POSITIVE IMPLICATIVE ENERGETIC SUBSETS OF
BCK-ALGEBRAS

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Abstract. The notion of PI-energetic subsets in BCK-algebras is introduced, and several properties are investigated. Characterizations of I-energetic subsets and PI-energetic subsets are discussed, and conditions for a subset to be an I-energetic subset and a PI-energetic subset are provided. Relations between I-energetic subsets and PI-energetic subsets are considered, and conditions for an I-energetic subset to be a PI-energetic subset are given. Using PI-energetic subset, a positive implicative ideal is constructed. The condensational property of PI-energetic subset is established.

Keywords: S-energetic subset, I-energetic subset, PI-energetic subset

1. Introduction

Jun et al. [2] introduced the notions of S-energetic subsets and I-energetic subsets in BCK/BCI-algebras, and investigated several properties.

In this paper, we introduce the notion of PI-energetic subsets in BCK-algebras, and investigate several properties. We consider characterizations of I-energetic subsets and PI-energetic subsets. We provide conditions for a subset to be an I-energetic subset and a PI-energetic subset. We discuss relations between I-energetic subsets and PI-energetic subsets. We give conditions for an I-energetic subset to be a PI-energetic subset. Using PI-energetic subset, we make a positive implicative ideal. We establish the condensational property of PI-energetic subset.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

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An algebra \((X; *, 0)\) of type \((2, 0)\) is called a \emph{BCI-algebra} if it satisfies the following conditions

\begin{enumerate}[(I)]
  \item \((\forall x, y, z \in X) ((x * y) * (x * z)) * (z * y) = 0)\),
  \item \((\forall x, y \in X) ((x * (x * y)) * y = 0)\),
  \item \((\forall x \in X) (x * x = 0)\),
  \item \((\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y)\).
\end{enumerate}

If a BCI-algebra \(X\) satisfies the following identity

\begin{enumerate}[(V)]
  \item \((\forall x \in X) (0 * x = 0)\),
\end{enumerate}
then \(X\) is called a \emph{BCK-algebra}. Any \emph{BCK/BCI}-algebra \(X\) satisfies the following conditions:

\begin{enumerate}[(2.1)]
  \item \((\forall x \in X) (x * 0 = x)\),
  \item \((\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)\),
  \item \((\forall x, y, z \in X) ((x * y) * z = (x * z) * y)\),
  \item \((\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y)\)
\end{enumerate}

where \(x \leq y\) if and only if \(x * y = 0\). A nonempty subset \(S\) of a \emph{BCK/BCI}-algebra \(X\) is called a \emph{subalgebra} of \(X\) if \(x * y \in S\) for all \(x, y \in S\). A subset \(I\) of a \emph{BCK/BCI}-algebra \(X\) is called an \emph{ideal} of \(X\) if it satisfies

\begin{enumerate}[(2.5)]
  \item \(0 \in I\),
  \item \((\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I)\).
\end{enumerate}

Every ideal \(I\) of a \emph{BCK/BCI}-algebra \(X\) satisfies the following condition:

\begin{enumerate}[(2.7)]
  \item \((\forall x, y \in X) (x \in I, y \leq x \Rightarrow y \in I)\).
\end{enumerate}

A subset \(I\) of a \emph{BCK}-algebra \(X\) is called a \emph{positive implicative ideal} (see [3]) of \(X\) if it satisfies (2.5) and

\begin{enumerate}[(2.8)]
  \item \((\forall x, y, z \in X) ((x * y) * z \in I, y * z \in I \Rightarrow x * z \in I)\).
\end{enumerate}

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [3]).

We refer the reader to the books [1, 3] for further information regarding \emph{BCK/BCI}-algebras.
3. Energetic subsets

In what follows, let $X$ denote a $BCK$-algebra unless otherwise specified.

