A NEW CHARACTERIZATION OF $L_2(p)$ WITH $p \in \{19, 23\}$ BY NSE

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Abstract. Let $G$ be a group. We denote by $\text{nse}(G) := \{m_k \mid k \in \pi_e(G)\}$, where $\pi_e(G)$ is the set of element orders of $G$ and $m_k$ is the number of elements of order $k$ in $G$. In this paper, we characterize simple linear group $L_2(p)$ uniquely by set $\text{nse}(L_2(p))$ when $p \in \{19, 23\}$.

Keywords: Finite groups, numbers of elements with the same order, linear groups.

1. Introduction

Throughout this paper, all groups are finite and $G$ denotes a group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, $\pi_e(G)$ the set of element orders of $G$. We call $G$ a simple $K_n$-group if $G$ is simple with $|\pi(G)| = n$. If $r \in \pi(G)$, then $P_r$ and $n_r(G)$ denote a Sylow $r$-subgroup of $G$ and the number of Sylow $r$-subgroups of $G$, respectively. Let $n$ be an integer. We denote by $\varphi(n)$ the Euler function of $n$.

The prime graph $\text{GK}(G)$ of a group $G$ is defined as a graph with vertex set $\pi(G)$ and two distinct primes $p, q \in \pi(G)$ are adjacent if $G$ contains an element of order $pq$. Moreover, the connected components of $\text{GK}(G)$ are denoted by $\pi_i, 1 \leq i \leq t(G)$, where $t(G)$ is the number of connected components of $G$. In particular, we define by $\pi_1$ the component containing the prime 2 for a group of even order.

Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. Set $\text{nse}(G) := \{m_k \mid k \in \pi_e(G)\}$. In 1987, Thompson gave an example showing that not all groups can be characterized by $\text{nse}(G)$ and $|G|$: Let $G_1 = (C_2 \times C_2 \times C_2)$.

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C_2 \times C_2) \rtimes A_7 and G_2 = L_3(4) \rtimes C_2 be two maximal subgroups of M_{23}. Then \text{nse}(G_1) = \text{nse}(G_2) and |G_1| = |G_2|, but G_1 \not\cong G_2.

So it is an interesting topic to study class of groups G which can be characterized by the group order |G| and the set \text{nse}(G). Then authors of [8] proved that all simple K_4-groups can be uniquely determined by \text{nse}(G) and |G|. Also in [9], it is proved that \text{L}_2(3) \cong \text{A}_4, \text{L}_2(4) \cong \text{L}_2(5) \cong \text{A}_5 and \text{L}_2(9) \cong \text{A}_6 are uniquely determined by \text{nse}(G). M. Khatami, B. Khosravi and Z. Akhlaghi ([4]) deduced that simple groups \text{L}_2(p) is characterizable by uniquely the set \text{nse}(\text{L}_2(p)) if \text{p} \in \{7, 8, 11, 13\}. Recently, the authors of this paper ([6]) proved that simple linear groups \text{L}_2(p) with \text{p} \in \{17, 27, 29\} can be uniquely determined by its set \text{nse}(\text{L}_2(q)).

By using prime graph of a group as a new skill, in this present paper we prove that simple linear group \text{L}_2(p) can be determined by exactly the set \text{nse}(\text{L}_2(p)) if \text{p} \in \{19, 23\}. Our main theorem is:

**Theorem A.** Let \text{G} be a group and \text{p} \in \{19, 23\}. Then \text{G} \cong \text{L}_2(p) if and only if \text{nse}(\text{G}) = \text{nse}(\text{L}_2(p)).

Throughout this paper, we denote \text{n}_{r}(\text{G}) by \text{n}_{r} and \text{m}_{k}(\text{G}) by \text{m}_{k} if there is no confusion. Further unexplained notation is standard, readers may refer to [2].

### 2. Preliminaries

In this section we give some lemmas which will be used in the sequel.

**Lemma 2.1.** Let \text{G} be a group. Let \text{k} \in \pi_{a}(\text{G}) and \text{m}_{k} be the number of elements of order \text{k} in \text{G}. Then \varphi(\text{k}) \mid \text{m}_{k}. In particular, if there exists some odd integer \text{n} \in \text{nse}(\text{G}), then \text{2} \in \pi(\text{G}) and \text{m}_{2} = \text{n}. Moreover, \text{m}_{k} \neq \text{n} for any \text{k} \geq 3.

