A NEW CHARACTERIZATION OF $L_2(p)$ WITH $p \in \{19, 23\}$ BY NSE

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Abstract. Let G be a group. We denote by $nse(G) := \{m_k \mid k \in \pi_e(G)\}$, where $\pi_e(G)$ is the set of element orders of G and m_k is the number of elements of order k in G. In this paper, we characterize simple linear group $L_2(p)$ uniquely by set $nse(L_2(p))$ when $p \in \{19, 23\}$.

Keywords: Finite groups, numbers of elements with the same order, linear groups.

1. Introduction

Throughout this paper, all groups are finite and G denotes a group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, $\pi_e(G)$ the set of element orders of G. We call G a simple K_n-group if G is simple with $|\pi(G)| = n$. If $r \in \pi(G)$, then P_r and $n_r(G)$ denote a Sylow r-subgroup of G and the number of Sylow r-subgroups of G, respectively. Let n be an integer. We denote by $\varphi(n)$ the Euler function of n.

The prime graph GK(G) of a group G is defined as a graph with vertex set $\pi(G)$ and two distinct primes $p, q \in \pi(G)$ are adjacent if G contains an element of order pq. Moreover, the connected components of GK(G) are denoted by $\pi_i, 1 \leq i \leq t(G)$, where $t(G)$ is the number of connected components of G. In particular, we define by π_1 the component containing the prime 2 for a group of even order.

Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Set $nse(G) := \{m_k \mid k \in \pi_e(G)\}$. In 1987, Thompson gave an example showing that not all groups can be characterized by $nse(G)$ and $|G|$: Let $G_1 = (C_2 \times C_2 \times$

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$C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be two maximal subgroups of M_{23}. Then $nse(G_1) = nse(G_2)$ and $|G_1| = |G_2|$, but $G_1 \ncong G_2$.

So it is an interesting topic to study class of groups G which can be characterized by the group order $|G|$ and the set $nse(G)$. Then authors of [8] proved that all simple K_4-groups can be uniquely determined by $nse(G)$ and $|G|$. Also in [9], it is proved that $L_2(3) \cong A_4$, $L_2(4) \cong L_2(5) \cong A_5$ and $L_2(9) \cong A_6$ are uniquely determined by $nse(G)$. M. Khatami, B. Khosravi and Z. Akhlaghi ([4]) deduced that simple groups $L_2(p)$ is characterizable by uniquely the set $nse(L_2(p))$ if $p \in \{7, 8, 11, 13\}$. Recently, the authors of this paper ([6]) proved that simple linear groups $L_2(p)$ with $p \in \{17, 27, 29\}$ can be uniquely determined by its set $nse(L_2(q))$.

By using prime graph of a group as a new skill, in this present paper we prove that simple linear group $L_2(p)$ can be determined by exactly the set $nse(L_2(p))$ if $p \in \{19, 23\}$. Our main theorem is:

Theorem A. *Let G be a group and $p \in \{19, 23\}$. Then $G \cong L_2(p)$ if and only if $nse(G) = nse(L_2(p))$.*

Throughout this paper, we denote $n_r(G)$ by n_r and $m_k(G)$ by m_k if there is no confusion. Further unexplained notation is standard, readers may refer to [2].

2. Preliminaries

In this section we give some lemmas which will be used in the sequel.

Lemma 2.1. *Let G be a group. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Then $\varphi(k) \mid m_k$. In particular, if there exists some odd integer $n \in nse(G)$, then $2 \in \pi(G)$ and $m_2 = n$. Moreover, $m_k \neq n$ for any $k \geq 3$.*

Proof. Let t be the number of the cyclic subgroups of order k of G, where $1 \neq k \in \pi_e(G)$. Then $m_k = t\varphi(k)$, yielding to $\varphi(k) \mid m_k$. In particular, if there exists some odd integer $n \in nse(G)$, then $m_2 = n$, since otherwise, $\varphi(k)$ is even for any $k \geq 3$. □

