A NOTE ON S-ACTS AND BOUNDED LINEAR OPERATORS

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Abstract. In this work, the properties of the certain operator have been studied by looking at the associated S-act of this operator, and conversely. Some operators, for example such operator, one to one, onto operators have been looked. On the other hand, basic mathematical interpretation understanding of S-acts, such as faithful, finitely generated, singular, separated, torsion free and noetherian acts. We have found out the properties may be associated with S-act which has any of these properties. Let V be a inner product space over a field $F$, $T$ be a bounded operator on $V$, and let $S = \{e^x + y \mid x, y \text{ are independent variables in } R\}$ be the semigroup. Define $\theta : S \times V \to V$ by $\theta(e^x + y, v) = e^{T + T^*}(v)$. This function makes $V$ a left $S$-act, denote by $V_{T+T^*}$ and we call it the associated $S$-act of $T + T^*$.

Keywords: Bounded linear operator, finite dimensional Banach space, $S$-act, faithful $S$-act, Noetherian $S$-act.

1. Introduction

A non-empty set $S$ with a binary operation $S \times S \to S$, $(s,s') \mapsto s.s'$, is called a groupoid. The operation of a groupoid is often called multiplication. Instead of $s.s'$ we usually write $ss'$. The multiplication on a groupoid $S$ is called associative if $a(bc) = (ab)c$ for all $a, b, c \in S$. A groupoid with associative multiplication or for short an associative groupoid is called a semi-group. A semigroup $S$ with 1 is called monoid (see [1]). Let $S$ be a monoid and $A$ a non empty set. If we have a mapping $\mu : A \times S \to A, (a, s) \mapsto as := \mu(a, s)$ Such that $a(st) = (as)t$, and $a.1 = a$, for $a \in A, s, t, \in S$. We call $A$ a right $S$-act or a right act over $S$ and write $A_S$, we can define a left $S$-act and write $S_A$. In [2], The module of an operator was study, let $V$ be a vector space over a field $F$. Let $T$ be a linear operator acting on the elements of $V$ on the left. Let $R = F[x]$ be the ring of polynomials in $x$ with coefficients in $F$. Define $\varphi : R \times V \to V$ by $\varphi(p,v) = p.v = p(T)v$. That $\varphi$ makes $V$ a left $R$-module denoted $V_T$, and calls the associated $R$- module. Let $H$ be a Hilbert space over a field $K$ ($K$ may be

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real or complex), let $T$ be a bounded linear operator on $H$. Define exponential operator $e^T$ as follows: $e^T = \sum_{n=0}^{\infty} T^n/n!$. Note the definition of exponential operator is well defined, this means the sum in the definition of the exponential operator exists [3]. Let $V$ be a Banach space over a field $F,T$ be a bounded operator on $V$ and $S = \{e^x : x \in R\}$ be the semi-group. Define $\mu : S \times V \to V$ by $\mu(e^x,v) = e^T(v)$. This function makes $V$ a left $S$-act, denoted by $V_T$. We call it the associated $S$-act of $T$ [4]. In this paper the associated $S$-act of $T+T^*$ have been study, let $S = \{e^{x+y}|x, y \text{ are independent variables in } R\}$ be the semi-group. Define $\theta : S \times V \to V$ by $\theta(e^{x+y},v) = e^{T+T^*}(v)$. This function makes $V$ a left $S$-act, denote by $V_{T+T^*}$, the form of every element in $V_{T+T^*}$ is $e^{T+T^*}(v)$, and if two operators $T$ and $S$ are similar then $V_{(T+T^*)}$ is isomorphic to $V_{S+S^*}$. The relation between a bounded linear operator $T$ and faithful $S$-act have been study. The relation between finite dimensional Banach space $V$ and Noetherian $S$-act, discussed have been study by if $V$ is finite dimension then $V_{S+S^*}$ act is Noetherian.

2. Main results

Definition 2.1. Let $S = \{e^{x+y}|x, y \text{ are independent variables in } R\}$ be the semi-group. Define $\theta : S \times V \to V$ by $\theta(e^{x+y},v) = e^{T+T^*}(v)$. This function makes $V$ a left $S$-act, denote by $V_{T+T^*}$.

