POINTWISE SLANT SUBMERSIONS FROM KENMOTSU MANIFOLDS INTO RIEMANNIAN MANIFOLDS

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Abstract. The purpose of this paper is to study pointwise slant submersions from Kenmotsu manifolds onto Riemannian manifolds admitting vertical and horizontal structure vector fields and find some results related to totally geodesic and harmonic properties.
Keywords: Riemannian submersion, almost contact manifold, pointwise slant submersion

1. Introduction

Immersions and submersions are special tools in Differential Geometry. Both play important role in Riemannian Geometry. O'Neill [19] and Gray [14] introduced Riemannian submersions between Riemannian manifolds. Submersions between Riemannian manifolds equipped with an additional structure of almost contact type on total space, firstly studied by Watson [23] and Chinea [11] independently. We know that Riemannian submersions are related to Mathematical Physics and have their applications in the Kaluza-Klein theory ([7], [16]) and the Yang-Mills theory [6] etc.

On the other hand, submersions have been studied by several authors. Some related research papers are: Geometry of slant submanifolds [10], Slant submersions from almost Hermitian manifolds [20], Slant submanifolds of Lorentzian Sasakian and para Sasakian manifolds [1], Riemannian submersions from almost

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contact metric manifolds [15], On quasi-slant submanifolds of an almost Hermitian manifold [12], Slant submanifolds in Sasakian manifolds [8], Pointwise slant submanifolds in almost Hermitian manifolds [9], Almost contact metric submersions [11], Riemannian Submersions and Related Topics [13], Pointwise slant submersions [17], Slant submanifolds of a Riemannian product manifold [2], Slant submanifolds in contact geometry [18], Point-wise slant submanifolds in almost contact geometry [3], Pointwise slant submersions from Cosymplectic manifolds [21] etc.

In this paper, we study pointwise slant submersions from Kenmotsu manifolds onto Riemannian manifolds. The paper is organized as follows. In section 2, we collect main notions and formulae which are needed for this paper. In section 3, we obtain some results of pointwise slant submersions from Kenmotsu manifolds onto Riemannian manifolds admitting vertical and horizontal structure vector fields.

2. Preliminaries

Let M be an almost contact metric manifold. So there exist on M, a $(1,1)$ tensor field ϕ, a vector field ξ, a $1-$form η and a Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \eta(\xi) = 1, \tag{2.2}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(\phi X, Y) = -g(X, \phi Y), \tag{2.3}$$

for any vector fields X and Y on M and I is the identity tensor field [5]. An almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) is denoted by (M, ϕ, ξ, η, g).

An almost contact metric manifold M is called a Kenmotsu manifold if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection of the Riemannian metric g. If (M, ϕ, ξ, η, g) be a Kenmotsu manifold, then the following equation holds:

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.5}$$

Let M be an $m-$dimensional Riemannian manifold and N be an $n-$dimensional Riemannian manifold $(m > n)$ with Riemannian metrics g_M and g_N respectively. Let $f : (M, g_M) \to (N, g_N)$ be a C^∞ map. We denote the kernel space of f_* by $\ker f_*$ and consider the orthogonal complementary space $\mathcal{H} = (\ker f_*)^\perp$ to $\ker f_*$. Then the tangent bundle of M has the following decomposition

$$TM = (\ker f_*) \oplus (\ker f_*)^\perp. \tag{2.6}$$

We also denote the *range* of f_* by $rangef_*$ and consider the orthogonal complementary space $(rangef_*)^\perp$ to $rangef_*$ in the tangent bundle TN of N. Thus the tangent bundle TN of N has the following decomposition

$$TN = (rangef_*) \oplus (rangef_*)^\perp. \tag{2.7}$$

A Riemannian submersion f is a C^∞ map from Riemannian manifold (M, g_M) onto (N, g_N) satisfying the following conditions:
(i) f has the maximal rank,
(ii) The differential f_* preserves the lengths of horizontal vectors.
For each $x \in N$, $f^{-1}(x)$ is fiber which is a $(m-n)$ dimensional submanifold of M. If a vector field on M is always tangent (resp. orthogonal) to fibers, then it is called vertical (resp. horizontal). A vector field X on M is said to basic if it is horizontal and $f-$related to a vector field X_* on N, i.e., $f_*X_p = X_{*f(p)}$ for all $p \in M$. We denote the projection morphisms on the distributions $\ker f_*$ and $(\ker f_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$ respectively.

A smooth map $f : (M, g_M) \to (N, g_N)$ between Riemannian manifolds is a Riemannian submersion if and only if

$$g_M(U, V) = g_N(f_*U, f_*V), \tag{2.8}$$

for every $U, V \in (\ker f_*)^\perp$.