**Proof.** Let \text{t} be the number of the cyclic subgroups of order \text{k} of \text{G}, where \text{1} \neq \text{k} \in \pi_{a}(\text{G}). Then \text{m}_{k} = \text{t} \varphi(\text{k}), yielding to \varphi(\text{k}) \mid \text{m}_{k}. In particular, if there exists some odd integer \text{n} \in \text{nse}(\text{G}), then \text{m}_{2} = \text{n}, since otherwise, \varphi(\text{k}) is even for any \text{k} \geq 3. \hfill \Box

**Lemma 2.2** ([3]). Let \text{G} be a group and \text{m} be a positive integer dividing |\text{G}|. If \text{L}_{m}(\text{G}) = \{\text{g} \in \text{G}|\text{g}^{\text{m}} = \text{1}\}, then \text{m} \mid |\text{L}_{m}(\text{G})|.

**Lemma 2.3** ([7, Lemma 2.3]). Let \text{G} be a group and \text{P} be a cyclic Sylow p-subgroup of \text{G}. Assume further that |\text{P}| = \text{p}^{\text{a}} and \text{r} is an integer such that \text{p}^{\text{a}} \text{r} \in \pi_{a}(\text{G}). Then \text{m}_{p^{\text{a}}r} = \text{m}_{r}(\text{C}_{G}(\text{P})) \text{m}_{p^{\text{a}}} \mid \text{m}_{p^{\text{a}}r}.

**Lemma 2.4** ([5]). Let \text{G} be a group and \text{p} \in \pi(\text{G}) be odd. Suppose that \text{P} is a Sylow p-subgroup of \text{G} and \text{n} = \text{p}^{\text{s}} \text{m}, where (\text{p}, \text{m}) = 1. If \text{P} is not cyclic and \text{s} > 1, then the number of elements of order \text{n} is always a multiple of \text{p}^{\text{s}}.
Recall that $G$ is a 2-Frobenius group, if there $G$ has a normal series $1 < H < K < G$ such that $G/H$ and $K$ are Frobenius groups with $K/H$ and $H$ as Frobenius kernels, respectively.

**Lemma 2.5** ([11, Theorem]). Let $G$ be a group such that $t(G) \geq 2$. Then $G$ has one of the following structures:

(a) $G$ is a Frobenius or 2-Frobenius group.

(b) $G$ has a normal series $1 < H < K < G$ such that $\pi(H) \cup \pi(K/H) \subseteq \pi_1$ and $K/H$ is a nonabelian simple group.

**Lemma 2.6** ([1, Theorem 2]). If $G$ is a 2-Frobenius group of even order, then $t(G) = 2$ and $G$ has a normal series $1 < H < K < G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, $|G/K| > |\text{Aut}(K/H)|$, $G/K$ and $K/H$ are cyclic. In particular, $|G/K| < |K/H|$ and $G$ is solvable.

**Lemma 2.7.** Let $G$ be a simple group. If $\pi(G) = \{2,3,5,19\}$, then $G \cong L_2(19)$; if $\pi(G) = \{2,3,11,23\}$, then $G \cong L_2(23)$.

**Proof.** It follows immediately from [10, Corollary 2] and [10, Corollary 4].

**Lemma 2.8** ([8, Lemma 2.5]). Let $G$ be a group with a normal series: $K \leq L \leq G$. Suppose that $P \in \text{Syl}_p(G)$, where $p \in \pi(G)$. If $P \leq L$ and $p \nmid |K|$, then the following statements hold:

1. $|G : N_G(P)| = 2^s |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;

2. $|L/K : N_{L/K}(PK/K)t| = |G : N_G(P)| = |L : N_L(P)| \cdot n_p(L/K)t = n_p(L)$ for some positive integer $t$. Furthermore, $|N_K(P)t| = |K|$.

3. **Proof of Theorem A**

**Proof.** The necessity of the theorem is obvious. We only prove the sufficiency and we will discuss it case by case.

**Case 1.** $nse(G) = \{1,171,684,380,1140,360\} = nse(L_2(19))$.

By Lemma 2.1 and Lemma 2.2, we see that $2 \in \pi(G) \subseteq \{2,3,5,7,19\}$ and $m_2 = 171$. Let $\exp(P_2) = 2^s$. Then $s \leq 4$ since $\varphi(2^s) \mid m_2^s$ by Lemma 2.1. Moreover, $|P_2| \leq 2^6$ by Lemma 2.2. Next we show that $\pi(G) = \{2,3,5,19\}$.