**Definition 3.1** ([2]). A nonempty subset $A$ of $X$ is said to be $S$-energetic if it satisfies

$$\forall a, b \in X \ (a * b \in A \Rightarrow \{a, b\} \cap A \neq \emptyset).$$

**Definition 3.2** ([2]). A nonempty subset $A$ of $X$ is said to be $I$-energetic if it satisfies

$$\forall x, y \in X \ (y \in A \Rightarrow \{x, y * x\} \cap A \neq \emptyset).$$

**Proposition 3.3.** Every $I$-energetic subset $A$ of $X$ which does not contain zero element $0$ satisfies the following properties:

$$\forall x, y \in X \ (x \leq y, \ x \in A \Rightarrow y \in A).$$

**Proof.** Let $x \leq y$ for $x \in A$ and $y \in X$. Then

$$\{y, 0\} \cap A = \{y, x * y\} \cap A \neq \emptyset.$$ Since $0 \notin A$, it follows that $y \in A$.

**Theorem 3.4.** For a nonempty subset $A$ of $X$ which does not contain zero element $0$, the following are equivalent:

1. $A$ is $I$-energetic.

2. $$\forall x, y, z \in X \ (z * y \leq x, \ z \in A \Rightarrow \{x, y\} \cap A \neq \emptyset).$$

**Proof.** Assume that $A$ is $I$-energetic and let $z * y \leq x$ for $z \in A$ and $x, y \in X$. Then $\{y, z * y\} \cap A \neq \emptyset$ by (3.2). Hence $y \in A$ or $z * y \in A$. If $y \in A$, then clearly $\{x, y\} \cap A \neq \emptyset$. If $z * y \in A$, then $\{x, 0\} \cap A = \{x, (z * y) * x\} \cap A \neq \emptyset$ by (3.2). Since $0 \notin A$, it follows that $x \in A$ and so $\{x, y\} \cap A \neq \emptyset$.

Conversely, suppose that (2) is valid and let $x \in A$. Since $x * (x * y) \leq y$, it follows that $\{y, x * y\} \cap A \neq \emptyset$. Therefore $A$ is $I$-energetic.

**Theorem 3.5.** If a nonempty subset $A$ of $X$ satisfies the following condition:

$$\forall x, y, z \in X \ (x * y \in A \Rightarrow \{z, ((x * y) * z) \cap A \neq \emptyset),$$

then $A$ is $I$-energetic.

**Proof.** Assume that $x \in A$. Then $x * 0 = x \in A$ by (2.1), which implies from (2.1) and (3.4) that

$$\{z, x * z\} \cap A = \{z, ((x * 0) * z) \cap A \neq \emptyset$$

for all $z \in X$. Hence $A$ is $I$-energetic.
Lemma 3.6 ([2]). Let $A$ be a nonempty subset of $X$ with $0 \notin A$. If $A$ is I-energetic, then $X \setminus A$ is an ideal of $X$.

Using Theorem 3.5 and Lemma 3.6, we have the following corollary.

Corollary 3.7. Let $A$ be a nonempty subset of $X$ with $0 \notin A$. If $A$ satisfies the condition (3.4), then $X \setminus A$ is an ideal of $X$.

Definition 3.8. A nonempty subset $A$ of $X$ is said to be positive implicative energetic (briefly, $PI$-energetic) if it satisfies

\[(\forall x,y,z \in X) (x \ast z \in A \Rightarrow \{(x \ast y) \ast z, y \ast z \} \cap A \neq \emptyset).\]

Example 3.9. (1) Let $X = \{0, 1, 2, 3, 4\}$ be a $BCK$-algebra with the following Cayley table:

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</table>

It is routine to verify that $A := \{2, 4\}$ is a $PI$-energetic subset of $X$.

(2) Consider a $BCK$-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

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</table>

Then $A_1 := \{1, 2, 4\}$, $A_2 := \{2, 3, 4\}$ and $A_3 := \{2, 4\}$ are $PI$-energetic subsets of $X$.

