GENERALIZED NUMERICAL RADIUS INEQUALITIES FOR $2 \times 2$ OPERATOR MATRICES

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Abstract. We prove some new generalized numerical radius inequalities for $2 \times 2$ operator matrices, which improve and generalize an earlier numerical radius inequalities.

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1. Introduction

Let $B (H)$ denote the $C^*$–algebra of all bounded linear operators on a complex Hilbert space $H$ with inner product $\langle . , . \rangle$. For $A \in B (H)$, let $\omega (A) = \sup \{ \langle Ax , x \rangle : x \in H , \| x \| = 1 \}$, $\| A \| = \sup \{ \| Ax \| : x \in H , \| x \| = 1 \}$, where $\| x \|^2 = \langle x , x \rangle$, $r (A) = \sup \{ | \lambda | : \lambda \in \sigma (A) \}$, where $\sigma (A)$ is the spectrum of $A$, and $| A | = (A^* A)^{\frac{1}{2}}$ denote the numerical radius of $A$, the usual operator norm of $A$, the spectral radius of $A$, and the absolute value of $A$, respectively.

It is well – known that $\omega (\cdot)$ is a norm on $B (H)$, which is equivalent to the usual operator norm $\| \cdot \|$. In fact, for every $A \in B (H)$, we have

$$1 \leq \| A \| \leq \omega (A) \leq \| A \| .$$

These inequalities are sharp. The first inequality becomes an equality if $A^2 = 0$, and the second inequality becomes an equality if $A$ is normal.

The inequalities in (1.1) have been improved considerably by Kittaneh in [10] and [11]. It has been shown in [10] and [11], respectively, that if $A \in \mathcal{H}$, then

$$\omega (A) \leq \frac{1}{2} \| A \| + | A^* | \leq \frac{1}{2} \left( \| A \| + \| A^2 \|^{\frac{1}{2}} \right)$$

and

$$\frac{1}{4} \| A^* A + A A^* \| \leq \omega^2 (A) \leq \frac{1}{2} \| A^* A + A A^* \| .$$

An important property of the numerical radius norm is its weak unitary invariance, that is, for $A \in B (H)$,

$$\omega (U A U^*) = \omega (A) ,$$
for every unitary $U \in B(H)$.

Several numerical radius inequalities improving the inequalities in (1.1) have
been recently given in [4], [10], [11], and [12].

Let $H_1, H_2, \ldots, H_n$ be complex Hilbert spaces, and consider $H = \bigoplus_{i=1}^{n} H_i$ with
respect to this decomposition, every an $n \times n$ operator matrix representation $A = [A_{ij}]$, with entries $A_{ij} \in B(H_j, H_i)$, the space of all bounded linear operators
from $H_j$ to $H_i$. Operator matrices provide a useful tool for studying Hilbert
space operators, which have been extensively studied in the literature (see, e.g.,
[5]). In [8], Hou and Du established useful estimates for the spectral radius,
the numerical radius, and the usual operator norm of an $n \times n$ operator matrix
$A = [A_{ij}]$. In particular, they proved that

$$r(A) \leq r\left(\|A_{ij}\|\right),$$

$$\omega(A) \leq \omega\left(\|A_{ij}\|\right),$$

and

$$\|A\| \leq \|\|A_{ij}\||\|.$$  

Recent numerical radius equalities and inequalities for operator matrices can be
found in [1, 2], and [6].

In this paper, we give new generalized numerical radius inequalities for $2 \times 2$
operator matrices. In section 2, we establish generalized upper bounds for the
numerical radii of the off-diagonal parts of $2 \times 2$ operator matrices, i.e., operator
matrices of the form $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Also we establish generalized upper bounds
for the numerical radii of other $2 \times 2$ operator matrices.

2. Main results

The aim of this section is to give generalized upper bounds for the numerical
radius of the off-diagonal part of a $2 \times 2$ operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ defined
on $H_1 \oplus H_2$. In order to state our results, we need the following well-known
lemmas.

The first lemma is a generalization of the mixed Schwarz inequality which
has been proved by Kittaneh [9].

**Lemma 1.** Let $T$ be an operator in $B(H)$ and let $f$ and $g$ be nonnegative
functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t) \cdot g(t) = t$ for all $t \in [0, \infty)$. Then $|\langle Tx, y \rangle| \leq \|f(|T|)x\| \cdot \|g(|T^*|)y\|$, for all $x, y \in H$.

