EXTENDED d-HOMOLOGY

S.N. Hosseini
Mathematics Department
Shahid Bahonar University of Kerman
Kerman
Iran
nhoseini@uk.ac.ir

M.Z. Kazemi Baneh*
Mathematics Department
University of Kurdistan
Sanandaj
Iran
zaherkazemi@uok.ac.ir

Abstract. In this article, in more general categories than the abelian categories, we define a homology functor with respest to a kernel transformation d, called the extended d-homology. We then compare the standard homology and the extended d-homology functors.
Keywords: (pre) abelian category, standard homology, extended d-homology, category of R-modules.

1. Introduction and preliminaries

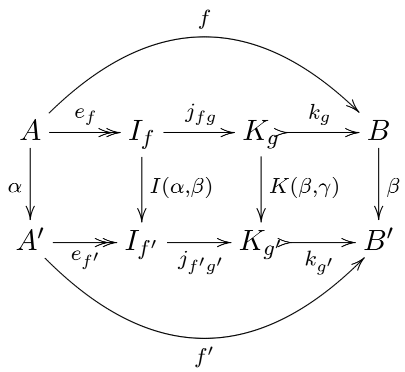
The definition of the standard homology functor has been extended from the category of R-modules to abelian categories in [7]. In [4] we have defined the homology with respect to a kernel transformation d, also called the d-homology, in more general categories. In this section we give the definition of d-homology and some of the results obtained in [4]. In Section 2, we define a second homology functor called the extended homology with respect to a kernel transformation d, or the extended d-homology. We furnish some illustrative examples and also prove the extended d-homology is the cokernel of a certain map. In Section 3, we compare the standard homology as given in [7] and the extended d-homology, by giving a natural transformation from the standard homology functor to the extended d-homology functor. We then consider conditions under which this natural transformation is a natural isomorphism. We also show, in abelian categories, the standard homology is the extended d-homology with respect to a particular kernel transformation d. Some other results are also given at the end of this section.

To this end, for a pointed category $\mathcal{C}$, following the notation of [4], we recall:

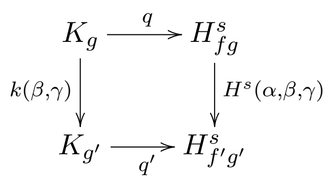
$*$. Corresponding author

- For $f : A \longrightarrow B$, the maps $K_f \xrightarrow{k_f} A$, $B \xrightarrow{c_f} C_f$ and $P_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A$ are respectively the kernel, the cokernel and the kernel pair of f.
- The image I_f of f is the coequalizer of the kernel pair of f. f can be factorized to $f = m_f e_f$ that e_f is the coequalizer of the kernel pair of f.
- For a pair of maps $A \underset{g}{\overset{f}{\rightrightarrows}} B$, the maps $Equ(f,g) \xrightarrow{equ(f,g)} A$ and $B \xrightarrow{coe(f,g)} Coe(f,g)$ are respectively the equalizer and the coequalizer of (f,g).

  For a pointed category $\mathcal{C}$ with pullbacks and pushouts, let $\bar{\mathcal{C}}$ be the arrow category and $\hat{\mathcal{C}}$ be the pair-chain category of $\mathcal{C}$. Let $K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the kernel functor and $I : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the image functor.
- The functor $j : \hat{\mathcal{C}} \longrightarrow \bar{\mathcal{C}}$ takes the object $(f,g) \in \hat{\mathcal{C}}$ to j_{fg} and the morphism (α, β, γ) to $(I(\alpha, \beta), K(\beta, \gamma))$ and we have the following commutative diagram



- The standard homology or s-homology functor H^s, takes $(f,g) \in \hat{\mathcal{C}}$ to $Coker(j_{fg})$, and for a pair chain map $(\alpha, \beta, \gamma) : (f,g) \longrightarrow (f', g')$, we have the following commutative diagram



  where $q = coker(j_{fg})$ and $q' = coker(j_{f'g'})$.

2. Extended homology with respect to a kernel transformation

In this section, unless stated otherwise, $\mathcal{C}$ is a pointed category with pullbacks and pushouts.