Example 3.10. Let $X = \{0, 1, 2, 3, 4\}$ be a $BCK$-algebra with the following Cayley table:

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The set $A := \{0, 3\}$ is not a $PI$-energetic subset of $X$ since $3 \ast 2 = 3 \in A$ but
\{(3 \ast 1) \ast 2, 1 \ast 2\} \cap A = \emptyset.

**Proposition 3.11.** Let \( A \) be a PI-energetic subset of \( X \). If \( A \) does not contain zero element \( 0 \), then the following implication is valid.

\[(\forall x, y \in X) (x \ast y \in A \Rightarrow (x \ast y) \ast y \in A).\]

**Proof.** Let \( x, y \in X \) be such that \( x \ast y \in A \). Then

\[\{0, (x \ast y) \ast y\} \cap A = \{y \ast y, (x \ast y) \ast y\} \cap A \neq \emptyset\]

by (III) and (3.5). Since \( 0 \notin A \), it follows that \((x \ast y) \ast y \in A\).

We provide conditions for a subset to be PI-energetic.

**Theorem 3.12.** Let \( A \) be a nonempty subset of \( X \) which does not contain zero element \( 0 \). If \( A \) satisfies the condition (3.4), then \( A \) is PI-energetic.

**Proof.** Let \( x, z \in X \) be such that \( x \ast z \in A \). If \( A \) is not PI-energetic, then there exists \( y \in X \) such that

\[\{y \ast z, (x \ast y) \ast z\} \cap A = \emptyset.\]

Hence \( y \ast z \in X \setminus A \) and \((x \ast y) \ast z \in X \setminus A \). Since

\[(x \ast z) \ast (y \ast z) \leq (x \ast y) \ast z\]

and \( X \setminus A \) is an ideal of \( X \), it follows from (2.7) that \((x \ast z) \ast (y \ast z) \in X \setminus A \). Thus

\[\{y \ast z, ((x \ast z) \ast (y \ast z)) \cap A = \emptyset,\]

which is contradictory to the condition (3.4). Therefore

\[\{(x \ast y) \ast z, y \ast z\} \cap A \neq \emptyset\]

whenever \( x \ast z \in A \) for all \( x, y, z \in X \), and so \( A \) is PI-energetic.

**Theorem 3.13.** For any nonempty subset \( A \) of \( X \), if \( X \setminus A \) satisfies the condition (2.8), then \( A \) is PI-energetic.

**Proof.** Assume that \( A \) is not PI-energetic. Then for any \( x, z \in X \) with \( x \ast z \in A \), there exists \( y \in X \) such that \( \{(x \ast y) \ast z, y \ast z\} \cap A = \emptyset \). It follows that

\[(x \ast y) \ast z \in X \setminus A \text{ and } y \ast z \in X \setminus A.\]

Since \( X \setminus A \) satisfies the condition (2.8), we have \( x \ast z \in X \setminus A \), that is, \( x \ast z \notin A \). This is a contradiction, and so \( A \) is a PI-energetic subset of \( X \).

**Corollary 3.14.** For any nonempty subset \( A \) of \( X \), if \( X \setminus A \) satisfies the condition (2.8), then \( A \) is I-energetic.
Corollary 3.15. For any nonempty subset $A$ of $X$, if $X \setminus A$ is a positive implicative ideal of $X$, then $A$ is $PI$-energetic and so $I$-energetic.

The converse of Corollary 3.15 is not true in general as seen in the following example.

Example 3.16. Consider a $BCK$-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table:

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It is routine to verify that $A := \{0, 1\}$ is a $PI$-energetic subset of $X$, but $X \setminus A = \{2, 3\}$ is not a positive implicative ideal of $X$.

We consider relations between an $I$-energetic subset and a $PI$-energetic subset.

Theorem 3.17. Every $PI$-energetic subset is $I$-energetic.

Proof. Let $A$ be a $PI$-energetic subset of $X$. Assume that $x \in A$. Since $x \ast 0 = x \in A$, it follows from (2.1) and (3.5) that $\{y, x \ast y\}\cap A = \{y \ast 0, (x \ast y) \ast 0\}\cap A \neq \emptyset$, for all $y \in X$. Hence $A$ is an $I$-energetic subset of $X$. $\square$

The converse of Theorem 3.17 is not true as seen in the following example.