The second lemma contains two parts. Part (a) is well known and can be
found in [3, p. 10]. Part (b) is also known (see, e.g., [1]).
**Lemma 2.** Let $X, Y \in B(H)$. Then

(a) $\omega \left( \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right) = \max \{ \omega(X), \omega(Y) \}$.

(b) $\omega \left( \begin{bmatrix} X & Y \\ Y & X \end{bmatrix} \right) = \max \{ \omega(X + Y), \omega(X - Y) \}$.

In particular, $\omega \left( \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \right) = \omega(X)$.

The third lemma is very useful in computing the numerical radius for matrices (see [7]).

**Lemma 3.** If $A = [a_{ij}] \in M_n(\mathbb{C})$, then

$$\omega(A) \leq \omega(\|a_{ij}\|) = \frac{1}{2} \omega(\|a_{ij} + |a_{ji}|\|).$$

*Our first result is a generalization of the first inequality in (1.2).*

**Theorem 1.** Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ be a $2 \times 2$ operator matrix in $B(H_1 \oplus H_2)$, and let $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$\omega(S) \leq \frac{1}{2} \max \{ \|f^2(|C|) + g^2(|B^*|)\|, \|f^2(|B|) + g^2(|C^*|)\| \}. \quad (2.1)$$

**Proof.** Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2)$, with $\|x\| = 1$. Then we have

\[
\langle \langle Sx, x \rangle \rangle \leq \frac{1}{2} \langle f^2(|S|)x, x \rangle \quad \text{by Lemma 1}
\]

\[
= \frac{1}{2} \langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} x, x \rangle \quad \text{by the arithmetic – geometric mean inequality}
\]

\[
\leq \frac{1}{2} \left( \frac{1}{2} \langle \begin{bmatrix} f^2(|C|) + g^2(|B^*|) \\ 0 \\ 0 & f^2(|B|) + g^2(|C^*|) \end{bmatrix} x, x \rangle \right).
\]

Thus,

$$\omega(S) = \sup \{ \langle \langle Sx, x \rangle \rangle : x \in (H_1 \oplus H_2), \|x\| = 1 \} \leq \frac{1}{2} \max \{ \|f^2(|C|) + g^2(|B^*|)\|, \|f^2(|B|) + g^2(|C^*|)\| \}.$$
as required.

Inequality (2.1) includes several numerical radius inequalities for operator matrices. Samples of inequalities are demonstrated in the following remarks.

**Remark 1.** For \( f(t) = t^x \) and \( g(t) = t^{1-x}, \) \( x \in (0, 1) \), in inequality (2.1), we get the following inequality

\[
\omega(S) \leq \frac{1}{2} \max \left\{ \|C^{2x} + |B^*|^{2(1-x)}\|, \|B^{2x} + |C^*|^{2(1-x)}\| \right\}.
\]

**Remark 2.** In Remark 1, if \( x = \frac{1}{2} \), then we get

\[
\omega(S) \leq \frac{1}{2} \max \{\|C| + |B^*|\|, \|B| + |C^*|\|\}.
\]

**Remark 3.** By letting \( H_1 = H_2 = B = C \) in Remark 2, and by using Lemma 2(b) it is easy to see that the inequality in Remark 2 generalizes the inequality (1.2), i.e.,

\[
\omega(S) = \omega(B) \leq \frac{1}{2} \|B| + |B^*|\|.
\]

In the next theorem, we employ the inequalities in (1.3) to generalize and improve the inequalities in (1.1).

**Theorem 2.** Let \( S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \) be a \( 2 \times 2 \) operator matrix in \( B(H_1 \oplus H_2) \). Then

\[
\frac{1}{2} \max \{\alpha, \beta\} \leq \omega \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{\sqrt{2}} \max \{\alpha, \beta\},
\]

where \( \alpha = \|C|^2 + |B^*|^2\|^\frac{1}{2} \) and \( \beta = \|B|^2 + |C^*|^2\|^\frac{1}{2} \).

In the following results, we establish generalized upper bounds for the numerical radii of a general \( 2 \times 2 \) operator matrices.