We put $p_n(T+T^*) = \sum_{i=0}^{n}(T+T^*)^i/(i!) = I + (T+T^*+ (T+T^*)^2/2! + (T+T^*)^3/3! + \cdots + (T+T^*)^n/n!$.

Proposition 2.2. If $K = \{V_j, j \in \Lambda\}$ is a basis for $V$, then each element of $V_{T+T^*}$ can be written in the form

\[ \lim_{n \to \infty} \sum_{i=0}^{n}(T+T^*)^i/(i!) \sum_{j \in \Lambda} a_jv_j = \lim_{n \to \infty} p_n(T+T^*).V \]

The symbol $\sum_{j \in \Lambda}$ means the sum is taken over a finite subset of $\Lambda$.

Proof. We define $\mu : S \times V \to V$, by $\mu(e^{x+y},v) = e^{T+T^*}(v) = \sum_{i=0}^{\infty}(T+T^*)^i/(i!)$. Let $w \in V_{T+T^*}$, then $w = \sum_{i=0}^{\infty}(T+T^*)^i/(i!)(v) = (I+(T+T^*+ (T+T^*)^2/2! + (T+T^*)^3/3! + \cdots + (T+T^*)^n/n!)(v)$. Since $K = \{v_j, j \in \Lambda\}$ is a basis for $V$ then $w = I + (T+T^*+ (T+T^*)^2/2! + (T+T^*)^3/3! + \cdots + (\sum_{j \in \Lambda} a_jv_j) = I(\sum_{j \in \Lambda} a_jv_j) + (T+T^*)(\sum_{j \in \Lambda} (a_jv_j) + (T+T^*)(\sum_{j \in \Lambda} (a_jv_j) + (T+T^*)(\sum_{j \in \Lambda} (a_jv_j) + \cdots But the series $\sum_{i=0}^{\infty}(T+T^*)^i/(i!)$ converges in $B(H)$, Where $T \in B(H)$. Therefore $\sum_{i=0}^{\infty}(T+T^*)^i/(i!)$ is converge in $B(H)$Then we get $w = \lim_{n \to \infty} \sum_{i=0}^{n}(T+T^*)^i/(i!) \sum_{j \in \Lambda} (a_jv_j) = \lim_{n \to \infty} p_n(T+T^*).V$.

Examples 2.3. 1. Let $\{v_j, j \in \Lambda\}$ be a basis for a Banach space $V$. Let $O$ be the zero operator on $V$, recall $O^o = I$. Let $w \in V_{0+0^o}$, then by proposition 2.2 $w = e^{T+T^*}(v) = e^{0+0^o}(v)$ then $w = I(\sum_{j \in \Lambda} a_jv_j) = \sum_{j \in \Lambda} a_jv_j$, since $e^0 = I$ [2], therefore, $e^{0+0^o} = I$. 
2. Let $I : V \rightarrow V$ be the identity operator on $V$. \{v_j : j \in \Lambda\} be a basis for $V$, and let $w \in V_{+I}T^*$, then by proposition 2.2 $w = e^{T+T^*}(v) = e^{I+I^*}((\sum_{j \in \Lambda} a_j v_j))$. Since $I^* = I$ therefore $w = e^{I+I}((\sum_{j \in \Lambda} a_j v_j)) = (I+I)\sum_{n=0}^{\infty} 1/n!((\sum_{j \in \Lambda} a_j v_j)) = 2 \sum_{n=0}^{\infty} 1/n!((\sum_{j \in \Lambda} a_j v_j))$, put $a_n = \sum_{n=0}^{\infty} 1/n!$ then $w = \lim_{n \rightarrow \infty} 2 \sum_{n=0}^{\infty} a_n((\sum_{j \in \Lambda} a_j v_j))$.