Definition 2.1. *Let $m : A \longrightarrow C$ and $j : B \longrightarrow C$ be two maps in $\mathcal{C}$. Define $A +_C B$ also denoted by $A + B$ by the pushout:*

$$\begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & A \\ {\scriptstyle\gamma}\downarrow & po & \downarrow{\scriptstyle h} \\ B & \xrightarrow[i]{} & A +_C B \end{array}$$

where $B \xleftarrow{\gamma} P_{jm} \xrightarrow{\alpha} A$ *is the pullback of (j, m).*

With $S : \mathcal{C} \longrightarrow \mathcal{C}$ the squaring functor, taking $a \xrightarrow{f} b$ to $a^2 \xrightarrow{f^2} b^2$, a kernel transformation is a natural transformation $d : S \circ K \longrightarrow K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$, such that for all $(f, g) \in \hat{\mathcal{C}}$ and diagonal map Δ we have the maps $m_{d_g\Delta_g} : I_{d_g\Delta_g} \longrightarrow K_g$ (such that $m_{d_g\Delta_g} e_{d_g\Delta_g} = d_g\Delta_g$) and $j_{fg} : I_f \longrightarrow K_g$. The sum $I_{d_g\Delta} + I_f$ is therefore obtained by the following diagrams:

$$\begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g\Delta_g} \\ {\scriptstyle\gamma}\downarrow & \text{pb} & \downarrow{\scriptstyle m_{d_g\Delta_g}} \\ I_f & \xrightarrow[j_{fg}]{} & K_g \end{array} \qquad \text{and} \qquad \begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g\Delta_g} \\ {\scriptstyle\gamma}\downarrow & \text{po} & \downarrow{\scriptstyle h} \\ I_f & \xrightarrow[i]{} & I_{d_g\Delta_g} + I_f \end{array}$$

Since $m_{d_g\Delta_g}\alpha = j_{fg}\gamma$, there is a unique map $\beta : I_{d_g\Delta_g} + I_f \longrightarrow K_g$ making the following triangles commutative

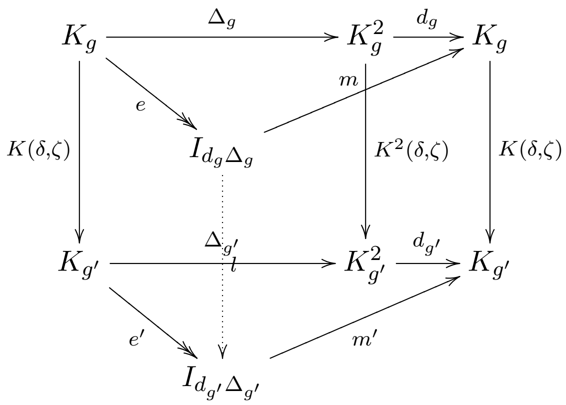
$$\begin{array}{cccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g\Delta_g} & \\ {\scriptstyle\gamma}\downarrow & \text{po} & \downarrow{\scriptstyle h} & \searrow{\scriptstyle m_{d_g\Delta_g}} \\ I_f & \xrightarrow[i]{} & I_{d_g\Delta_g} + I_f & \\ & \searrow{\scriptstyle j_{fg}} & & \searrow{\scriptstyle \beta} \\ & & & K_g \end{array}$$

Letting $m = m_{d_g\Delta_g}$, $e = e_{d_g\Delta_g}$ and $j = j_{fg}$ for simplicity and setting $\bar{H}^d_{fg} = C_\beta$, the cokernel of β, we have:

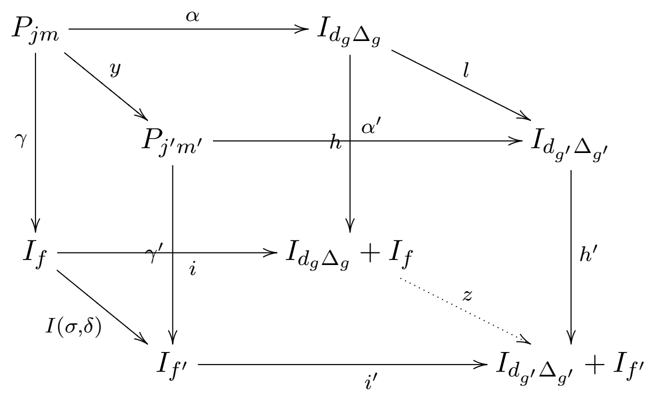
Lemma 2.2. *For each morphism $(\sigma, \delta, \zeta) : (f, g) \longrightarrow (f', g')$ in $\hat{\mathcal{C}}$, there is a unique map $\bar{H}^d(\sigma, \delta, \zeta) : \bar{H}^d_{fg} \longrightarrow \bar{H}^d_{f'g'}$, such that the following diagram commutes:*

$$\begin{array}{ccc} K_g & \xrightarrow{c_\beta} & \bar{H}^d_{fg} \\ {\scriptstyle K(\delta,\zeta)}\downarrow & & \downarrow{\scriptstyle \bar{H}^d(\sigma,\delta,\zeta)} \\ K_{g'} & \xrightarrow[c_{\beta'}]{} & \bar{H}^d_{f'g'} \end{array}$$