**Theorem 3.** Let \( T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) be a \( 2 \times 2 \) operator matrix in \( B(H_1 \oplus H_2) \), and let \( f \) and \( g \) be nonnegative functions on \( [0, \infty) \) which are continuous and satisfying the relation \( f(t)g(t) = t \), for all \( t \in [0, \infty) \). Then

\[
\omega(T) \leq \frac{1}{2} \max \{\|a\|, \|b\|\},
\]

where,

\[
a = f^2(|A|) + g^2(|A^*|) + f^2(|C|) + g^2(|B^*|)
\]

and

\[
b = f^2(|D|) + g^2(|D^*|) + f^2(|B|) + g^2(|C^*|).
\]
Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2)$, with $\|x\| = 1$. Then we have

\[
\begin{align*}
&\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} x, x \right\rangle \leq \left\langle f^2 \left( \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2 \left( \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}^* \right) x, x \right\rangle^{\frac{1}{2}} \\
&+ \left\langle f^2 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2 \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}^* \right) x, x \right\rangle^{\frac{1}{2}} \text{(by Lemma 1)} \\
&= \left\langle f^2 \left( \begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2 \left( \begin{bmatrix} |A^*| & 0 \\ 0 & |D^*| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \\
&+ \left\langle f^2 \left( \begin{bmatrix} 0 & |B| \\ 0 & |C| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2 \left( \begin{bmatrix} 0 & |B^*| \\ 0 & |C^*| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2} \left( \left\langle f^2 \left( \begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix} \right) x, x \right\rangle + \left\langle f^2 \left( \begin{bmatrix} 0 & |B^*| \\ 0 & |C^*| \end{bmatrix} \right) x, x \right\rangle \right)
\end{align*}
\]

(by the arithmetic-geometric mean inequality)

\[
= \frac{1}{2} \left\langle \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x, x \right\rangle.
\]

Thus,

\[
\omega(T) = \sup \{ |\langle Tx, x \rangle| : x \in (H_1 \oplus H_2), \|x\| = 1 \} \leq \frac{1}{2} \max \{ \|a\|, \|b\| \},
\]

as required.

Remark 4. If $f(t) = t^x$ and $g(t) = t^{1-x}$, $\alpha \in (0, 1)$, in Theorem 3, then we get the following inequality

\[
\omega \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \frac{1}{2} \max \left\{ \left\| A^{2x} + |A^*|^{2(1-x)} + |C|^{2x} + |B^*|^{2(1-x)} \right\|, \right. \\
\left. \left\| D^{2x} + |D^*|^{2(1-x)} + |B|^{2x} + |C^*|^{2(1-x)} \right\| \right\}.
\]

Remark 5. From Remark 4 with $\alpha = \frac{1}{2}$ and Lemma 2(b), If $A = B = C = D$, then we get the following inequality

\[
\omega \left( \begin{bmatrix} A & A \\ A & A \end{bmatrix} \right) = 2\omega(A) \leq \frac{1}{2} \|2|A| + 2|A^*|| = \| |A| + |A^*| \|,
\]
and so,\[
\omega (A) \leq \frac{1}{2} \|A\| + \|A^*\|.
\]

**Remark 6.** From Remark 4 with \(\alpha = \frac{1}{2}\) and the first inequality in (1.1), if \(A = C = D = 0\), then we get the following equality
\[
\frac{1}{2} \|B\| = \frac{1}{2} \left\| \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right\| \leq \omega \left( \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) \leq \frac{1}{2} \max \{ \|B^*\|, \|B\| \} = \frac{1}{2} \|B\|.
\]

Hence,
\[
\omega \left( \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) = \omega \left( \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} \right) = \frac{1}{2} \|B\|.
\]

In the following theorem, we present an improvement of the inequality (1.6) when \(n = 2\).

**Theorem 4.** Let \(T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\) be a \(2 \times 2\) operator matrix in \(B(H_1 \oplus H_2)\), and let \(f\) and \(g\) be nonnegative functions on \([0, \infty)\) which are continuous and satisfying the relation \(f(t) g(t) = t\) for all \(t \in [0, \infty)\). Then
\[
\omega (T) \leq \omega \left( \begin{bmatrix} \omega (A) & f (|C|) \|g(|C^*|)\| \\ \|f(|B|)\| \|g(|B^*|)\| & \omega (D) \end{bmatrix} \right).
\]