3. Let \{v_j : j \in \Lambda\} be a basis for a Banach space $V$, and $T$ be the nilpotent operator on $V$ (i.e. $T^n = 0$ and $T^{n-1} \neq 0$ for some positive integer $n$) then, by proposition 2.2 $w = e^{T+T^*}(v) = (I+(T + T^*)2/2! + ((T + T^*)3/3! + \cdots + ((T + T^*)n/n!))(v) = ([I + (T + T^*)2/2! + ((T + T^*)3/3! + \cdots + ((T + T^*)n/(n-1)])/n!)((\sum_{j \in \Lambda} a_j v_j)) + (T + T^*)n/n!((\sum_{j \in \Lambda} a_j v_j)) = C_n n^2 + \cdots + (n(n-1))T^n/n!)((\sum_{j \in \Lambda} a_j v_j) + a_n((\sum_{j \in \Lambda} a_j v_j)) = \lim_{n \rightarrow \infty} p_{n-1}(T + T^*)(\sum_{j \in \Lambda} a_j v_j) + a_n((\sum_{j \in \Lambda} a_j v_j) = \lim_{n \rightarrow \infty} p_{n-1}(T + T^*) + a_n)v$, where $a_n = (\lim_{n \rightarrow \infty} T^{n+1} + (n+1)T^n/n!)$.

Proposition 2.4. Let $T$ and $S$ be two bounded operators on $V$. If $S$ and $T$ are similar. Then $V_{S+S^*}$ and $V_{T+T^*}$ are isomorphic.

Proof. Assume that $T$ and $S$ are similar, and i.e. there exist an invertible operator $h$ on $V$, such that $hTh^{-1} = S$ [6], then $(hSh^{-1} = T).h$, this gives $h(S+S^*) = (T+T^*)h$. Since $hS = Th$, then $h(S+S^*)n = h(S+S^*)(S+S^*)n-1 = (T+T^*)h(S+S^*)(S+S^*)n-2 = (T+T^*)(T+T^*)h(S+S^*)(S+S^*)n-3 = \ldots = (T+T^*)n h$. therefore

\[
he^{S+S^*} = e^{T+T^*}h.
\]

Define $h' : V_{S+S^*} \rightarrow V_{T+T^*}$ by

\[
(e^{S+S^*}(v))h' = e^{T+T^*}(h(v)).
\]

To prove $h'$ is isomorphism we must prove:

1. $h'$ is well defined

\[
\begin{align*}
e^{S+S^*}(v_1) = e^{S+S^*}(v_2) \text{ then } h(e^{S+S^*}(v_1)) = h(e^{S+S^*}(v_2)) (\text{ Since } h \text{ is well defined). Then by equation 1 we get } e^{T+T^*}(h(v_1)) = h(e^{S+S^*}(v_1)) = h(e^{S+S^*}(v_2)) = e^{T+T^*}(h(v_2)).
\end{align*}
\]

2. $h'$ is linear

\[
\begin{align*}
&\forall u, v \in V_{S+S^*}, \text{ then } h'(u + v) = h'(u) + h'(v) = e^{T+T^*}(h(\text{u})) + e^{T+T^*}(h(\text{v})) = e^{T+T^*}(h(\text{u}) + h(\text{v})) = e^{T+T^*}(h(\text{u} + \text{v})).
\end{align*}
\]

3. $h'$ is injective

\[
\begin{align*}
&\forall u \in V_{S+S^*}, \text{ then } h'(u) = e^{T+T^*}(h(u)) \text{ and } h'(u) = 0 \Rightarrow e^{T+T^*}(h(u)) = 0 \Rightarrow h(u) = 0 \Rightarrow u = 0.
\end{align*}
\]

4. $h'$ is surjective

\[
\begin{align*}
&\forall v \in V_{T+T^*}, \text{ then } h'(v) = e^{T+T^*}(h(v)) \text{ and } h'(v) = v \Rightarrow e^{T+T^*}(h(v)) = v \Rightarrow h(v) = v.
\end{align*}
\]

Therefore $h'$ is an isomorphism.
Then, by equations 2, 3 we get \((e^{S+S^*}(v_1))h' = (e^{S+S^*}(v_2))h'\). Thus \(h'\) is well defined.

2. \(h'\) is one to one.

Let \((e^{S+S^*}(v_1))h' = (e^{S+S^*}(v_2))h'\) Then by equation 2, we get \(e^{T+T^*}(h(v_1)) = e^{T+T^*}(h(v_2))\) Then by equation 1 we get \(h(e^{S+S^*}(v_1)) = h(e^{S+S^*}(v_2))\) But \(h\) is invertible then \(h^{-1}(1)h(e^{S+S^*}(v_1)) = h^{-1}(1)h(e^{S+S^*}(v_2))\). This give \((e^{S+S^*}(v_1)) = (e^{S+S^*}(v_2))\), therefore \(h'\) is one to one.