If $7 \in \pi(G)$, then Lemma 2.2 gives $m_7 = 1140$. More, $7^2 \nmid \pi_p(G)$ since $\varphi(7^2) \mid m_{7^2}$ by Lemma 2.1. Therefore, $|P_7| = 7$ as $|P_7| \mid (1+m_7)$ by Lemma 2.2.

If $14 \in \pi(G)$, then $m_7 = 14$ by Lemma 2.3, against $14 \nmid (1+m_2+m_7+m_{14})$ by Lemma 2.2. As a consequence, $P_7$ acts fixed-point-freely on $\Omega_2 := \{\text{all elements of order 2 in } G\}$, leading to $7 \nmid m_2$. This contradiction forces $\pi(G) \subseteq \{2,3,5,19\}$.

Similarly, if $3 \in \pi(G)$, then $m_3 = 380$ and $\exp(P_3) \leq 3^2$. Assume that $\exp(P_3) = 9$. Then $m_9 = 1140$ by Lemma 2.1 and Lemma 2.2, which implies that $P_3$ is cyclic by Lemma 2.4. In this case, $n_3 = m_9/\varphi(9) = 2 \cdot 5 \cdot 19$ and thus
\[ \pi(G) = \{2, 3, 5, 19\}, \] we are done. On the other hand, if \( \exp(P_3) = 3 \), we also have \( \pi(G) = \{2, 3, 5, 19\} \).

Assume now \( 5 \in \pi(G) \). By a similar argument, we obtain that \( m_5 = 684 \) and \( \exp(P_5) \leq 2^4 \). More, if \( \exp(P_5) = 5^2 \), then \( m_{25} = 1140 \) and \( |P_5| = 5^2 \) by Lemma 2.2. In this case, \( n_5 = m_{25}/\varphi(25) = 3 \cdot 19 \), implying \( \pi(G) = \{2, 3, 5, 19\} \); if \( \exp(P_5) = 5 \), then \( |P_5| = 5 \), leading to \( n_5 = m_5/\varphi(5) = 3^2 \cdot 19 \), which also deduces that \( \pi(G) = \{2, 3, 5, 19\} \), as we need.

As a consequence, we may consider \( G \) as a \( \{2, 19\}\)-group. Note that \( G \) is not a 2-group since \( \exp(P_2) \leq 2^4 \) and \( |\text{nse}(G)| = 6 \). Thus \( 19 \in \pi(G) \). Lemma 2.2 shows that \( 19^2 \not\in \pi_e(G) \). Moreover, \( |P_{19}| | 19^2 \). Assume that \( |P_{19}| = 19 \). Then \( n_{19} = m_{19}/\varphi(19) = 2 \cdot 5 \), a contradiction. Hence \( |P_{19}| = 19^2 \). Since \( \pi_e(G) \subseteq \{1, 2, \cdots, 2^4\} \cup \{19, 19 \cdot 2, \cdots, 19 \cdot 2^3\} \), we obtain that

\[ |G| = 2736 + 684k_1 + 380k_2 + 1140k_3 + 360k_4 \]

with \( \sum_{i=1}^{4} k_i \leq 3 \). That is,

\[ 2^a \cdot 19^2 = 2^4 \cdot 3^2 \cdot 19 + 2^2 \cdot 3^2 \cdot 19k_1 + 2^2 \cdot 5 \cdot 19k_2 + 2^2 \cdot 3 \cdot 5 \cdot 19k_3 + 3^3 \cdot 3^2 \cdot 5k_4. \]

Except \( 2^3 \cdot 3^2 \cdot 5k_4 \), \( 19 \) divides both sides of the equation (2), we get \( k_4 = 0 \) as \( \sum_{i=1}^{4} k_i \leq 3 \). Thus,

\[ 2^{a-2} \cdot 19 = 2^2 \cdot 3^2 + 2^2k_1 + 5k_2 + 3 \cdot 5k_3. \]

 Recall that \( a \leq 6 \). Then the unique equation of (3) is \( k_1 = k_4 = 0, k_2 = k_3 = 2 \) with \( a = 4 \), against \( \sum_{i=1}^{4} k_i \leq 3 \). Consequently, \( \pi(G) = \{2, 3, 5, 19\} \), as required.