The O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ define by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.9}$$

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.10}$$

for arbitrary vector fields E and F on M, where ∇ is the Riemannian connection on M [19].

Lemma 1. *Let f be a Riemannian submersion between Riemannian manifolds (M, g_M) and (N, g_N). If X and Y are basic vector fields on M, then:*

(i) $g_M(X, Y) = g_N(f_*X, f_*Y)$,

(ii) *the horizontal part $[X, Y]^{\mathcal{H}}$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$ i.e., $f_*([X, Y]^{\mathcal{H}}) = [X_*, Y_*]$,*

(iii) $[V, X]$ *is vertical for any vector field V of* $\ker f_*$,

(iv) $(\nabla^M_X Y)^{\mathcal{H}}$ *is vertical for any vector field corresponding to $\nabla^N_{X_*} Y_*$ where ∇^M and ∇^N are the Riemannian connection on M and N respectively.*

Now, from equations (2.9) and (2.10), we get

$$\nabla_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y, \tag{2.11}$$

$$\nabla_X V = \mathcal{H}\nabla_X V + \mathcal{T}_X V, \tag{2.12}$$

$$\nabla_V X = \mathcal{A}_V X + \mathcal{V}\nabla_V X, \tag{2.13}$$

$$\nabla_V W = \mathcal{H}\nabla_V W + \mathcal{A}_V W, \tag{2.14}$$

for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, where $\widehat{\nabla}_X Y = \mathcal{V}\nabla_X Y$. Moreover, if V is basic, then $\mathcal{H}\nabla_X V = \mathcal{A}_V X$.

On the other hand, for any $E \in \Gamma(TM)$, it is seen that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$.

The tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy the equations:

$$\mathcal{T}_X Y = \mathcal{T}_Y X, \tag{2.15}$$

$$\mathcal{A}_V W = -\mathcal{A}_W V = \frac{1}{2}\mathcal{V}[V, W], \tag{2.16}$$

for $X, Y \in \Gamma(\ker F_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$.

It can be easily seen that a Riemannian submersion $f : (M, g_M) \to (N, g_N)$ has totally geodesic fibers if and only if $\mathcal{T}$ identically vanishes.

Now, we consider the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that f is a C^∞ mapping between them. Then the differential f_* of f can be considered as a section of the bundle $Hom(TM, f^{-1}TN) \to M$, where $f^{-1}TN$ is the pullback bundle that has fibers $(f^{-1}TN)_q = T_{f(q)}N, q \in M$. If $Hom(TM, f^{-1}TN)$ has a connection ∇ induced from the Riemannian connection ∇^M, then the second fundamental form of f is given by

$$(\nabla f_*)(X, Y) = \nabla^f_X f_* Y - f_*(\nabla^M_X Y), \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$, where ∇^f is the pullback connection.

If f is a Riemannian submersion, then we can easily see that

$$(\nabla f_*)(V, W) = 0, \tag{2.18}$$

for any $V, W \in \Gamma(\ker f_*)^\perp$ [4].

3. The pointwise slant submersions from almost contact metric manifolds

Let f be a Riemannian submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). If for each $x \in M$, the angle $\theta(X)$ between ϕX and the space $\ker f_*$ is independent of the choice of the non-zero vector field $X \in \Gamma(\ker f_*) - \{\xi\}$, then f is called a pointwise slant submersion and the angle θ is said to be slant function of the pointwise slant submersion.

A pointwise slant submersion is called slant if its slant function θ is independent of the choice of the point on $(M, \phi, \xi, \eta, g_M)$. Then the constant θ is called the slant angle of the slant submersion [21].

3.1 Pointwise slant submersion for $\xi \in \Gamma(\ker f_*)$

Let f be a Riemannian submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N).

For any $X \in \Gamma(\ker f_*)$, we get

$$\phi X = \psi X + \omega X, \tag{3.1}$$

where ψX and ωX are vertical and horizontal components of ϕX respectively.

For any $V \in \Gamma(\ker f_*)^\perp$, we have

$$\phi V = BV + CV, \tag{3.2}$$

where BV and CV are vertical and horizontal components of ϕV respectively

By using equations $(2.3), (3.1)$ and (3.2), we get

$$g_M(\psi X, Y) = -g_M(X, \psi Y), \tag{3.3}$$

$$g_M(\omega X, V) = -g_M(X, BV), \tag{3.4}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$.

Again using the equations $(2.5), (2.11), (2.13), (3.1)$ and (3.2), we get

$$\widehat{\nabla}_X \xi = X - \eta(X)\xi, \mathcal{T}_X \xi = 0, \tag{3.5}$$

for any $X \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$.