3. \(h'\) is onto.

Let \(e^T(v) \in V_T\) since \(v \in V\) then \(h^{-1}(1) \in V\) and \(e^{S+S^*}(h^{-1}(v)) \in e^{S+S^*}\), then by equation 2 we get \((e^{S+S^*}(h^{-1}(v)))h' = e^{T+T^*}(h(h^{-1}(v))) = e^{T+T^*}(v)\), then \(h'\) is onto. Note that \((e^{S+S^*}(v))h' = e^{T+T^*}(h(v)) = h(e^{S+S^*}(v))\), thus \((e^{S+S^*}(v))h' = h(e^{S+S^*}(v))\) But \(h\) is an operator (linear) on \(V\), thus \(h\) is \(S\)-homomorphism, this give \(h'\) is \(S\)-homomorphism. Then \(h'\) is an \(S\)-isomorphism. \(\square\)

Recall that the left \(S\)-act \(A_s\) is faithful if for \(s, t \in S\) the equality \(sa = ta\) for all \(a \in A_s\), implies \(s = t\). The relation between faithful \(S\)-act and bounded linear operator \(T\) have been explained in the following proposition.

**Proposition 2.5.** For any bounded linear operator \(T\) then \(V_{T+T^*}\) is a faithful \(S\)-act.

**Proof.** We want to show that \(V_{T+T^*}\) is a faithful \(S\)-act, for any bounded operator \(T\). Let \(e^{x_1+y_1}e^{T+T^*}(v) = e^{x_2+y_2}e^{T+T^*}(v)\). Since \(e^T\) is operator then \(e^T\) is linear transformation, this give \(e^{x_1+y_1}e^{T+T^*}(v) = e^{x_2+y_2}e^{T+T^*}(v)\) = \(e^{T+T^*}(e^{x_1+y_1}v) = e^{T+T^*}(e^{x_2+y_2}v)\). Since \(e^T\) is one to one, therefore \(e^{T+T^*}\) is one to one. Hence \(e^{x_1+y_1}v = e^{x_2+y_2}v\), then \(e^{x_1+y_1}v - e^{x_2+y_2}v = 0\). Then \((e^{x_1+y_1} - e^{x_2+y_2})v = 0\), thus \(e^{x_1+y_1} = e^{x_2+y_2}\). Therefore \(V_{T+T^*}\) is faithful \(S\)-act. \(\square\)

**Remark 2.6.** If \(V\) is a finite dimensional Banach space, then \(V_{T+T^*}\) is finitely generated \(S\)-act.

In [7], show that a subspace \(W\) of \(V\) is said to be an invariant subspace of \(V\) under \(T\) if \(Tw \subseteq W\) for all \(w \in W\).

The following proposition shows under what condition the vector space \(V\) is finite dimension.

**Proposition 2.7.** If \(T\) is one to one and onto and \(V_{T+T^*}\) is finitely generated, then \(V\) is finite dimensional.

**Proof.** Assume that \(V\) is not finite dimensional. Let \(K(T) = \{w \in V | (T + T^*)w = 0\}\). It is clear that \(K\) is as invariant subspace of \(V\) (since \(K \subseteq V\) and \(\forall w \in K,(T + T^*)(w) = 0\) but \(0 \in K\) then \((T + T^*)(K) \subseteq K\) and by the first isomorphic theorem of \(S\)-act, then \((T + T^*)V \cong V/K\) [1], since \(T\) is one to one then \(T^*\) is onto and \(T\) is onto then \((T + T^*)V = V\), therefore \(V \cong V/K\). By assuming that \(V\) is not finite dimensional then either \(K\) is infinite dimensional
For any bounded linear operator \(T\) and \(T^*\), if \(T_{T+T^*}\) is singular \(S\)-act then \(V\) is generated by one element.