We claim that \( |P_5| = 5 \), \( |P_{19}| = 19 \), where \( P_5 \) and \( P_{19} \) are Sylow 5-subgroup and Sylow 19-subgroup of \( G \), respectively. Further, \( 19r \not\in \pi_e(G) \) and \( 5s \not\in \pi_e(G) \) for \( 2 \neq s \in \pi(G) \).

Suppose that \( |P_{19}| = 19^2 \). If \( |P_5| = 25 \), then \( \exp(P_5) = 5^2 \). One has \( n_5 = m_{25}/\varphi(25) = 3 \cdot 19 \), implying \( 19 \mid |N_G(P_5)| \). Let \( N_19 \in \text{Syl}_{19}(N_G(P_5)) \). By Sylow’s Theorem, it follows that \( P_5 \times N_19 \leq G \), leading to \( 5^2 \cdot 19 \in \pi_e(G) \), against \( \varphi(19)m_{52} \mid m_{5^2} \cdot 19 \) by Lemma 2.3. Assume that \( |P_5| = 5 \). In this case, \( n_5 = m_5/\varphi(5) = 3^2 \cdot 19 \), implying \( 19 \mid |N_G(P_5)| \). By Sylow’s Theorem, \( P_5 \times P_{19} \leq G \) with \( P_{19} \) a Sylow 19-subgroup of \( N_G(P_5) \). Hence \( 5 \cdot 19 \in \pi_e(G) \), contrary to \( \varphi(19)m_5 \mid m_{5^2} \cdot 19 \) by Lemma 2.3. Consequently, \( |P_{19}| = 19 \). Furthermore, if there exists some prime \( r \) such that \( 19r \in \pi_e(G) \), then \( \varphi(r)m_{19} \mid m_{19r} \). This forces \( r = 2 \) and \( m_{19} = m_{38} \). However, under this situation, \( 38 \mid (1 + m_2 + m_{19} + m_{38}) = 892 \), against Lemma 2.2.

Analogously, we may prove that \( |P_3| = 5 \). If not, \( |P_3| = 5^2 \) and \( P_3 \) acts fixed-point-freely on \( \Omega_{19} := \{ \text{all elements of order 19 of } G \} \). Hence, \( 25 \mid m_{19} \), a contradiction. As a result, \( |P_3| = 5 \). Further, and \( 5s \not\in \pi_e(G) \) for any \( 2 \neq s \in \pi(G) \), as required.

As a result, \( t(G) \geq 2 \). Assume that \( G = K \times H \) is a Frobenius group. Then either \( 19 \mid |K| \) or \( 19 \mid |H| \). If the former holds, then \( |K| = 19 \) as \( t(G) = 2 \). In
this case, \( m_{19} = 18 \in nse(G) \). This contradiction shows that \( 19 \mid |H| \). Let \( K_r \) be a Sylow \( r \)-subgroup of \( K \) with \( r \in \pi(K) \) and \( H_{19} \) be a Sylow 19-subgroup of \( H \). Then \( K_r \times H_{19} \) is also a Frobenius group, which implies that \( 19 \mid (|K_r| - 1) \), a contradiction if we consider the order of \( K \).

Suppose then that \( G \) is a 2-Frobenius group, then \( G \) has the following normal series:

\[
1 \triangleleft H \triangleleft K \triangleleft G
\]

with \( |K/H| = 19 \) and \( |G/K| \mid |\Aut(K/H)| \) by Lemma 2.6. Hence \( 5 \mid |H| \) and thus \( m_5 = 4 \), a contradiction.

By Lemma 2.5, \( G \) has the following normal series:

\[
1 \triangleleft H \triangleleft K \triangleleft G
\]

with \( K/H \) a non-solvable simple group and \( \pi(H) \cup \pi(G/K) \subseteq \pi \). If \( K/H \) is a simple \( K_3 \)-group, then \( 19 \mid |K/H| \) by [2], which leading to \( 19 \mid |H| \) or \( 19 \mid |G/K| \).

Assume that the former holds. Notice that \( 5 \mid |K/H| \). Let \( K_5 \) be a Sylow 5-subgroup of \( K \) and \( H_{19} \) be a Sylow 19-subgroup of \( H \). Then \( K_5 \times H_{19} \leq G \), yielding to \( 5 \cdot 19 \in \pi_e(G) \), which is a contradiction to the argument above. Similarly, it will also deduce a contradiction if \( 19 \mid |G/H| \).