For any $X, Y \in \Gamma(\ker f_*)$, define

$$(\nabla_X \psi)Y = \widehat{\nabla}_X \psi Y - \psi \widehat{\nabla}_X Y, \tag{3.6}$$

and

$$(\nabla_X \omega)Y = \mathcal{H}\nabla_X \omega Y - \omega \widehat{\nabla}_X Y, \tag{3.7}$$

where ∇ is the Riemannian connection on M. Next, we say that the ω is parallel if

$$(\nabla_X \omega)Y = 0. \tag{3.8}$$

Then we easily have:

Lemma 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a pointwise slant submersion, then*

$$(\nabla_X \psi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y - g(\psi X, Y)\xi + \eta(Y)\psi X, \tag{3.9}$$

and

$$(\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \psi Y + \eta(Y)\omega X, \tag{3.10}$$

for any $X, Y \in \Gamma(\ker f_*)$.

In the same way with the proof of Theorem 1 in [21], we have the following theorem:

Theorem 1. *If* $(M, \phi, \xi, \eta, g_M)$ *be a Kenmotsu manifold and* (N, g_N) *be a Riemannian manifold. If* $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ *is a pointwise slant submersion, then*

$$\psi^2 = \cos^2\theta(-I + \eta \otimes \xi).$$

Simlarly, as in the proof of Lemma 2 *in* [21], *we have*

Corollary 1. Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a pointwise slant submersion, then we have

$$g_M(\psi X, \psi Y) = \cos^2\theta(g_M(X, Y) - \eta(X)\eta(Y)),$$
$$g_M(\omega X, \omega Y) = \sin^2\theta(g_M(X, Y) - \eta(X)\eta(Y)),$$

for any $X, Y \in \Gamma(\ker f_*)$.

In the same way with the proof of Lemma 3 in [21], we can state the following theorem:

Theorem 2. *Let* $(M, \phi, \xi, \eta, g_M)$ *be a Kenmotsu manifold and* (N, g_N) *be a Riemannian manifold. Assume that* $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ *is a pointwise slant submersion. If* ω *is parallel, then we have*

$$T_{\psi X}\psi X = \cos^2\theta(T_X X - \eta(X)\omega\psi X),$$

for any $X \in \Gamma(\ker f_*)$.

Let f be a C^∞ map from Riemannian manifold (M, g_M) onto (N, g_N), then the adjoint ${}^*f_*$ map of f_* is characterized by

$$g_M(X, {}^*f_{*q}Y) = g_N(f_{*q}X, Y), \tag{3.11}$$

for any $X \in T_qM, Y \in T_{f(q)}N$ and $q \in M$. For each $q \in M$, f_*^h is a C^∞ map defined by

$$f_*^h : ((\ker f_*)^\perp(q), g_M(\ker f_*)^\perp(q)) \to (range f_*(q), g_N(range f_*)(q)),$$

where denote the adjoint of f_*^h by ${}^*f_*^h$. Let ${}^*f_{*q}$ be the adjoint of f_{*q} that is defined by $f_{*q} : (T_qM, g_M) \to (T_{f(q)}N, g_N)$.

The linear transformation $({}^*f_*)^h : range f_*(q) \to (\ker f_*)^\perp(q)$, defined as $({}^*f_*)^h Y = {}^* f_* Y$, where $Y \in \Gamma(range f_*)$, is an isomorphism and $(f_{*q}^h)^{-1} = ({}^*f_{*q})^h = {}^*(f_{*q}^h)$.

Theorem 3. *Let* $(M, \phi, \xi, \eta, g_M)$ *be a Kenmotsu manifold and* (N, g_N) *be a Riemannian manifold. If* $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ *is a pointwise slant* *submersion with non-zero slant function θ, then the fibers are totally geodesic submanifolds of M if and only if*

$$\begin{aligned}&g_N(\nabla^N_{V'} f_*(\omega X), f_*(\omega Y))\\&= -g_M([X,V],Y)\sin^2\theta + V(\theta) g_M(\phi X, \phi Y)\sin 2\theta + g_M(\mathcal{A}_V \omega\psi X, Y)\\&- g_M(\mathcal{A}_V \omega X, \psi Y) - \eta(Y) g_M(BV, \psi X) - \eta(\nabla_V X)\eta(Y)\sin^2\theta,\end{aligned}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$, where V and V' are $f-$related vector fields and ∇^N is the Riemannian connection on N.