Example 3.18. Let $X = \{0, 1, 2, 3, 4\}$ be a $BCK$-algebra with the following Cayley table:

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The set $A := \{1, 2, 3\}$ is an $I$-energetic subset of $X$, but it is not $PI$-energetic since $3 \ast 2 = 1 \in A$ but $\{(3 \ast 2) \ast 2, 2 \ast 2\}\cap A = \emptyset$.

We know that the $I$-energetic subset $A := \{1, 2, 3\}$ in Example 3.18 does not satisfy the condition (3.6). We consider conditions for an $I$-energetic subset to be $PI$-energetic.

Theorem 3.19. Let $A$ be an $I$-energetic subset of $X$ which does not contain zero element $0$. If $A$ satisfies the condition (3.6), then $A$ is $PI$-energetic.
Proof. Let $x, z \in X$ be such that $x \ast z \in A$. We first show that

\begin{equation}
(x \ast z) \ast (y \ast z) \in A \Rightarrow (x \ast y) \ast z \in A
\end{equation}

for all $x, y, z \in X$. Let $(x \ast z) \ast (y \ast z) \in A$ and assume that $(x \ast y) \ast z \notin A$, that is, $(x \ast y) \ast z \in X \setminus A$. Since

\begin{equation}
((x \ast (y \ast z)) \ast z) \ast z \leq (x \ast y) \ast z
\end{equation}

and $X \setminus A$ is an ideal of $X$ (see Lemma 3.6), we have $((x \ast (y \ast z)) \ast z) \in X \setminus A$ by (2.7). It follows from (2.3) and (3.6) that

\begin{equation}
(x \ast z) \ast (y \ast z) = (x \ast (y \ast z)) \ast z \in X \setminus A,
\end{equation}

which is a contradiction. Hence (3.7) is valid. Now suppose that \{\$y \ast z$, $(x \ast y) \ast z\} \cap A = \emptyset$ for some $y \in X$. Then $y \ast z \in X \setminus A$ and $(x \ast y) \ast z \in X \setminus A$. It follows from (3.7) that $(x \ast z) \ast (y \ast z) \in X \setminus A$ and so that $x \ast z \in X \setminus A$. This is a contradiction, and so $\{y \ast z$, $(x \ast y) \ast z\} \cap A \neq \emptyset$ whenever $x \ast z \in A$ for all $x, y, z \in X$. Therefore $A$ is a PI-energetic subset of $X$.

Corollary 3.20. Every PI-energetic subset $A$ which does not contain zero element 0 satisfies the condition (3.7).

Proposition 3.21. Let $A$ be a nonempty subset of $X$ which does not contain zero element 0. If $A$ satisfies the condition (3.7), then the condition (3.4) is valid.

Proof. Let $x \ast y \in A$ for $x, y \in X$. Assume that \{\$z$, $(x \ast y) \ast z\} \cap A = \emptyset$ for some $z \in X$. Then $z \in X \setminus A$ and $((x \ast z) \ast y) \ast y = ((x \ast y) \ast y) \ast z \in X \setminus A$. It follows from (III), (2.1), (2.3) and (3.7) that $(x \ast y) \ast z = ((x \ast z) \ast y) \ast (y \ast y) \in X \setminus A$. Since $X \setminus A$ is an ideal of $X$, we have $x \ast y \in X \setminus A$, which is a contradiction. Therefore (3.4) is valid.

Corollary 3.22. Let $A$ be an I-energetic subset of $X$ which does not contain zero element 0. If $A$ satisfies the condition (3.6), then $A$ also satisfies the condition (3.4).

For any nonempty subset $A$ of $X$, consider a set

\begin{equation}
E_a := \{x \in X \mid x \ast a \in A\}.
\end{equation}

Theorem 3.23. Let $A$ be an I-energetic subset of $X$. Then $A$ is PI-energetic if and only if $E_a$ is an I-energetic subset of $X$ for all $a \in X$.