**Proof.** Since \(V_{T+T^*}\) is singular \(S\)-act, then

\[
\psi_V=\{(e^{T+T^*}(v_1), e^{T+T^*}(v_2)) \in V_{T+T^*} \times V_{T+T^*} | e^{x+y}e^{T+T^*}(v_1) = e^{x+y}e^{T+T^*}(v_2) \text{ for some } e^x \in H \text{ for some reductive subset } H \text{ of } S\}
\]

then \(e^{x+y}e^{T+T^*}(v_1) = e^{x+y}e^{T+T^*}(v_2) \text{ .... (2-1)}\). This gives \(e^{T+T^*}(e^{x+y}v_1) = e^{T+T^*}(e^{x+y}v_2) \text{ since } e^{T+T^*} \text{ is one to one, therefore } e^{x+y}v_1 = e^{x+y}v_2 \text{ then } e^yv_1 + (-1)e^{x+y}v_2 = 0 \text{ .... (2-2)}\) but \(H \text{ is reductive subset of } S\), then by (2-1) find \(e^{T+T^*}(v_1) = e^{T+T^*}(v_2) \text{ thus } v_1 = v_2\). We replace \(v_1 = v_2\) on (2-2) then \(e^{x+y}v_1 + (-1)e^{x+y}v_1 = 0\), therefore \(V\) is generated by one element this gives \(V\) is a finite dimensional.

Recall that an \(S\)-act is separated if for each \(a \neq b\) in \(A\) there exists \(s \neq e\) such that \(sa \neq sb\) [1]. In the following proposition, we explain the relationship between a bounded linear operator \(T\) and separated \(S\)-act.

**Proposition 2.9.** For any bounded linear operator \(T\) then \(V_{T+T^*}\) is separated \(S\)-act.

**Proof.** Let \(a \neq b\) in \(V_{T+T^*}\) to prove \(V_{T+T^*}\) is separated we have to show that there exist \(s, e \in S\), \(s \neq e\) where \(e\) is the identity element such that \(sa \neq sb\). Assume \(sa = sb\), \(e \neq s \in S\), such that \(s = e^{x+y}\), \(e\) is the identity element, \(a, b \in V_{T+T^*}\), this gives \(e^{x+y}e^{T+T^*}(v_1) = e^{x+y}e^{T+T^*}(v_2), v_1, v_2 \in V_{T+T^*}\).
If \( e^{T+T^*} \) is operator then \( e^{T+T^*} \) is linear transformation, this gives \( e^{x+y}.e^{T+T^*}(v_1) = e^{x+y}.e^{T+T^*}(v_2) \), thus \( e^{T+T^*}(e^{x+y}.v_1) = e^{T+T^*}(e^{x+y}.v_2) \), but \( e^{T+T^*} \) is one to one, then \( e^{x+y}.v_1 = e^{x+y}.v_2 \), hence \( (v_1 - v_2)e^{x+y} = 0 \), since \( e^{x+y} \neq 0 \). Then \( v_1 = v_2 \), this gives either \( e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2) \) or \( e^{T+T^*}(v_1) = e^{T+T^*}(v_2) \), but if \( e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2) \), this give \( v_1 = v_2 \)) this contradiction with \( v_1 = v_2 \), then \( e^{T+T^*}(v_1) = e^{T+T^*}(v_2) \), means \( a = b \) which a contradiction, then \( V_{T+T^*} \) is separated S-act.

The converse of the proposition 2.9 have been study in the following proposition.

