(ε)-KENMOTSU MANIFOLDS ADMITTING A
SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The object of the present paper is to study some properties of quasi-conformal and concircular curvature tensor on (ε)-Kenmotsu manifolds with respect to a semi-symmetric metric connection.

Keywords: Quasi-conformally flat, φ-concircularly flat, (ε)-Kenmotsu manifold, semi-symmetric metric connection, η-Einstein manifold.

1. Introduction

Takahashi [19] studied Sasakian manifold with associated pseudo-Riemannian metrics and are known as (ε)-Sasakian manifolds. Bejancu and Duggal [3] shows the existence of (ε)-almost contact metric structures and provide an example of (ε)-Sasakian manifolds. Further investigation on this manifold was taken up by Xufeng and Xiaoli [23] and Rakesh kumar et al [13].

The study of manifolds with indefinite metric has a great relevance from the standpoint of geometrization of physics and relativity. Recently De and Sarkar [7] introduced indefinite metrics on Kenmotsu manifold, and are called as (ε)-Kenmotsu manifolds. Here they studied conformally flat, Weyl semisymmetric, φ-recurrent (ε)-Kenmotsu manifolds. Further, Singh et al [18] and Haseeb et al [9] established the relation between Levi-Civita connection and semi-symmetric metric connection and obtained the relation between curvature tensors of Levi-Civita connection and semi-symmetric metric connection in an (ε)-Kenmotsu manifold.

After the introduction of an idea of semi-symmetric linear connection in a differentiable manifold [8], Hayden [10] defined a semi-symmetric metric connection on a Riemannian manifold and this was further studied by Yano [24], Barua et al [2], De and Biswas [5].

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Quasi-conformal curvature tensor and concircular curvature tensor are the important curvature tensors from the differential geometry point of view. A quasi-conformal transformation is the one which transforms infinitesimal circles into infinitesimal ellipses, whereas a concircular transformation transforms every geodesic circle of an \( n \)-dimensional Riemannian manifold \( M \) into a geodesic circle.

The quasi-conformal curvature tensor \( \tilde{C} \) [25] and concircular curvature tensor \( \tilde{Z} \) [26] with respect to the semi-symmetric metric connection are respectively given by

\[
\tilde{C}(X, Y)Z = a\tilde{R}(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y
+ g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y]
- \frac{\tilde{r}}{n}\left\{\frac{a}{(n - 1)} + 2b\right\}[g(Y, Z)X - g(X, Z)Y],
\]

(1.1)

\[
\tilde{Z}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{\tilde{r}}{n(n - 1)}[g(Y, Z)X - g(X, Z)Y].
\]

(1.2)

The paper is organized as follows: Section 2 contains the preliminaries of \((\epsilon)\)-Kenmotsu manifold and a semi-symmetric metric connection on an \((\epsilon)\)-Kenmotsu manifold. Section 3 and 4 are devoted to the study of quasi-conformally flat and quasi-conformally semisymmetric \((\epsilon)\)-Kenmotsu manifold admitting a semi-symmetric metric connection. In the next section we study \(\phi\)-concircularly flat \((\epsilon)\)-Kenmotsu manifold admitting a semi-symmetric metric connection and shown that \(\phi\)-concircularly flat \((\epsilon)\)-Kenmotsu manifold admitting a semi-symmetric metric connection is an \(\eta\)-Einstein manifold. Further in section 6, we prove \((\epsilon)\)-Kenmotsu manifold admitting a semi-symmetric metric connection satisfying \(\tilde{Z}(X, Y) \cdot S(U, W) = 0\) is an \(\eta\)-Einstein manifold.