Lemma 2.2 ([3])**.** *Let G be a group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.3 ([7, Lemma 2.3])**.** *Let G be a group and P be a cyclic Sylow p-subgroup of G. Assume further that $|P| = p^a$ and r is an integer such that $p^a r \in \pi_e(G)$. Then $m_{p^a r} = m_r(C_G(P))m_{p^a}$. In particular, $\varphi(r)m_{p^a} \mid m_{p^a r}$.*

Lemma 2.4 ([5])**.** *Let G be a group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Recall that G is a 2-Frobenius group, if there G has a normal series $1 \lhd H \lhd K \lhd G$ such that G/H and K are Frobenius groups with K/H and H as Frobenius kernels, respectively.

Lemma 2.5 ([11, Theorem]). *Let G be a group such that $t(G) \geq 2$. Then G has one of the following structures:*

(*a*) *G is a Frobenius or 2-Frobenius group.*

(*b*) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(K/H) \subseteq \pi_1$ and K/H is a nonabelian simple group.*

Lemma 2.6 ([1, Theorem 2]). *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \bigcup \pi(G/K) = \pi_1$, $|G/K| \mid |Aut\ (K/H)|$, G/K and K/H are cyclic. In particular, $|G/K| < |K/H|$ and G is solvable.*

Lemma 2.7. *Let G be a simple group. If $\pi(G) = \{2, 3, 5, 19\}$, then $G \cong L_2(19)$; if $\pi(G) = \{2, 3, 11, 23\}$, then $G \cong L_2(23)$.*

Proof. It follows immediately from [10, Corollary 2] and [10, Corollary 4]. □

Lemma 2.8 ([8, Lemma 2.5]). *Let G be a group with a normal series: $K \trianglelefteq L \trianglelefteq G$. Suppose that $P \in Syl_p(G)$, where $p \in \pi(G)$. If $P \leq L$ and $p \nmid |K|$, then the following statements hold:*

(1) *$|G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;*

(2) *$|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(L/K)t = n_p(G) = n_p(L)$ for some positive integer t. Furthermore, $|N_K(P)|t = |K|$.*

3. Proof of Theorem A

Proof. The necessity of the theorem is obvious. We only prove the sufficiency and we will discuss it case by case.

Case 1. $nse(G) = \{1, 171, 684, 380, 1140, 360\} = nse(L_2(19))$.

By Lemma 2.1 and Lemma 2.2, we see that $2 \in \pi(G) \subseteq \{2, 3, 5, 7, 19\}$ and $m_2 = 171$. Let $\exp(P_2) = 2^s$. Then $s \leq 4$ since $\varphi(2^s) \mid m_{2^s}$ by Lemma 2.1. Moreover, $|P_2| \leq 2^6$ by Lemma 2.2. Next we show that $\pi(G) = \{2, 3, 5, 19\}$.

If $7 \in \pi(G)$, then Lemma 2.2 gives $m_7 = 1140$. More, $7^2 \notin \pi_e(G)$ since $\varphi(7^2) \mid m_{7^2}$ by Lemma 2.1. Therefore, $|P_7| = 7$ as $|P_7| \mid (1 + m_7)$ by Lemma 2.2. If $14 \in \pi_e(G)$, then $m_7 = m_{14}$ by Lemma 2.3, against $14 \mid (1 + m_2 + m_7 + m_{14})$ by Lemma 2.2. As a consequence, P_7 acts fixed-point-freely on $\Omega_2 :=$ {all elements of order 2 in G}, leading to $7 \mid m_2$. This contradiction forces $\pi(G) \subseteq \{2, 3, 5, 19\}$.