**Proposition 2.10.** If \( V_{T+T^*} \) is separated S-act then \( T \) is one to one.

**Proof.** Assume that \( V_{T+T^*} \) is separated, we want to prove \( T \) is one to one. let \( v_1 \neq v_2 \), we must prove \( T(v_1) \neq T(v_2) \). since \( v_1 \neq v_2 \) then either \( e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2) \) or \( e^{T+T^*}(v_1) = e^{T+T^*}(v_2) \). If \( e^{T+T^*}(v_1) = e^{T+T^*}(v_2) \), this contradiction with \( v_1 = v_2 \), hence \( e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2) \), but \( V_{T+T^*} \) is separated S-act then \( 3e \neq s \), \( s = e^{x+y} \in S \) such that \( e^{x+y}.e^{T+T^*}(v_1) \neq e^{x+y}.e^{T+T^*}(v_2) \), since \( e^{T+T^*} \) is an operator, then \( e^{T+T^*} \) is linear transformation, and hence \( e^{T+T^*}(e^{x+y}.v_1) \neq e^{T+T^*}(e^{x+y}.v_2) \), this means \( (I + (T + T^*)) + (T + T^*)^2/2! + (T + T^*)^3/3! + \cdots (e^{x+y}.v_1) \neq ((I + (T + T^*)) + (T + T^*)^2/2! + (T + T^*)^3/3! + \cdots (e^{x+y}.v_2) \) we get, \( e^{x+y}.v_1 + (T + T^*)(e^{x+y}.v_1) + (T + T^*)^2/2!(e^{x+y}.v_1) + ((T + T^*)^3/3!(e^{x+y}.v_2) + \cdots e^{x+y}.v_1 \neq e^{x+y}.v_2 \), this gives \( v_1 \neq v_2 \) and \( T + T^*) \neq (e^{x+y}.v_1) \neq (e^{x+y}.v_2) \), but \( T + T^* \) is an operator, this gives \( e^{x+y}.(T + T^*)(v_1) \neq e^{x+y}.(T + T^*)(v_2) \), since \( e^{x+y} \neq 0 \), then \( (T + T^*) \neq 0 \), and since \( (T + T^*)^2/2!(e^{x+y}.v_1) \neq (T + T^*)^2/2!(e^{x+y}.v_2) \), therefore \( (T + T^*)^2/2!(T + T^*)(e^{x+y}.v_1) \neq (T + T^*)^2/2!(T + T^*)(e^{x+y}.v_2) \) this gives \( (T + T^*)(e^{x+y}.v_1) \neq (T + T^*)(e^{x+y}.v_2) \) by using the same way, we get \( (T + T^*)(v_1) \neq (T + T^*)(v_2) \). Then we proof \( T + T^* \) is one to one, thus \( T \) is one to one.

Recall that An act \( A_S \) is torsion free if for any \( x, y \in A_S \), and for any right cancellable element \( c \in S \), the equality \( xc = yc \) this implies \( x = y \) (see [1]). In the following proposition the relation between a bounded linear operator \( T \) and torsion free S-act have been explain.

**Proposition 2.11.** For any bounded linear operator \( T \) then \( V_{T+T^*} \) is torsion free S-act.

**Proof.** Assume \( e^{x+y}.e^{T+T^*}(v_1) = e^{x+y}.e^{T+T^*}(v_2) \), \( \forall e^{x+y} \) is cancellable element in \( S \), this give \( e^{T+T^*}(e^{x+y}.v_1) = e^{T+T^*}(e^{x+y}.v_2) \), since \( e^{T+T^*} \) is one to one, therefore \( e^{x+y}.v_1 = e^{x+y}.v_2 \), then \( v_1 = v_2 \), thus either \( e^{T+T^*}(v_1) = e^{T+T^*}(v_2) \) or \( e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2) \), if \( e^{T+T^*}(v_1) = e^{T+T^*}(v_2) \), we get contradiction with \( v_1 = v_2 \), then \( e^{T+T^*}(v_1) = e^{T+T^*}(v_2) \), this give \( V_{T+T^*} \) is torsion free.
Recall that a monid $S$ is right Noetherian if and only if it satisfies the ascending chain condition for right ideals, this mean for every ascending chain $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_n \subseteq K_{n+1} \subseteq \cdots$, of its right subacts, there exists $n \in \mathbb{N}$ such that $K_n = K_{n+1} = \cdots$

**Theorem 2.12** ([8]). If $S$ is Noetherian and $A$ is finitely generated $S$-act then $A$ is Noetherain $S$-act.

**Proposition 2.13.** Let $V$ be a finite dimensional normed space and $T$ is similar to any operator $J$ from $R$ to $R$ then is Noetherian $S$-act if and only if $S$ is Noetherian.

**Proof.** Since $V$ is finite dimensional then it is finitely generated $S$-act by remark 2.3, therefore is Noetherian $S$-act, by theorem 2.12. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_n \subseteq K_{n+1}$ be any ascending sequence ideals of $S$, then it is a sequence of subacts of $S_S$ denoted by $S_J$, where $J$ any operator from $R$ to $R$, since $T$ is similar to $J$, then by proposition 2.5, $V_{T+J^*}$ is isomorphic $S_{J+J^*}$, thus $S_{J+J^*}$ is Noetherian $S$-act, therefore this sequence is finite, then $S$ is Noetherian. 

**References**


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