### 2. Preliminaries

An almost contact structure on a \( n \)-dimensional differentiable manifold \( M \) is a triple \((\phi, \xi, \eta)\), where \( \phi \) is a tensor field of type \((1, 1)\), \( \eta \) is a 1-form and \( \xi \) is a vector field such that

\[
\phi^2 = -I + \eta \circ \xi,
\]

(2.1)

\[
\eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0.
\]

(2.2)

A differential manifold with an almost contact structure is called an almost contact manifold. An almost contact metric manifold is an almost contact manifold endowed with a compatible metric \( g \). An almost contact metric manifold \( M \) is said to be an \((\epsilon)\)-almost contact metric manifold if

\[
g(\xi, \xi) = \pm 1 = \epsilon,
\]

(2.3)

\[
\eta(X) = \epsilon g(X, \xi), \quad \text{rank}(\phi) = n - 1,
\]

(2.4)

\[
g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(TM),
\]

(2.5)
holds, where $\xi$ is space-like or time-like but it is never a light like vector field. We say that $(\phi, \xi, \eta, g)$ is an $(\epsilon)$-contact metric structure if we have

\begin{equation}
(2.6) \quad d\eta(X, Y) = g(X, \phi Y).
\end{equation}

In this case, $M$ is an $(\epsilon)$-contact metric manifold. An $(\epsilon)$-contact metric manifold is called an $(\epsilon)$-Kenmotsu manifold [7] if

\begin{equation}
(2.7) \quad (\nabla_X \phi)Y = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X,
\end{equation}

holds, where $\nabla$ is the Riemannian connection of $g$. An $(\epsilon)$-almost contact metric manifold is a $(\epsilon)$-Kenmotsu manifold if and only if

\begin{equation}
(2.8) \quad \nabla_X\xi = \epsilon(X - \eta(X)\xi).
\end{equation}

The following conditions holds in an $(\epsilon)$-Kenmotsu manifold [7]:

\begin{align}
(2.9) & \quad (\nabla_X\eta)(Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \\
(2.10) & \quad \eta(R(X, Y)Z) = \epsilon\{g(X, Z)Y - g(Y, Z)X\}, \\
(2.11) & \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi, \\
(2.12) & \quad S(X, \xi) = -(n-1)\eta(X), \quad Q\xi = -\epsilon(n-1)\xi, \\
(2.13) & \quad S(\phi X, \phi Y) = S(X, Y) + \epsilon(n-1)\eta(X)\eta(Y).
\end{align}

A semi-symmetric metric connection $\tilde{\nabla}$ on an $n$-dimensional $(\epsilon)$-Kenmotsu manifold is given by [18],

\begin{equation}
(2.14) \quad \tilde{\nabla}_XY = \nabla_XY + \eta(Y)X - g(X, Y)\xi.
\end{equation}

A relation between the curvature tensor $\bar{R}$, Ricci curvature $\bar{S}$ and the scalar curvature $\bar{r}$ of $M$ with respect to semi-symmetric metric connection $\tilde{\nabla}$ and $R, S$ and $r$ of $M$ with respect to the Riemannian connection $\nabla$ are given by

\begin{align}
\bar{R}(X, Y)Z & = R(X, Y)Z + (2 + \epsilon)[g(X, Z)Y - g(Y, Z)X] \\
& + (1 + \epsilon)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \\
& + (1 + \epsilon)\eta(Z)\eta(Y)X - \eta(X)Y, \\
\bar{S}(Y, Z) & = S(Y, Z) + [(\epsilon + 2)\eta(Y) + 2]g(Y, Z) \\
& + (1 + \epsilon)(n-2\epsilon)\eta(Y)\eta(Z), \\
\bar{r} & = r + n[(\epsilon + 2)\eta(Y) + 2] + \epsilon(n + 2\epsilon).
\end{align}

3. Quasi-conformally flat $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection

**Definition 3.1.** An $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection is said to be quasi-conformally flat if $\bar{C}(X, Y)Z = 0$. 
Suppose $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection is quasi-conformally flat. Then from (1.1) we have
\[
a \bar{R}(X, Y)Z = b[S(X, Z)Y - S(Y, Z)X + g(X, Z)\bar{Q}Y - g(Y, Z)\bar{Q}X]
\]
\[
+ \frac{\bar{r}}{n} \left\{ \frac{a}{n-1} + 2b \right\} [g(Y, Z)X - g(X, Z)Y].
\]
(3.1)

Taking an inner product of the above equation with $\xi$, we get
\[
a g(R(X, Y)Z, \xi) = a\epsilon\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} + b[\epsilon S(X, Z)\eta(Y)
\]
\[
- \epsilon S(Y, Z)\eta(X) + \{g(X, Z)Y - g(Y, Z)X\} \{4\epsilon - 3n\epsilon - 2n + 3\}]
\]
\[
+ \epsilon \left\{ \frac{\bar{r}}{n} \left\{ \frac{a}{n-1} + 2b \right\} [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \right\}.
\]
(3.2)