Proof. Consider the following diagram in which the back squares commute by naturality of d and Δ



It follows that $(K(\delta,\zeta),K(\delta,\zeta)) : d_g\Delta_g \longrightarrow d_{g'}\Delta_{g'}$ is a map in $\bar{\mathcal{C}}$ and so we get the map $l = I(K(\delta,\zeta),K(\delta,\zeta)) : I_{d_g\Delta_g} \longrightarrow I_{d_{g'}\Delta_{g'}}$ such that $le = e'K(\delta,\zeta)$. Since e is epic, $K(\delta,\zeta)m = m'l$. On the other hand $j'I(\sigma,\delta) = K(\delta,\zeta)j$ and we get $m'l\alpha = K(\delta,\zeta)m\alpha = K(\delta,\zeta)j\gamma = j'I(\sigma,\delta)\gamma$. So there is a unique map $y : P_{jm} \longrightarrow P_{j'm'}$ such that $\gamma'y = I(\sigma,\delta)\gamma$ and $\alpha'y = l\alpha$. So in the diagram:



the front and back squares are pushouts, and the left and top squares are commutative. It follows that there is a unique map z such that: $zi = i'I(\sigma,\delta)$ and $zh = h'l$. We then get

$$\beta'zi = \beta'iI(\sigma,\delta) = j'I(\sigma,\delta) = K(\delta,\zeta)j = K(\delta,\zeta)\beta i$$

and

$$K(\delta,\zeta)\beta h = K(\delta,\zeta)m = m'l\gamma = \beta'h'l = \beta'zh$$

Pushoutness of $I_{d_g\Delta_g} + I_f$ implies $K(\delta,\zeta)\beta = \beta'z$.

Now $c_{\beta'}K(\delta,\zeta)\beta = c_{\beta'}\beta' z = 0$ and so there is a unique $\bar{H}^d(\sigma,\delta,\zeta) : \bar{H}^d{}_{fg} \longrightarrow \bar{H}^d{}_{f'g'}$ making the right square in the following diagram commutative:

$$\begin{array}{ccccc}
I_{d_g\Delta_g} + I_f & \overset{\beta}{\underset{0}{\rightrightarrows}} & K_g & \xrightarrow{c_\beta} & \bar{H}^d{}_{fg} \\
{\scriptstyle z}\downarrow & & \downarrow{\scriptstyle K(\delta,\zeta)} & & \vdots\,{\scriptstyle \bar{H}^d(\sigma,\delta,\zeta)} \\
I_{d_{g'}\Delta_{g'}} + I_{f'} & \overset{\beta'}{\underset{0}{\rightrightarrows}} & K_{g'} & \xrightarrow[c_{\beta'}]{} & \bar{H}^d{}_{f'g'}
\end{array}$$

□

Theorem 2.3. *The mapping $\bar{H}^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ that takes the object $(f,g) \in \hat{\mathcal{C}}$ to $\bar{H}^d_{fg}$ and the morphism (α,β,γ) to $\bar{H}^d(\alpha,\beta,\gamma)$ is a functor.*

The functor $\bar{H}^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ is called the extended d-homology or the extended homology functor with respect to the kernel transformation d.

Example 2.4. Let $\mathcal{C} = Rmod$ and $d = +(r \times s) = rpr_1 + spr_2$ with $r,s \in R$. If $A \xrightarrow{f} B \xrightarrow{g} C$ with $gf = 0$, Then:

$$\bar{H}^d_{fg} = \frac{K_g}{(r+s)K_g + I_f}$$

Example 2.5. As a special case of the above example, let $\mathcal{C} = Abgrp$, for $d = +(r \times s)$ with $r,s \in \mathbb{Z}$, $K_g = \mathbb{Z}$, $I_f = n\mathbb{Z}$, we have:

$$\bar{H} = \frac{\mathbb{Z}}{(r+s)\mathbb{Z} + n\mathbb{Z}} = \frac{\mathbb{Z}}{(r+s,n)\mathbb{Z}} = \mathbb{Z}_{(r+s,n)}$$

where $(r+s,n)$ is the greatest common divisor of $r+s$ and n.