Similarly, if $3 \in \pi(G)$, then $m_3 = 380$ and $\exp(P_3) \leq 3^2$. Assume that $\exp(P_3) = 9$. Then $m_9 = 1140$ by Lemma 2.1 and Lemma 2.2, which implies that P_3 is cyclic by Lemma 2.4. In this case, $n_3 = m_9/\varphi(9) = 2 \cdot 5 \cdot 19$ and thus

$\pi(G) = \{2, 3, 5, 19\}$, we are done. On the other hand, if $\exp(P_3) = 3$, we also have $\pi(G) = \{2, 3, 5, 19\}$.

Assume now $5 \in \pi(G)$. By a similar argument, we obtain that $m_5 = 684$ and $\exp(P_5) \leq 5^2$. More, if $\exp(P_5) = 5^2$, then $m_{25} = 1140$ and $|P_5| = 5^2$ by Lemma 2.2. In this case, $n_5 = m_{25}/\varphi(25) = 3 \cdot 19$, implying $\pi(G) = \{2, 3, 5, 19\}$; if $\exp(P_5) = 5$, then $|P_5| = 5$, leading to $n_5 = m_5/\varphi(5) = 3^2 \cdot 19$, which also deduces that $\pi(G) = \{2, 3, 5, 19\}$, as we need.

As a consequence, we may consider G as a $\{2, 19\}$-group. Note that G is not a 2-group since $\exp(P_2) \leq 2^4$ and $|nse(G)| = 6$. Thus $19 \in \pi(G)$. Lemma 2.2 shows that $19^2 \notin \pi_e(G)$. Moreover, $|P_{19}| \mid 19^2$. Assume that $|P_{19}| = 19$. Then $n_{19} = m_{19}/\varphi(19) = 2^2 \cdot 5$, a contradiction. Hence $|P_{19}| = 19^2$. Since $\pi_e(G) \subseteq \{1, 2, \cdots, 2^4\} \cup \{19, 19 \cdot 2, \cdots, 19 \cdot 2^3\}$, we obtain that

$$|G| = 2736 + 684k_1 + 380k_2 + 1140k_3 + 360k_4 \tag{1}$$

with $\sum_{i=1}^{4} k_i \leq 3$. That is,

$$2^a \cdot 19^2 = 2^4 \cdot 3^2 \cdot 19 + 2^2 \cdot 3^2 \cdot 19k_1 + 2^2 \cdot 5 \cdot 19k_2 + 2^2 \cdot 3 \cdot 5 \cdot 19k_3 + 2^3 \cdot 3^2 \cdot 5k_4. \tag{2}$$

Except $2^3 \cdot 3^2 \cdot 5k_4$, 19 divides both sides of the equation (2), we get $k_4 = 0$ as $\sum_{i=1}^{4} k_i \leq 3$. Thus,

$$2^{a-2} \cdot 19 = 2^2 \cdot 3^2 + 3^2 k_1 + 5k_2 + 3 \cdot 5k_3. \tag{3}$$

Recall that $a \leq 6$. Then the unique equation of (3) is $k_1 = k_4 = 0, k_2 = k_3 = 2$ with $a = 4$, against $\sum_{i=1}^{4} k_i \leq 3$. Consequently, $\pi(G) = \{2, 3, 5, 19\}$, as required.

We claim that $|P_5| = 5$, $|P_{19}| = 19$, where P_5 and P_{19} are Sylow 5-subgroup and Sylow 19-subgroup of G, respectively. Further, $19r \notin \pi_e(G)$ and $5s \notin \pi_e(G)$ for $2 \neq s \in \pi(G)$.

