).

This shows that \(K \subseteq R_*(K) \Rightarrow K = R_*(K)\).

Hence \(R_*(K) = K = R^*(K)\).

This proves (a)\(\Rightarrow\)(b).

Suppose that \(R_*(K) = K = R^*(K)\).

Then \(R_*(K)\) is a subring of \(U\).

This proves (b)\(\Rightarrow\)(c).

Suppose that \(R_*(K)\) is a subring of \(U\).

\(\Rightarrow 0 \in R_*(K)\)

\(\Rightarrow I = 0 + I \subseteq K\)

This proves (c)\(\Rightarrow\)(a).

**Proposition 4.4.** If \(K\) is a subring of \(U\) then \(R^*(K)\) is a subring of \(U\).

**Proof.** Let \(K\) be a subring of \(U\).

Let \(x \in R^*(K)\).

\(\Rightarrow x + I \cap K \neq \phi\)
there exists a point \( z_1 \) such that \( z_1 \in x + I \cap K \)
\( z_1 \in x + I \) and \( z_1 \in K \)
\( z_1 - x \in I \) and \( z_1 \in K \)
Let \( y \in R^*(K) \).
\( y + I \cap K \neq \phi \)
there exists a point \( z_2 \) such that \( z_2 \in y + I \cap K \)
\( z_2 \in y + I \) and \( z_2 \in K \)
\( z_2 - y \in I \) and \( z_2 \in K \)
Since \( I \) is an ideal of \( U \), \( z_1 - x \in I \) and \( z_2 - y \in I \) we have \( z_1 - z_2 - x + y \in I \).
\( z_1 - z_2 \in x - y + I \).
Since \( K \) is a subring of \( U \), \( z_1 \in K \) and \( z_2 \in K \) we have \( z_1 - z_2 \in K \).
Hence \( z_1 - z_2 \in x - y + I \cap K \).
This shows that \( x - y + I \cap K \) is non-empty.
\( x - y \in R^*(K) \)
Since \( z_2 \in U \) and \( z_1 - x \in I \), it follows that \( z_1 z_2 - x z_2 \in I \).
Since \( x \in U \) and \( z_2 - y \in I \), it follows that \( x z_2 - x y \in I \).
Then \( z_1 z_2 - x z_2 + x z_2 - x y \in I \).
\( z_1 z_2 - xy \in I \)
\( z_1 z_2 \in xy + I \)
Since \( K \) is a subring of \( U \), \( z_1 \in K \) and \( z_2 \in K \) we have \( z_1 z_2 \in K \).
\( xy + I \cap K \) is non-empty.
\( xy \in R^*(K) \)
Hence \( R^*(K) \) is a subring of \( U \).

**Proposition 4.5.** If \( K \) is an ideal of \( U \) then \( R^*(K) \) is an ideal of \( U \).

**Proof.** Since \( K \) is an ideal of \( U \), it is a subring of \( U \).
Then \( R^*(K) \) is a subring of \( U \).
Let \( u \in U \) and \( x \in R^*(K) \). Then \( (x + I) \cap K \neq \phi \).
\( uy \in K \) and \( uy - ux \in I \).
\( uy \in K \) and \( uy + ux + I \).
\( uy \in ux + I \cap K \).
\( ux + I \cap K \neq \phi \).
\( ux \in R^*(K) \).
Similarly \( xu \in R^*(K) \). Hence \( R^*(K) \) is an ideal of \( U \).

**Remark 4.2.** The proof of the proposition 4.6 follows easily from the definitions and hence can be omitted.

**Proposition 4.6.** If \( K \) is a maximal ideal of \( U \) then \( R^*(K) \) is a maximal ideal of \( U \).

**References**


Accepted: 1.11.2016