Example 2.6 ([7, 4])**.** Let $\mathcal{C}$ be the category, Sh_R, of short exact sequences of R-modules, $(F,G) \in \hat{\mathcal{C}}$ and $d = +(r \times s)$ $r,s \in R$. Then:

$$\begin{array}{ccccccccccc}
M & \quad & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
{\scriptstyle F}\downarrow & & & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
N & & 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \\
{\scriptstyle G}\downarrow & & & & {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \gamma'} & & \\
P & & 0 & \longrightarrow & A'' & \xrightarrow[f'']{} & B'' & \xrightarrow[g'']{} & C'' & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccccc}
I_F & \quad & 0 & \longrightarrow & \frac{I_f + K_\beta}{K_\beta} & \longrightarrow & \frac{B}{K_\beta} & \longrightarrow & \frac{C}{I_{g\beta}} & \longrightarrow & 0 \\
{\scriptstyle j}\downarrow & & & & \downarrow{\scriptstyle \bar{\alpha}} & & \downarrow & & \downarrow & & \\
K_G & & 0 & \longrightarrow & K_{\alpha'} & \longrightarrow & K_{\beta'} & \longrightarrow & I_{g'\beta'} & \longrightarrow & 0
\end{array}$$

where $I_{g\beta}$ is the image of the restriction of g to K_β and $\bar{\alpha}(f(a)+K_\beta)=\alpha(a)$. We also have:

$$\begin{array}{ccccccccccc}
I_{d_G\Delta_G} & \qquad & 0 & \longrightarrow & \frac{I_{f'\alpha'}+K_2}{K_2} & \longrightarrow & \frac{K_{\beta'}}{K_2} & \longrightarrow & \frac{I_{g'\beta'}}{K_3} & \longrightarrow & 0 \\
{\scriptstyle m}\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
K_G & & 0 & \longrightarrow & K_{\alpha'} & \longrightarrow & K_{\beta'} & \longrightarrow & I_{g'\beta'} & \longrightarrow & 0
\end{array}$$

where $K_2=\{a\in K_{\beta'}|(r+s)a=0\}$ and K_3 is the image of the restriction of g' to K_2. So the extended homology is:

$$\bar{H}^d_{FG} \qquad 0 \longrightarrow K_{\hat{g}} \longrightarrow \frac{K_{\beta'}}{I_\beta+(r+s)K_{\beta'}} \xrightarrow{\hat{g}} \frac{I_{g'\beta'}}{I_\gamma+(r+s)I_{g'\beta'}} \longrightarrow 0$$

where $\hat{g}$ is the quotient of the restriction of g' to $K_{\beta'}$.

Example 2.7. Let $d=0$. For any $(f,g)\in\hat{\mathcal{C}}$, $\bar{H}^d_{fg}=H^s_{fg}$.

Lemma 2.8. *Let $d=pr_1$ or $d=pr_2$. For any $(f,g)\in\hat{\mathcal{C}}$, $\bar{H}^d_{fg}=0$.*

Proof. Since $pr_1\Delta=1$, $I_{d_g\Delta_g}=K_g$, $P_{jm}=I_f$ and $I_f+I_{d_g\Delta_g}=K_g$, $\beta=1$ and $\bar{H}^d_{fg}=0$. Similar argument holds for $d=pr_2$. $\square$

Calling the projection transformations, pr_1 and pr_2, and the zero transformation, the trivial transformations we have:

Proposition 2.9 ([4, 5])**.** *The only kernel transformations in the categories, Top_*, of pointed topological spaces, Set_*, of pointed sets and the category, $\overrightarrow{Set}$, of partial sets, are the trivial ones.*

Example 2.10. Let $\mathcal{C}$ be the category, Set_*, of pointed sets ($\overrightarrow{Set}$, of partial sets or Top_*, of pointed topological spaces). For $d=pr_1$ and $d=pr_2$, $\beta=1$ and for $d=0$, $\beta=j$. Therefore $\bar{H}^d_{fg}=0$ or $\bar{H}^d_{fg}=H^s_{fg}$.