Suppose that $|P_{19}| = 19^2$. If $|P_5| = 25$, then $\exp(P_5) = 5^2$. One has $n_5 = m_{25}/\varphi(25) = 3 \cdot 19$, implying $19 \mid |N_G(P_5)|$. Let $N_{19} \in \mathrm{Syl}_{19}(N_G(P_5))$. By Sylow's Theorem, it follows that $P_5 \times N_{19} \leq G$, leading to $5^2 \cdot 19 \in \pi_e(G)$, against $\varphi(19)m_{5^2} \mid m_{5^2 \cdot 19}$ by Lemma 2.3. Assume that $|P_5| = 5$. In this case, $n_5 = m_5/\varphi(5) = 3^2 \cdot 19$, implying $19 \mid |N_G(P_5)|$. By Sylow's Theorem, $P_5 \times P_{19} \leq G$ with P_{19} a Sylow 19-subgroup of $N_G(P_5)$. Hence $5 \cdot 19 \in \pi_e(G)$, contrary to $\varphi(19)m_5 \mid m_{5 \cdot 19}$ by Lemma 2.3. Consequently, $|P_{19}| = 19$. Furthermore, if there exists some primer r such that $19r \in \pi_e(G)$, then $\varphi(r)m_{19} \mid m_{19r}$. This forces $r = 2$ and $m_{19} = m_{38}$. However, under this situation, $38 \nmid (1 + m_2 + m_{19} + m_{38}) = 892$, against Lemma 2.2.

Analogously, we may prove that $|P_5| = 5$. If not, $|P_5| = 5^2$ and P_5 acts fixed-point-freely on $\Omega_{19} := \{$ all elements of order 19 of $G\}$. Hence, $25 \mid m_{19}$, a contradiction. As a result, $|P_5| = 5$. Further, and $5s \notin \pi_e(G)$ for any $2 \neq s \in \pi(G)$, as required.

As a result, $t(G) \geq 2$. Assume that $G = K \rtimes H$ is a Frobenius group. Then either $19 \mid |K|$ or $19 \mid |H|$. If the former holds, then $|K| = 19$ as $t(G) = 2$. In this case, $m_{19} = 18 \in nse(G)$. This contradiction shows that $19 \mid |H|$. Let K_r be a Sylow r-subgroup of K with $r \in \pi(K)$ and H_{19} be a Sylow 19-subgroup of H. Then $K_r \rtimes H_{19}$ is also a Frobenius group, which implies that $19 \mid (|K_r| - 1)$, a contradiction if we consider the order of K.

Suppose then that G is a 2-Frobenius group, then G has the following normal series:

$$1 \lhd H \lhd K \lhd G$$

with $|K/H| = 19$ and $|G/K| \mid |\mathrm{Aut}(K/H)|$ by Lemma 2.6. Hence $5 \mid |H|$ and thus $m_5 = 4$, a contradiction.

By Lemma 2.5, G has the following normal series:

$$1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$$

with K/H a non-solvable simple group and $\pi(H) \cup \pi(G/K) \subseteq \pi_1$. If K/H is a simple K_3-group, then $19 \nmid |K/H|$ by [2], which leading to $19 \mid |H|$ or $19 \mid |G/K|$. Assume that the former holds. Notice that $5 \mid |K/H|$. Let K_5 be a Sylow 5-subgroup of K and H_{19} be a Sylow 19-subgroup of H. Then $K_5 \times H_{19} \leq G$, yielding to $5 \cdot 19 \in \pi_e(G)$, which is a contradiction to the argument above. Similarly, it will also deduce a contradiction if $19 \mid |G/H|$.

As a consequence, K/H is a non-solvable simple K_4-group, yielding to $K/H \cong L_2(19)$ by Lemma 2.7. Moreover, it follows by Lemma 2.8 that $n_{19}(K/H) = n_{19}$ and $|N_H(P_{19})|t = |H|$. Since $m_{19} = m_{19}(K/H)$, we obtain $n_{19}(K/H) = n_{19}$, which follows that $t = 1$ and thus $H = N_H(P_{19})$. Note that $H \trianglelefteq G$. Then $HP_{19} = H \times P_{19} \leq G$. As $19r \notin \pi_e(G)$ for any prime $r \in \pi(G)$, we get $H = 1$. Hence $K \cong L_2(19)$ and $|G/K| \mid 2$. If $G = K.2$, then by [2], $m_2 = 361 \notin nse(G)$. This contradiction implies that $G \cong L_2(19)$.