Setting $X = \xi$ in (3.2) and then using (2.2), (2.3), (2.10) and (2.12), we obtain
\[
S(Y, Z) = Ag(Y, Z) + B\eta(Y)\eta(Z),
\]
(3.3)

where
\[
A = \frac{2a}{b} - (4 - 3n) + (2n - 3)\epsilon
\]
\[
+ \left\{ \frac{r + n[(\epsilon + 2)(\epsilon - n) + 2] + \epsilon(1 + \epsilon)(n - 2\epsilon)}{n} \right\} \left\{ \frac{a}{n-1} + 2b \right\}
\]

and
\[
B = -\frac{2a}{b} + (4 - 3n)(1 + \epsilon)
\]
\[
+ \left\{ \frac{\epsilon(r + n[(\epsilon + 2)(\epsilon - n) + 2] + \epsilon(1 + \epsilon)(n - 2\epsilon))}{n} \right\} \left\{ \frac{a}{n-1} + 2b \right\}.
\]

Thus, we can state the following;

**Theorem 3.2.** A quasi conformally flat $n$-dimensional $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection is an $\eta$-Einstein manifold.

4. Quasi-conformally semisymmetric $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection

**Theorem 4.1.** A quasi-conformally semisymmetric $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection is an $\eta$-Einstein manifold.

**Proof.** Suppose $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection satisfies
\[
\bar{R}(\xi, Y) \cdot \bar{C}(U, V)W = 0.
\]
Which implies
\[ \ddot{R}(\xi, Y)\bar{C}(U, V)W - \bar{C}(\ddot{R}(\xi, Y)U, V)W \]
\[ - \bar{C}(U, \ddot{R}(\xi, Y)V)W - \bar{C}(U, V)\ddot{R}(\xi, Y)W = 0. \]

By virtue of (2.15), (4.1) takes the form
\[ (1 + \epsilon)[\eta(\bar{C}(U, V)W)Y - g(Y, \bar{C}(U, V)W)\xi - \eta(U)\bar{C}(Y, V)W + g(Y, U)\bar{C}(U, Y)W - \eta(W)\bar{C}(U, Y)Y + g(Y, W)\bar{C}(U, V)\xi]. \]

Replacing \( Y \) by \( U \) in the above equation and then taking inner product with \( \xi \), one can obtain
\[ (1 + \epsilon)[g(U, \bar{C}(U, V)W) - g(U, U)\eta(\bar{C}(\xi, V)W) - \epsilon g(U, V)g(\bar{C}(\xi, W)\xi, U) - \epsilon\eta(W)g(\bar{C}(U, V)\xi, U)]. \]

Now putting \( U = e_i \) in (4.3), where \( \{e_i\}, i = 1, 2, \cdots, n \) is an orthonormal basis of the tangent space at each point of the manifold and sum up with respect to \( i \) and using (1.1), (2.11), (2.12), (2.15), (2.16) and (2.17), we get
\[ S(V, W) = \alpha g(V, W) + \beta \eta(V)\eta(W), \]
where \( \alpha = -[(\epsilon + 2)(\epsilon - n) + 2] + \frac{1}{(a+b)}[(a + b(n - 2))\frac{r}{n} - (n - 1)\{(a + (n - 1)b)(1 + \epsilon) - \frac{r}{n(n-1)}(a + 2b(n - 1))\}] \)
and \( \beta = -(\epsilon + 1)(n - 2\epsilon) + \frac{1}{(a+b)}[(a + 2b(n - 1))\frac{r}{n(n-1)} - \epsilon\{(a + (n - 2)b)\{(n - 1)(1 + \epsilon) + \frac{r}{n}\}\}]. \)

Hence the proof.