As a consequence, \( K/H \) is a non-solvable simple \( K_4 \)-group, yielding to \( K/H \cong L_2(19) \) by Lemma 2.7. Moreover, it follows by Lemma 2.8 that \( n_{19}(K/H) = n_{19} \) and \( |N_H(P_{19})| = |H| \). Since \( m_{19} = m_{19}(K/H) = n_{19} \), which follows that \( t = 1 \) and thus \( H = N_H(P_{19}) \). Note that \( H \leq G \). Then \( H \leq G \). As \( 19r \notin \pi_e(G) \) for any prime \( r \in \pi(G) \), we get \( H = 1 \). Hence \( K \cong L_2(19) \) and \( |G/K| \mid 2 \). If \( G = K_2 \), then by [2], \( m_2 = 361 \notin nse(G) \). This contradiction implies that \( G \cong L_2(19) \).

**Case 2.** \( nse(G) = \{1, 253, 506, 2760, 1012, 528\} = nse(L_2(23)) \).

By Lemma 2.1 and Lemma 2.2, we see that \( 2 \in \pi(G) \subseteq \{2, 3, 11, 23, 1013\} \) and \( m_2 = 253 \). Moreover, \( \exp(P_2) \leq 2^5 \) and \( |P_2| \leq 2^6 \).

If \( 1013 \in \pi(G) \), then \( m_{1013} = 1012 \), leading to \( |P_{1013}| = 1013 \) and \( P_{1013} \leq G \), where \( P_{1013} \) is a Sylow 1013-subgroup of \( G \). Let \( \Omega_2 := \{ \text{all elements of order 2 in } G \} \). We claim that \( P_{1013} \) acts fixed-point-freely on \( \Omega \). Otherwise, \( 2026 \in \pi_e(G) \) and \( m_{2026} = 1012 \) since \( \varphi(2026) \mid m_{2026} \), against \( m_{2026} \mid (1 + m_2 + m_{1013} + m_{1013}) \) by Lemma 2.2. As a consequence, \( |P_{1013}| \) divides \( |\Omega_2| = m_2 \), again a contradiction. Therefore, \( \pi(G) \subseteq \{2, 3, 11, 23\} \).

We assert that \( \pi(G) = \{2, 3, 11, 23\} \). Assume first that \( 11 \in \pi(G) \). Then Lemma 2.2 gives \( m_{11} = 2700, 11^2 \notin \pi_e(G) \) and \( |P_{11}| = 11 \). By Sylow’s Theorem, \( |G : N_G(P_{11})| = m_{11}/\varphi(11) = 2^2 \cdot 3 \cdot 23 \). This indicates that \( \pi(G) = \{2, 3, 11, 23\} \), we are done.

Suppose then that \( 3 \in \pi(G) \). We see that \( m_3 = 506 \) by Lemma 2.2. Let \( \exp(P_3) = 3^s \). Then \( s \leq 3^2 \) as \( \varphi(3^s) \mid m_{3^s} \). If \( \exp(P_3) = 3^2 \), then \( m_9 = 528 \). Lemma 2.4 implies that \( P_3 \) is cyclic. Hence \( n_3 = m_9/\varphi(9) = 2^4 \cdot 11 \); if \( \exp(P_3) = 3 \), the same arguments gives \( n_3 = 11 \cdot 23 \). As a result, we always have
$11 \in \pi(G)$ if $3 \in \pi(G)$. By the argument in the previous paragraph, we also have $\pi(G) = \{2, 3, 11, 23\}$.

Consequently, we may consider $G$ as a $\{2, 23\}$-group. Note that $\exp(P_2) \leq 2^5$. If $G$ is a 2-group, then $|G| = 5060$, which is not a power of 2, a contradiction. Thus $23 \in \pi(G)$. In this case, $m_{23} = 528$ and $\exp(P_{23}) \leq 23^2$. If $23^2 \in \pi_e(G)$, then $m_{23^2} = 506$ or 1012, against $23^2 \mid (1 + m_{23} + m_{23^2}) = 1035$ or 1541 by Lemma 2.2. Hence $\exp(P_{23}) = 23$. It follows by Lemma 2.2 that $|P_{23}| \mid 23^2$.