Case 2. $nse(G) = \{1, 253, 506, 2760, 1012, 528\} = nse(L_2(23))$**.**

By Lemma 2.1 and Lemma 2.2, we see that $2 \in \pi(G) \subseteq \{2, 3, 11, 23, 1013\}$ and $m_2 = 253$. Moreover, $\exp(P_2) \leq 2^5$ and $|P_2| \leq 2^6$.

If $1013 \in \pi(G)$, then $m_{1013} = 1012$, leading to $|P_{1013}| = 1013$ and $P_{1013} \trianglelefteq G$, where P_{1013} is a Sylow 1013-subgroup of G. Let $\Omega_2 := \{$ all elements of order 2 in $G\}$. We claim that P_{1013} acts fixed-point-freely on Ω. Otherwise, $2026 \in \pi_e(G)$ and $m_{2026} = 1012$ since $\varphi(2026) \mid m_{2026}$, against $m_{2026} \mid (1 + m_2 + m_{1013} + m_{2\cdot 1013})$ by Lemma 2.2. As a consequence, $|P_{1013}|$ divides $|\Omega_2| = m_2$, again a contradiction. Therefore, $\pi(G) \subseteq \{2, 3, 11, 23\}$.

We assert that $\pi(G) = \{2, 3, 11, 23\}$. Assume first that $11 \in \pi(G)$. Then Lemma 2.2 gives $m_{11} = 2760$, $11^2 \notin \pi_e(G)$ and $|P_{11}| = 11$. By Sylow's Theorem, $|G : N_G(P_{11})| = m_{11}/\varphi(11) = 2^2 \cdot 3 \cdot 23$. This indicates that $\pi(G) = \{2, 3, 11, 23\}$, we are done.

Suppose then that $3 \in \pi(G)$. We see that $m_3 = 506$ by Lemma 2.2. Let $\exp(P_3) = 3^s$. Then $s \leq 3^2$ as $\varphi(3^s) \mid m_{3^s}$. If $\exp(P_3) = 3^2$, then $m_9 = 528$. Lemma 2.4 implies that P_3 is cyclic. Hence $n_3 = m_9/\varphi(9) = 2^3 \cdot 11$; if $\exp(P_3) = 3$, the same arguments gives $n_3 = 11 \cdot 23$. As a result, we always have

$11 \in \pi(G)$ if $3 \in \pi(G)$. By the argument in the previous paragraph, we also have $\pi(G) = \{2, 3, 11, 23\}$.

Consequently, we may consider G as a $\{2, 23\}$-group. Note that $\exp(P_2) \leq 2^5$. If G is a 2-group, then $|G| = 5060$, which is not a power of 2, a contradiction. Thus $23 \in \pi(G)$. In this case, $m_{23} = 528$ and $\exp(P_{23}) \leq 23^2$. If $23^2 \in \pi_e(G)$, then $m_{23^2} = 506$ or 1012 , against $23^2 \mid (1 + m_{23} + m_{23^2}) = 1035$ or 1541 by Lemma 2.2. Hence $\exp(P_{23}) = 23$. It follows by Lemma 2.2 that $|P_{23}| \mid 23^2$. Assume that $|P_{23}| = 23$, then $n_{23} = m_{23}/\varphi(23) = 2^3 \cdot 3$, a contradiction. Hence $|P_{23}| = 23^2$. Moreover, $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5\} \cup \{23, 2 \cdot 23, 2^2 \cdot 23, 2^3 \cdot 23, 2^4 \cdot 23\}$. We show that $2^5 \notin \pi_e(G)$. Otherwise, P_{23} act fixed-point-freely on $\Omega_{2^5} :=$ { all elements of order 2^5 in G}, which implies that $|P_{23}| \mid m_{2^5}$, which is a contradiction since $m_{2^5} = 528$. Thus $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4\} \cup \{23, 2 \cdot 23, 2^2 \cdot 23, 2^3 \cdot 23, 2^4 \cdot 23\}$. Further,