**Proof.** Let \(x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2), \) with \(\|x\| = 1\). Then we have
\[
\|Tx, x\| = |\langle Ax_1, x_1 \rangle + \langle Bx_2, x_1 \rangle + \langle Cx_1, x_2 \rangle + \langle Dx_2, x_2 \rangle|
\leq |\langle Ax_1, x_1 \rangle| + |\langle Bx_2, x_1 \rangle| + |\langle Cx_1, x_2 \rangle| + |\langle Dx_2, x_2 \rangle|
\leq \omega (A) \|x_1\|^2 + \|f (|B|) x_2\| \|g (|B^*|) x_1\|
+ \|f (|C|) \|g (|C^*|) x_2\| + \omega (D) \|x_2\|^2
\]
(by definition of \(\omega (.)\) and Lemma 1.1)
\[
\leq \omega (A) \|x_1\|^2 + \|f (|B|)\| \|g (|B^*|)\| \|x_1\| \|x_2\|
+ \|f (|C|)\| \|g (|C^*|)\| \|x_1\| \|x_2\| + \omega (D) \|x_2\|^2
\]
\[
= \left( \begin{bmatrix} \omega (A) & f (|B|) \|g(|B^*|)\| \\ \|f(|C|)\| \|g(|C^*|)\| & \omega (D) \end{bmatrix} \right) \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix}.
\]

Now, the result follows by taking the supremum over all unit vectors in \((H_1 \oplus H_2)\).

Here, a weaker version of Theorem 4 has been also given in [2] when \(n = 2\).
Remark 7. From Theorem 4, if \( f(t) = \sqrt{t} = g(t) \), then we get the following inequality

\[
\omega \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \omega \left( \begin{bmatrix} \omega(A) & \frac{1}{2} \left\| B \right\| \left\| B^* \right\| \\ \left\| C \right\| \left\| C^* \right\| & \omega(D) \end{bmatrix} \right)
\]

\[
\leq \omega \left( \begin{bmatrix} \omega(A) & \frac{1}{2} \left\| B \right\| \\ \left\| C \right\| & \omega(D) \end{bmatrix} \right)
\]

\[
\leq \omega \left( \begin{bmatrix} \frac{1}{2} \left\| A \right\| & \frac{1}{2} \left\| B \right\| \\ \left\| C \right\| & \frac{1}{2} \left\| D \right\| \end{bmatrix} \right).
\]

Now, from Remark 7 and Lemma 3 we get the inequality

\[
\omega \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \frac{1}{2} \left( \omega(A) + \omega(D) + \sqrt{(\omega(A) - \omega(D))^2 + (\left\| B \right\| + \left\| C \right\|)^2} \right),
\]

which is a generalized for the second inequality in (1.1) when we take \( A = B = C = D \) and use Lemma 2(a). Also this inequality can be employed to give new bounds for the zeros of polynomials (see, e.g., [2, 10], and references therein).

Based on Lemma 2(a), the inequality (2.1) and the property \( \omega(X + Y) \leq \omega(X) + \omega(Y) \), we can prove the following corollary.

Corollary 1. Let \( T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) be a \( 2 \times 2 \) operator matrix in \( B(H_1 \oplus H_2) \), and let \( f \) and \( g \) be nonnegative functions on \([0, \infty)\) which are continuous and satisfying the relation \( f(t)g(t) = t \) for all \( t \in [0, \infty) \). Then

\[
\omega(T) \leq \max \{ \omega(A), \omega(D) \}
\]

\[
+ \frac{1}{2} \max \{ \left\| f^2(|C|) + g^2(|B^*|) \right\|, \left\| f^2(|B|) + g^2(|C^*|) \right\| \}.
\]

Remark 8. From Corollary 1 and Lemma 2(b), if \( f(t) = \sqrt{t} = g(t) \) and if \( A = B = C = D \), then we get the following inequality

\[
\omega \left( \begin{bmatrix} A & A \\ A & A \end{bmatrix} \right) = 2\omega(A) \leq \omega(A) + \frac{1}{2} \left\| A \right\| + \left\| A^* \right\|,
\]

and so,

\[
\omega(A) \leq \frac{1}{2} \left\| A \right\| + \left\| A^* \right\|.
\]

Remark 9. From Remark 4, Lemma 2(b), \( \alpha = \frac{1}{2} \) and if \( A = D, B = C \), then we get the following

\[
\max \{ \omega(A - B), \omega(A + B) \} \leq \frac{1}{2} \left\| A \right\| + \left\| A^* \right\| + \left\| B \right\| + \left\| B^* \right\|.
\]
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References


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