Given $A\xrightarrow{f}C$ and $B\xrightarrow{g}C$ in $\mathcal{C}$, let $A\xleftarrow{i}P_{fg}\xrightarrow{j}B$ be the pullback of (f,g), and $A\xrightarrow{m}A+B\xleftarrow{n}B$ be the pushout of (i,j). We know there is a unique $\beta: A+B\longrightarrow C$ such that $\beta m=f$ and $\beta n=g$. Let $A\coprod B$ be the coproduct of A and B with injections l_1 and l_2. Then $\beta(m\oplus n)=f\oplus g$.

Lemma 2.11. *Let $A\xrightarrow{f}C$ and $B\xrightarrow{g}C$ be in $\mathcal{C}$. Then:*

(i) There is a regular epi $\sigma: A+B\longrightarrow I_{f\oplus g}$ such that $\sigma(m\oplus n)=e_{f\oplus g}$.

(ii) If β is monic, then $A+B\cong I_{f\oplus g}$ and $m_{f\oplus g}$ is monic.

(iii) $c_{f\oplus g}\cong c_\beta$.

Proof. (i) Since $A+B$ is the pushout of (i,j), $m\oplus n: A\coprod B \longrightarrow A+B$ is the coequalizer of $P_{fg} \underset{l_2 j}{\overset{l_1 i}{\rightrightarrows}} A\coprod B$. On the other hand, since $(f\oplus g)l_1 i = fi = gj = (f\oplus g)l_2 j$, and $P_{f\oplus g}$ is the kernel pair of $f\oplus g$, there is a unique $\xi : P_{fg} \longrightarrow P_{f\oplus g}$ such that $\pi_1\xi = l_1 i$ and $\pi_2 \xi = l_2 j$. So we have the following diagram in which the top and bottom rows are coequalizers and the square commutes

$$\begin{array}{ccccc} P_{fg} & \underset{l_2 j}{\overset{l_1 i}{\rightrightarrows}} & A\coprod B & \xrightarrow{m\oplus n} & A+B \\ \xi\downarrow & & \downarrow 1_{A\coprod B} & & \vdots\ \sigma \\ P_{f\oplus g} & \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} & A\coprod B & \xrightarrow[e_{f\oplus g}]{} & I_{f\oplus g} \end{array}$$

So there is a unique map $\sigma : A+B \longrightarrow I_{f\oplus g}$ such that $\sigma(m\oplus n) = e_{f\oplus g}$ and σ is regular epic.

(ii) Since $\beta(m\oplus n) = f\oplus g = m_{f\oplus g}e_{f\oplus g} = m_{f\oplus g}\sigma(m\oplus n)$ and $m\oplus n$ is epic, $\beta = m_{f\oplus g}\sigma$. If β is monic, then σ will be monic and so is an isomorphism. Then $m_{f\oplus g}$ is monic.

(iii) Since $e_{f\oplus g}$ and σ are epic, we have $c_{f\oplus g} = coker(f\oplus g) \cong coker(m_{f\oplus g}e_{f\oplus g})$ $\cong coker(m_{f\oplus g}) \cong coker(m_{f\oplus g}\sigma) \cong coker(\beta) = c_\beta$. $\square$

Theorem 2.12. *Let $(f,g)\in\hat{\mathcal{C}}$ and d be a kernel transformation. Then $\bar{H}^d_{fg} \cong C_{j\oplus m}$.*

Proof. Replace the maps f and g of the previous lemma respectively by the maps $j: I_f \longrightarrow K_g$ and $m: I_{d_g\Delta_g} \longrightarrow K_g$. $\square$

Lemma 2.13. *Let $\mathcal{C}$ be a pre abelian category and $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be in $\mathcal{C}$. Then:*

(*i*) *There is a regular epi $\delta : I_{f\oplus g} \longrightarrow A+B$ such that $\delta e_{f\oplus g} = m\oplus n$.*

(*ii*) *If $m_{f\oplus g}$ is monic, then $A+B \cong I_{f\oplus g}$ and β is monic.*

Proof. Since $f\oplus g = fpr_1 + gpr_2$, $fpr_1(\pi_1 - \pi_2) = gpr_2(\pi_2 - \pi_1)$. The result then follows similar to the proof given in Lemma 2.11. $\square$

Theorem 2.14. *Let $\mathcal{C}$ be a pre abelian category and $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be in $\mathcal{C}$. Then β is monic if and only if $m_{f\oplus g}$ is monic. In this case $A+B \cong I_{f\oplus g}$.*

Proof. Follows from Lemmas 2.11 and 2.13. $\square$

Example 2.15 ([4])**.** Let $\mathcal{C}$ be the category, Sh_R, of short exact sequences of R-modules. Then for any pair $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be in Sh_R, β is monic, since $m_{f\oplus g}$ is.