Proof. Assume that $A$ is PI-energetic and let $y \in E_a$. Then $y \ast a \in A$, and so $\{x \ast a$, $(y \ast x) \ast a\} \cap A \neq \emptyset$ by (3.6). Thus $x \ast a \in A$ or $(y \ast x) \ast a \in A$, that is, $x \in E_a$ or $y \ast x \in E_a$. It follows that $\{x$, $y \ast x\} \cap E_a \neq \emptyset$. Therefore $E_a$ is an I-energetic subset of $X$ for all $a \in X$. 

Conversely, suppose that $E_a$ is an $I$-energetic subset of $X$ for all $a \in X$. Let $x \ast z \in A$ for $x, z \in X$. Then $x \in E_z$, and so $\{y, x \ast y\} \cap E_z \neq \emptyset$. It follows that $y \in E_z$ or $x \ast y \in E_z$ and so that $y \ast z \in A$ or $(x \ast y) \ast z \in A$. Hence \{y \ast z, (x \ast y) \ast z\} \cap A \neq \emptyset$, and therefore $A$ is $PI$-energetic.

Given a $PI$-energetic subset $A$ of $X$, we provide a condition that $X \setminus A$ is a positive implicative ideal of $X$.

**Theorem 3.24.** Let $A$ be a nonempty subset of $X$ with $0 \notin A$. If $A$ is $PI$-energetic, then $X \setminus A$ is a positive implicative ideal of $X$.

**Proof.** Since $0 \notin A$, we have $0 \in X \setminus A$. Let $x, y, z \in X$ be such that $y \ast z \in X \setminus A$ and $(x \ast y) \ast z \in X \setminus A$. Assume that $x \ast z \in A$. Then $(x \ast y) \ast z \cap A \neq \emptyset$ by (3.5), which implies that $y \ast z \in A$ or $(x \ast y) \ast z \in A$. This is a contradiction, and so $x \ast z \in X \setminus A$. This shows that $X \setminus A$ is a positive implicative ideal of $X$. \qed

**Corollary 3.25.** Let $A$ be a nonempty subset of $X$ with $0 \notin A$. If $A$ is $PI$-energetic, then $X \setminus A$ is an ideal and hence a subalgebra of $X$.

Let $A$ be an $I$-energetic subset and $B$ a $PI$-energetic subset of $X$ such that $A \subseteq B$ and $0 \notin B$. Assume that $(x \ast z) \ast (y \ast z) \in A$ for $x, y, z \in X$. Then $(x \ast z) \ast (y \ast z) \in B$ and so $(x \ast y) \ast z \in B$ by Corollary 3.20. If $(x \ast y) \ast z \notin A$, then $((x \ast ((x \ast y) \ast z)) \ast y) \ast z = ((x \ast y) \ast z) \ast ((x \ast y) \ast z) = 0 \in X \setminus B$ by (2.3) and (III). Since $X \setminus B$ is a positive implicative ideal of $X$, it follows that $((x \ast z) \ast (y \ast z)) \ast ((x \ast y) \ast z) = ((x \ast ((x \ast y) \ast z)) \ast (y \ast z) \in X \setminus B \subseteq X \setminus A$. Since $X \setminus A$ is an ideal of $X$, we have $(x \ast z) \ast (y \ast z) \in X \setminus A$ which is a contradiction. Therefore $(x \ast y) \ast z \in A$, and so $A$ satisfies the condition (3.7). Consequently, $A$ is a $PI$-energetic subset of $X$.

We summarize this as a theorem, so called the Condensational Property of $PI$-energetic set.

**Theorem 3.26 (Condensational Property of $PI$-energetic set).** Let $A$ and $B$ be $I$-energetic subsets of $X$ such that $A \subseteq B$ and $0 \notin B$. If $B$ is $PI$-energetic, then so is $A$.

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**References**


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