5. \( \phi \)-concircularly flat \((\epsilon)\)-Kenmotsu manifold admitting a

semi-symmetric metric connection

**Definition 5.1.** An \((\epsilon)\)-Kenmotsu manifold admitting a semi-symmetric metric connection is said to be \( \phi \)-concircularly flat if \( \bar{Z}(\phi X, \phi Y)\phi Z = 0. \)

Assume that \((\epsilon)\)-Kenmotsu manifold admitting a semi-symmetric metric connection is \( \phi \)-concircularly flat. Then from (1.2) we have
\[ g(R(\phi X, \phi Y)\phi Z, \phi W) \]
\[ = \frac{r}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)]. \]

Let \( \{e_1, \cdots, e_{n-1}, \xi\} \) be a local orthonormal basis of vector fields in \( M \). By using the fact that \( \{\phi e_1, \cdots, \phi e_{n-1}, \xi\} \) is also a local orthonormal basis, if we
put $X = W = e_i$ in (5.1) and sum up with respect to $i$, $1 \leq i \leq n - 1$, we get

$$\sum_{i=1}^{n-1} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{r}{n(n-1)} \sum_{i=1}^{n-1} [g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)].$$

(5.2)

It is easy to see that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) = S(Y, Z) + g(Y, Z),$$

(5.3)

$$\sum_{i=1}^{n-1} g(\phi e_i, Y)S(\phi e_i, Z) = S(Y, Z),$$

(5.4)

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1,$$

(5.5)

$$\sum_{i=1}^{n-1} g(\phi e_i, Y)g(Z, \phi e_i) = g(Y, Z).$$

(5.6)

And by making use of (5.3)-(5.6), the equation (5.2) turns into

$$S(\phi Y, \phi Z) + g(\phi Y, \phi Z) = [2 + \epsilon + \frac{r}{n(n-1)}](n - 2)g(\phi Y, \phi Z).$$

(5.7)

Thus, by applying (2.5) and (2.13) into (5.7), we get

$$S(Y, Z)$$

$$= [-1 + (2 + \epsilon + \frac{r + n[(\epsilon + 2)(\epsilon - n) + 2] + \epsilon(1 + \epsilon)(n - 2\epsilon)}{n(n-1)})(n - 2)]g(Y, Z)$$

$$- [3 + \epsilon + \frac{r + n[(\epsilon + 2)(\epsilon - n) + 2] + \epsilon(1 + \epsilon)(n - 2\epsilon)}{n(n-1)}](n - 2)e\eta(Y)e\eta(Z).$$

(5.8)

Hence, we have the following:

**Theorem 5.2.** A $\phi$-concircularly flat $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection is an $\eta$-Einstein manifold.

6. $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $Z(X, Y) \cdot S(U, W) = 0$

**Theorem 6.1.** An $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $Z(X, Y) \cdot S(U, W) = 0$ is an $\eta$-Einstein manifold.
**Proof.** Let us assume that $(\epsilon)$-Kenmotsu manifold admitting a semi-symmetric metric connection satisfies

(6.1) \[ \mathcal{Z}(X, Y) \cdot \mathcal{S}(U, W) = 0. \]

Which implies that

(6.2) \[ \mathcal{S}(\mathcal{Z}(\xi, Y)U, W) + \mathcal{S}(U, \mathcal{Z}(\xi, Y)W) = 0. \]

Using (1.2), (2.11) and (2.15) in (6.2), one can get

(6.3) \[ - [(1 + \epsilon) + \frac{\bar{\epsilon}}{n(n-1)}]g(Y, Z)\mathcal{S}(\xi, U) + [(1 + \epsilon) + \frac{\bar{\epsilon}}{n(n-1)}]\eta(Z)\mathcal{S}(Y, U) \]

\[ - [(1+\epsilon)+\frac{\bar{\epsilon}}{n(n-1)}]g(Y, U)\mathcal{S}(Z, \xi)+[(1+\epsilon)+\frac{\bar{\epsilon}}{n(n-1)}]\eta(U)\mathcal{S}(Z, Y)=0. \]

Plugging $U = \xi$ in (6.3) and then taking into an account of (2.1), (2.3), (2.4), (2.12) and (2.16), we obtain

\[ \mathcal{S}(Y, Z) = \left\{ - ((2 + \epsilon)(\epsilon - n) + 2) + (n - 1)(1 + \epsilon) \right\}g(Y, Z) \]

- $\left(1 + \epsilon\right)(n - 2\epsilon)\eta(Y)\eta(Z).$ \[ \square \]

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**References**


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