Example 2.16. Let $\mathcal{C}$ be any abelian category. Then for any pair $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be in $\mathcal{C}$, β is monic, since $m_{f\oplus g}$ is.

3. Standard homology versus extended d-homology

In this section, unless stated otherwise, we assume $\mathcal{C}$ is a category with a zero object, pullbacks and pushouts, and we investigate the relation between the standard homology and the extended d-homology.

Lemma 3.1. *Let d be a kernel transformation in $\mathcal{C}$. There is a natural transformation $p : H^s \longrightarrow \bar{H}^d$. Furthermore p is pointwise regular epic.*

Proof. For $(f, g) \in \hat{\mathcal{C}}$, p_{fg} is obtained by applying Lemma 2.2 of [4] to the following diagram

$$\begin{array}{ccccc} I_f & \xrightarrow{\ j\ } & K_g & \xrightarrow{\ c_j\ } & H^s_{fg} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle 1_{K_g}} & & \vdots\,{\scriptstyle p_{fg}} \\ I_f + I_{d_g\Delta_g} & \xrightarrow[\beta]{} & K_g & \xrightarrow[c_\beta]{} & \bar{H}^d_{fg} \end{array}$$

To show naturality, given $(\sigma, \delta, \zeta) : (f, g) \to (f', g')$ in $\hat{\mathcal{C}}$, we have $p_{f'g'} H^s(\sigma, \delta, \zeta) c_j = p_{f'g'} c_{j'} K(\delta, \zeta) = c_{\beta'} K(\delta, \zeta) = \bar{H}^d(\sigma, \delta, \zeta) c_\beta = \bar{H}^d(\sigma, \delta, \zeta) p_{fg} c_j$. Since c_j is epic, $p_{f'g'} H^s(\sigma, \delta, \zeta) = \bar{H}^d(\sigma, \delta, \zeta) p_{fg}$. □

Lemma 3.2. *If for $(f, g) \in \hat{\mathcal{C}}$, $i : I_f \longrightarrow I_f + I_{d_g\Delta_g}$ is epic, then $p_{fg} : H^s_{fg} \cong \bar{H}^d_{fg}$ is an isomorphism.*

Proof. $\bar{H}^d_{fg} = Coker(\beta) \cong Coker(\beta i) = Coker(j) = H^s_{fg}$. □

Corollary 3.3. *If $d\Delta = 0$ (hence if $d = 0$), then $\bar{H}^d \cong_n H^s$, i.e., $\bar{H}^d$ is naturally isomorphic to H^s.*

Proof. Let $(f, g) \in \hat{\mathcal{C}}$. Since $d_g \Delta_g = 0$, $I_{d_g\Delta_g} = I_0 = 0$. It can be easily shown that $I_f + I_{d_g\Delta_g} = I_f + 0 = I^c_j$, where I^c_j is the cokernel of kernel of j, see [4], and $i : I_f \longrightarrow I_f + 0 = I^c_j$ is the map c_{k_j} and is therefore epic. The result then follows from Lemma 3.2. □

Corollary 3.4. *Let $\mathcal{C}$ be an abelian category, $d = +(r \times s)$. If $(r + s)K_g = 0$, then $\bar{H}^d_{fg} \cong H^s_{fg}$.*

Proof. Since $I_{d_g\Delta_g} = (r + s)K_g = 0$, the result follows. □

Corollary 3.5. *Let $\mathcal{C}$ be an abelian category and $d = +(r \times -r)$. Then $\bar{H}^d \cong_n H^s$. In particular $\bar{H}^- \cong_n H^s$.*

Proof. Follows from the Corollary 3.3. □

Theorem 3.6. *Let $\mathcal{C}$ be an abelian category. For $(f, g) \in \hat{\mathcal{C}}$, $p_{fg} : H^s_{fg} \longrightarrow \bar{H}^d_{fg}$ is an isomorphism if and only if $i : I_f \longrightarrow I_f + I_{d_g\Delta_g}$ is an isomorphism.*