$$|G| = 2^a \cdot 23^2 = 5060 + 506k_1 + 2760k_2 + 1012k_3 + 528k_4 \tag{4}$$

with $\sum_{i=1}^{4} k_i \leq 4$. That is,

$$2^a \cdot 23^2 = 2^2 \cdot 5 \cdot 11 \cdot 23 + 2 \cdot 11 \cdot 23k_1 + 2^3 \cdot 3 \cdot 5 \cdot 23k_2 + 2^2 \cdot 11 \cdot 23k_3 + 2^4 \cdot 3 \cdot 11k_4. \tag{5}$$

Since 23 divides both sides of equation (5) except $2^4 \cdot 3 \cdot 11k_4$, we have $k_4 = 0$. Then the equation becomes

$$2^{a-1} \cdot 23 = 2 \cdot 5 \cdot 11 + 11k_1 + 2^2 \cdot 3 \cdot 5k_2 + 2 \cdot 11k_3. \tag{6}$$

Recall that $a \leq 6$ and $\sum_{i=1}^{4} k_i \leq 4$. It follows that $44 \leq 11k_1 + 2^2 \cdot 3 \cdot 5k_2 + 2 \cdot 11k_3 \leq 240$. Hence the unique possibility is $a = 4$, $k_1 = k_4 = 0$, $k_2 = 1, k_3 = 4$, contrary to $\sum_{i=1}^{4} k_i \leq 4$. This contradiction shows that $\pi(G) = \{2, 3, 11, 23\}$.

Recall that $|P_{11}| = 11$. If there exists some prime r such that $11r \in \pi_e(G)$, then $(r - 1) \cdot m_{11} \mid m_{11r}$ by Lemma 2.3, yielding that $r = 2$ and $m_{22} = 2760$. In this case, $22 \nmid (1 + m_2 + m_{11} + m_{22}) = 5774$, against Lemma 2.2. Therefore, $11 \parallel |G|$ and $11r \notin \pi_e(G)$ for any $r \in \pi(G)$.

Now we prove that $|P_{23}| = 23$. If not, $|P_{23}| = 23^2$. Moreover, P_{23} acts fixed-point-freely on $\Omega_{11} :=$ { all elements of order 11 in G}. As a consequence, $|P_{23}| \mid m_{11}$, which is a contradiction. Hence $|P_{23}| = 23$. Further, if there exists some prime r such that $23r \in \pi_e(G)$, then Lemma 2.3 shows that $r = 2$. In this case, $2 \cdot 23 \nmid (1 + m_2 + m_{23} + m_{2 \cdot 23})$, against Lemma 2.2. That is to say, for any prime $r \in \pi(G)$, we always have $11r \notin \pi_e(G)$, yielding that $t(G) \geq 3$.

By Lemma 2.5, we see that G is non-solvable and has the following normal series: $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ with K/H a simple K_4-group and $\pi(H) \cup \pi(G/K) \subseteq \pi_1$.

We see from Lemma 2.7 that $K/H \cong L_2(23)$. Moreover, Lemma 2.8 gives $|N_H(P_{23})|t = |H|$ and $n_{23}(K/H)t = n_{23}$. Since $K/H \cong L_2(23)$, we have $m_{23}(K/H) = m_{23}$, yielding $n_{23}(K/H) = n_{23}(K)$ and thus $t = 1$. Further, $H \leq N_G(P_{23})$, and hence $H \times P_{23} \leq G$. Note that $23r \notin \pi_e(G)$. Then $H = 1$. As a result, $K \cong L_2(23)$ and $|G/K| \mid 2$. Assume that $G = K.2$, then by [2], $m_2 = 529 \notin nse(G)$. This contradiction indicates that $G = K \cong L_2(23)$. □

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