Proof. For any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$, using equations (2.3), (2.4), (2.11) and (3.1), we get

$$\begin{aligned}g_M(T_X Y, V) &= -g_M([X,V],Y) + g_M(\nabla_V \psi^2 X, Y) + g_M(\nabla_V \omega\psi X, Y)\\&\quad -g_M(\nabla_V \omega X, \phi Y) - \eta(\nabla_V X)\eta(Y).\end{aligned}$$

From theorem 1 and using equations (2.11), (2.14) and (3.2), we get

$$\begin{aligned}&g_M(T_X Y, V)\sin^2\theta\\&= -g_M([X,V],Y)\sin^2\theta + V(\theta) g_M(\phi X, \phi Y)\sin 2\theta + g_M(\mathcal{A}_V \omega\psi X, Y)\\&- g_N(\nabla^N_{V'} f_*(\omega X), f_*(\omega Y)) - g_M(\mathcal{A}_V \omega X, \psi Y)\\&- \eta(\nabla_V X)\eta(Y))\sin^2\theta - \eta(Y) g_M(BV, \psi X).\end{aligned}$$

By considering the fibers as totally geodesic, we derive the formula in the above theorem. Conversely, it can be directly verified. □

Theorem 4. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a pointwise slant submersion with non-zero slant function θ, then f is a totally geodesic map if and only if*

$$\begin{aligned}&g_N(\nabla^N_{V'} f_*(\omega X), f_*(\omega Y))\\&= -g_M([X,V],Y)\sin^2\theta + V(\theta) g_M(\phi X, \phi Y)\sin 2\theta + g_M(\mathcal{A}_V \omega\psi X, Y)\\&- g_M(\mathcal{A}_V \omega X, \psi Y) - \eta(Y) g_M(BV, \psi X) - \eta(\nabla_V X)\eta(Y)\sin^2\theta,\end{aligned}$$

and

$$\begin{aligned}g_M(\mathcal{A}_V \omega X, BW) &= g_N(\nabla^f_V f_*(\omega\psi X), f_*(W)) - g_N(\nabla^f_{V'} f_*(\omega X), f_*(CW))\\&\quad - \eta(X) g_M(V, W)\sin^2\theta,\end{aligned}$$

for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, where V and V' are $f-$related vector fields and ∇^f is the pullback connection along f.

Proof. By definition, it follows that f is totally geodesic if and only if $(\nabla f_*)(X,Y) = 0$, for any $X, Y \in \Gamma(TM)$.

From theorem 3, we obtain the first equation. On the other hand, for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, using equations (2.3), (2.4) and (3.1) we get

$$g_M(\nabla_V X, W) = -g_M(\nabla_V \psi^2 X, W) - g_M(\nabla_V \omega\psi X, W) + g_M(\nabla_V \omega X, \phi W) + \eta(X) g_M(V, W).$$

From theorem 1 and using equations (2.8), (2.11), (2.14), (2.17) and (3.2), we get

$$\begin{aligned} & g_N((\nabla f_*)(V,X), f_*(W)) \sin^2\theta \\ & = -g_N(\nabla^f_V f_*(\omega X), f_*(W)) + g_N(\nabla^f_V f_*(\omega X), f_*(CW)) \\ & + g_M(A_V \omega X, BW) + \eta(X) g_M(V,W) \sin^2\theta. \end{aligned}$$

Converse is obvious. □

Theorem 5. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a pointwise slant submersion with non-zero slant function θ, then f is harmonic if and only if*

$$trace^* f_*((\nabla f_*)((.)\omega\psi(.))) - trace T_{(.)}\omega(.) + trace C^* f_*(\nabla f_*)((.)\omega(.)) = 0.$$

Proof. For any $X \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$, using equations (2.1), (2.3), (2.11), (3.1) and (3.2), we get

$$g_M(\mathcal{T}_X X, V) = -g_M(\phi \nabla_X \psi X, V) + g_M(\nabla_X \omega X, \phi V).$$

From theorem 1 and using equations (2.3), (2.4) and (3.1), we get

$$g_M(\mathcal{T}_X X, V) = g_M(\nabla_X X, V)\cos^2\theta - g_M(\nabla_X \omega\psi X, V) + g_M(\nabla_X \omega X, \phi V).$$

Using equations (2.12), (2.17), (3.2) and (3.11), we have

$$\begin{aligned} g_M(\mathcal{T}_X X, V)\sin^2\theta &= g_N({}^* f_*(\nabla f_*)(X, \omega\psi X), V) - g_M(\omega \mathcal{T}_X \omega X, V) \\ & \quad - g_N(C^* f_*(\nabla f_*)(X, \omega X), V). \end{aligned}$$

Conversely, a direct computation gives the proof. □

3.2 Pointwise slant submersions for $\xi \in \Gamma((\ker f_*)^\perp)$

In this section, we give the basic equations of pointwise slant submersions from Kenmotsu manifolds onto Riemannian manifolds for $\xi \in \Gamma(\ker f_*)^\perp$.