Proof. In an abelian category j and β are monic and so $\beta = ker(coker(\beta)) = ker(c_\beta)$ and $j = ker(coker(j)) = ker(c_j)$. So in the following diagram the rows are equalizers and the right square commutes

$$\begin{array}{ccccc} I_f & \xrightarrow{\ j\ } & K_g & \xrightarrow{\ c_j\ } & H^s_{fg} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle 1_{K_g}} & & \vdots\,{\scriptstyle p_{fg}} \\ I_f + I_{d_g \Delta_g} & \xrightarrow[\beta]{} & K_g & \xrightarrow[c_\beta]{} & \bar{H}^d_{fg} \end{array}$$

By dual of Lemma 2.2 in [4], i is a regular mono and if p_{fg} is a mono, i is an isomorphism. □

The converse has been shown previously.

Lemma 3.7. *Let $\mathcal{C}$ be an abelian category. For $(f, g) \in \hat{\mathcal{C}}$, $i : I_f \longrightarrow I_f + I_{d_g \Delta_g}$ is an isomorphism if and only if $I_{d_g \Delta_g}$ is a subobject of I_f, i.e., $m : I_{d_g \Delta_g} \longrightarrow K_g$ factors through $j : I_f \longrightarrow K_g$.*

Proof. Straightforward. □

Corollary 3.8. *Let $\mathcal{C}$ be an abelian category. For $(f, g) \in \hat{\mathcal{C}}$, $p_{fg} : H^s_{fg} \cong \bar{H}^d_{fg}$ is an isomorphism if and only if $I_{d_g \Delta_g}$ is a subobject of I_f.*

Theorem 3.9. *Let $\mathcal{C}$ be the category, Rmod. If $H^s_{fg} \cong H^s_{f'g'}$, then $\bar{H}^d_{fg} \cong \bar{H}^d_{f'g'}$ and $H^d_{fg} \cong H^d_{f'g'}$.*

Proof. Let $(f, g), (f', g') \in \hat{\mathcal{C}}$ and suppose $\psi : \frac{K_g}{I_f} \cong \frac{K_{g'}}{I_{f'}}$. Since in *Rmod*, $d = +(r \times s)$ for some $r, s \in Rmod$, $H^s_{fg} = \frac{K_g}{I_f}$ and $\bar{H}^d_{fg} = \frac{K_g}{(r+s)K_g + I_f}$, we have an epi $\phi : K_g \longrightarrow \frac{K_{g'}}{(r+s)K_{g'} + I_{f'}}$, which is the composition of the epis $q : K_g \longrightarrow \frac{K_g}{I_f}$, $\psi : \frac{K_g}{I_f} \longrightarrow \frac{K_{g'}}{I_{f'}}$ and $p_{f'g'} : \frac{K_{g'}}{I_{f'}} \longrightarrow \frac{K_{g'}}{(r+s)K_{g'} + I_{f'}}$. Some computations show that $K_\phi = (r + s)K_g + I_f$. The result then follows.

The proof of the second equality follows from the fact that $H^d_{fg} = \{[a] | a \in K_g\}$, where $[a] = \{b \in K_g | r(a - b) \in (r + s)K_g + I_f\} = \{b \in K_g | s(a - b) \in (r + s)K_g + I_f\}$, see [4]. □

References

[1] J. Adamek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories,* Wiley, New York, 1990.

[2] F. Borceux, D. Bourn, *MalCev, Protomodular, Homological and Semi-Abelian Categories*, Kluwer Academic Publishers, 2004.

[3] P. Freyd, *Abelian Categories,* Harper and Row, 1964.

[4] S.N. Hosseini, M.Z. Kazemi Baneh, *Homology with respect to a Kernel Transformation*, Turk. J. of Math., 34 (2010), 1-18, TÜBITAC.

[5] S.N. Hosseini, M.V. Mielke, *Universal Monos in Partial Morphism Categories*, Appl. Categor. Struct., 17 (2009), 435-444.

[6] S. MacLane, *Categories for the Working Mathematician*, 2nd edition, Springer-Verlag, 1998.

[7] M.S. Osborne, *Basic Homological Algebra*, Springer-Verlag, 2000.

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