Assume that $|P_{23}| = 23$, then $n_{23} = m_{23}/\varphi(23) = 2^3 \cdot 3$, a contradiction. Hence $|P_{23}| = 23^2$. Moreover, $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5\} \cup \{23, 2 \cdot 23, 2^2 \cdot 23, 2^3 \cdot 23, 2^4 \cdot 23\}$. We show that $2^5 \notin \pi_e(G)$. Otherwise, $P_{23}$ act fixed-point-freely on $\Omega_{2^5} := \{\text{all elements of order } 2^5 \text{ in } G\}$, which implies that $|P_{23}| \mid m_{2^5}$, which is a contradiction since $m_{2^5} = 528$. Thus $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4\} \cup \{23, 2 \cdot 23, 2^2 \cdot 23, 2^3 \cdot 23, 2^4 \cdot 23\}$. Further,

$$|G| = 2^a \cdot 23^2 = 5060 + 506k_1 + 2760k_2 + 1012k_3 + 528k_4$$

with $\sum_{i=1}^{4} k_i \leq 4$. That is,

$$2^a \cdot 23^2 = 2^2 \cdot 5 \cdot 11 \cdot 23 + 2 \cdot 11 \cdot 23k_1 + 2^3 \cdot 3 \cdot 5 \cdot 23k_2 + 2^2 \cdot 11 \cdot 23k_3 + 2^2 \cdot 3 \cdot 11k_4.$$

Since 23 divides both sides of equation (5) except $2^1 \cdot 3 \cdot 11k_4$, we have $k_4 = 0$. Then the equation becomes

$$2^{a-1} \cdot 23 = 2 \cdot 5 \cdot 11 + 11k_1 + 2^2 \cdot 3 \cdot 5k_2 + 2 \cdot 11k_3.$$

Recall that $a \leq 6$ and $\sum_{i=1}^{4} k_i \leq 4$. It follows that $44 \leq 11k_1 + 2^2 \cdot 3 \cdot 5k_2 + 2 \cdot 11k_3 \leq 240$. Hence the unique possibility is $a = 4$, $k_1 = k_4 = 0$, $k_2 = 1$, $k_3 = 4$, contrary to $\sum_{i=1}^{4} k_i \leq 4$. This contradiction shows that $\pi(G) = \{2, 3, 11, 23\}$.

Recall that $|P_{11}| = 11$. If there exists some prime $r$ such that $11r \in \pi_e(G)$, then $(r - 1) \cdot m_{11} \mid m_{11}$, by Lemma 2.3, yielding that $r = 2$ and $m_{22} = 2760$. In this case, $22 \mid (1 + m_2 + m_{11} + m_{22}) = 5774$, against Lemma 2.2. Therefore, $11 \parallel |G|$ and $11r \notin \pi_e(G)$ for any $r \in \pi(G)$.

Now we prove that $|P_{23}| = 23$. If not, $|P_{23}| = 23^2$. Moreover, $P_{23}$ acts fixed-point-freely on $\Omega_{11} := \{\text{all elements of order } 11 \text{ in } G\}$. As a consequence, $|P_{23}| \mid m_{11}$, which is a contradiction. Hence $|P_{23}| = 23$. Further, if there exists some prime $r$ such that $23r \in \pi_e(G)$, then Lemma 2.3 shows that $r = 2$. In this case, $2 \cdot 23 \mid (1 + m_2 + m_{23} + m_{22})$, against Lemma 2.2. That is to say, for any prime $r \in \pi(G)$, we always have $11r \notin \pi_e(G)$, yielding that $t(G) \geq 3$.

By Lemma 2.5, we see that $G$ is non-solvable and has the following normal series: $1 \leq H \leq K \leq G$ with $K/H$ a simple $K_4$-group and $\pi(H) \cup \pi(G/K) \subseteq \pi_1$.

We see from Lemma 2.7 that $K/H \cong L_2(23)$. Moreover, Lemma 2.8 gives $|N_H(P_{23})|/|H| = n_{23}$ and $n_{23}(K/H)/t = n_{23}$. Since $K/H \cong L_2(23)$, we have $m_{23}(K/H) = m_{23}$, yielding $n_{23}(K/H) = n_{23}(K)$ and thus $t = 1$. Further, $H \leq N_G(P_{23})$, and hence $H \times P_{23} \leq G$. Note that $23r \notin \pi_e(G)$. Then $H = 1$.

As a result, $K \cong L_2(23)$ and $|G/K| \parallel 2$. Assume that $G = K \cdot 2$, then by [2], $m_2 = 529 \notin nse(G)$. This contradiction indicates that $G = K \cong L_2(23)$. \qed
Acknowledgements. The authors are supported by the NNSF of China (No. 11301218) and the Nature Science Fund of Shandong Province (No. ZR2014AM020).

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Accepted: 22.06.2017