From equations (2.1) and (2.2), we get

$$\phi^2 X = -X, \tag{3.12}$$

and

$$g(\phi X, \phi Y) = g(X, Y), \tag{3.13}$$

for any $X, Y \in \Gamma(\ker f_*)$. Moreover, from equations (2.12), (2.14), (2.5), (3.1) and (3.2), we get

$$T_X \xi = X, \tag{3.14}$$

$$A_V \xi = 0, \tag{3.15}$$

and

$$\eta(\nabla_X Y) = -g_M(X, Y), \tag{3.16}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^{\perp}$.

Theorem 6. *If $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a pointwise slant submersion, then*

$$\psi^2 = -(\cos^2 \theta) I.$$

Corollary 2. Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a pointwise slant submersion, then

$$g_M(\psi X, \psi Y) = \cos^2 \theta g_M(X, Y),$$

$$g_M(\omega X, \omega Y) = \sin^2 \theta (g_M(X, Y),$$

for any $X, Y \in \Gamma(\ker f_*)$.

Theorem 7. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Assume that $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a pointwise slant submersion with slant function θ. If ω is parallel, then*

$$T_{\psi X} \psi X = \cos^2 \theta T_X X,$$

for any $X \in \Gamma(\ker f_*)$.

Theorem 8. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a pointwise slant submersion with non-zero slant function θ, then the fibers are totally geodesic submanifolds of M if and only if*

$$\begin{aligned} & g_N(\nabla^N_{V'} f_*(\omega X), f_*(\omega Y)) \\ & = -g_M([X, V], Y) \sin^2 \theta + V(\theta) g_M(X, Y) \sin 2\theta + g_M(\mathcal{A}_V \omega \psi X, Y) \\ & - g_M(\mathcal{A}_V \omega X, \psi Y), \end{aligned}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^{\perp}$, where V and V' are $f-$related vector fields and ∇^N is the Riemannian connection on N.

Proof. For any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$, using equations (2.2), (2.3), (2.4), (2.11), (2.14), (3.1), (3.2), (2.14) and Theorem 6, we get

$$\begin{aligned} & g_M(\mathcal{T}_X Y, V)\sin^2\theta \\ & = -g_M([X,V],Y)\sin^2\theta + V(\theta)g_M(X,Y)\sin 2\theta + g_M(\mathcal{A}_V\omega\psi X, Y) \\ & - g_N(\nabla^N_{V'} f_*(\omega X), f_*(\omega Y)) - g_M(\mathcal{A}_V\omega X, \psi Y). \end{aligned}$$

By considering the fibers as totally geodesic, we derive the formula. Conversely, it can be directly verified. □

Theorem 9. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a pointwise slant submersion with non-zero slant function θ, then f is totally geodesic map if and only if*

$$\begin{aligned} & g_N(\nabla^N_{V'} f_*(\omega X), f_*(\omega Y)) \\ & = -g_M([X,V],Y)\sin^2\theta + V(\theta)g_M(X,Y)\sin 2\theta + g_M(\mathcal{A}_V\omega\psi X, Y) \\ & - g_M(\mathcal{A}_V\omega X, \psi Y), \end{aligned}$$

and

$$\begin{aligned} g_M(\mathcal{A}_V\omega X, BW) \;=\; & g_N(\nabla^f_V f_*(\omega\psi X), f_*(W)) - g_N(\nabla^f_V f_*(\omega X), f_*(CW)) \\ & -\eta(W)g_M(BV, \psi X), \end{aligned}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, where V and V' are $f-$related vector fields and ∇^f is the pullback connection along f.

Proof. By definition, it follows that f is totally geodesic if and only if $(\nabla f_*)(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$.

From Theorem 3, we obtain the first equation. On the other hand, for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, using equations (2.2), (2.3), (2.4), (2.10), (2.11), (3.1), (3.2), (2.14), and Theorem 6, we obtain

$$\begin{aligned} & g_N((\nabla f_*)(V, X), f_*(W))\sin^2\theta \\ & = -g_N(\nabla^f_V f_*(\omega X), f_*(W)) + g_N(\nabla^f_V f_*(\omega X), f_*(CW)) \\ & + g_M(\mathcal{A}_V\omega X, BW) + \eta(W)g_M(BV, \psi X). \end{aligned}$$

Conversely, it can be easily proved. □

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