SOME PROPERTIES OF NEAR LEFT ALMOST RINGS BY USING IDEALS

Fawad Hussain*
Department of Mathematics
AUST, Havelian
Abbottabad, KPK
Pakistan
fawadhussain998@hotmail.com

Zeenat Jadoon
Department of Mathematics
Hazara University
Mansehra, KPK
Pakistan
zjadoon85@yahoo.com

Saleem Abdullah
Department of Mathematics, Abdul Wali Khan University
Mardan, KPK
Pakistan
saleemabdullah81@yahoo.com

Nazia Sadiq
Department of Mathematics, Hazara University
Mansehra, KPK
Pakistan
nazia.sadiq06@gmail.com

Abstract. The aim of this paper is to characterize near left almost rings by using ideals. We define a fully idempotent near left almost ring and discuss some of their properties. We then define prime, fully prime and irreducible ideals in a near left almost ring and explore some of their properties. Lastly we define M-system, P-system and I-system in a near left almost ring and discuss some of their properties.

Keywords: Ideal, fully idempotent near left almost ring, M-system, P-system and I-system.

1. Introduction

The notion of a left almost semigroup was introduced by Kazim and Naseeruddin [5]. A groupoid $(G, *)$ is said to be a left almost semigroup, abbreviated as LA-semigroup, if its elements satisfy the left invertive law, that is $(y * x) * z = (z * x) * y \ \forall \ x, \ y, \ z \in G$. In general a left almost semigroups doesn't satisfy

$*$. Corresponding author

associative and commutative laws but, it possesses the properties which are usually valid for associative and commutative algebraic structures. Further in 1993, Kamran [6] extended the concept of an LA-semigroup to a left almost group (abbreviated as LA-group). An LA-semigroup $(G, *)$ is said to be an LA-group if G has a left identity element with respect to "$*$" and every element of G has a left inverse with respect to "$*$". It is non-associative and non-commutative algebraic structure and is the generalization of a commutative group. In 2006, Yusuf [13] introduced a left almost ring. A non-empty set R together with two binary operations "$+$" and "$\cdot$" is said to be a left almost ring (abbreviated as LA-ring) if $(R, +)$ is a left almost group, $(R, \cdot)$ is a left almost semigroup and distributive laws of "$\cdot$" over "$+$" hold. Further different peoples in [2], [3], [7], [8], [9] and [12] worked on an LA-ring and explored many useful properties of an LA-ring. In 2011, Shah, Rehman and Raees [11] gave the concept of a near left almost ring (abbreviated as nLA-ring). A non-empty set T together with two binary operations "$+$" and "$\cdot$" is said to be a near left almost ring if $(T, +)$ is a left almost group, $(T, \cdot)$ is a left almost semigroup and left distributive property of "$\cdot$" over "$+$" holds. It is the generalization of a left almost ring. The authors of [11] discussed many interesting properties of a near left almost ring. There is another interesting paper [10] in which the authors discussed direct sum of ideals in an nLA-ring. Further in the paper [4], the authors defined direct product of two near left almost rings and explored some useful properties. In this paper, we explore some properties of near left almost rings by using ideals.

2. Preliminaries

In this section, we discuss some of the basic definitions and fundamental results which will be used later. We start with the following definition of a near left almost subring (abbreviated as nLA-subring) which has been taken from [11].

Definition 2.1. *Let $(T, +, \cdot)$ be a near left almost ring and S a non-empty subset of T. If S is itself a near left almost ring under the same binary operations of T, then S is called a near left almost subring of T.*

Let us state some properties of near left almost subrings. The following result gives necessary and sufficient conditions for near left almost subrings. The result has been taken from [11].

Proposition 2.2. *Let S be a non-empty subset of a near left almost ring $(T, +, \cdot)$. Then, S is a near left almost subring of T if and only if $m - n \in S$ and $m \cdot n \in S$ for all $m,\ n \in S$.*

We are now going to define left (right) ideals. The following definition is taken from the source [11].

Definition 2.3. *A non-empty subset I of a near left almost ring T is called a left ideal of T if $a - b \in I$ and $t \cdot a \in I$ $\forall$ $a,\ b \in I$ and $\forall$ $t \in T$ and I is called a right ideal of T if $a - b \in I$ and $(i + s) \cdot t - s \cdot t \in I$ $\forall$ $a,\ b,\ i \in I$ and $\forall$ $s,\ t \in T$.*

Let us state some properties of ideals which have been taken from [11].

Proposition 2.4. *Assume that I and J are two left (right) ideals of a near left almost ring T. Then $I \cap J$ is also a left (right) ideal of T.*

We are now going to define sum and product of two ideals of a near left almost ring. The definitions have been taken from the paper [10].

Definition 2.5. *Let $(T, +, \cdot)$ be an nLA-ring. Let I and J be two ideals of T, then we define*

$$I + J = \{x + y : x \in I and\ y \in J\}.$$

Definition 2.6. *If I and J are two ideals of an nLA-ring $(T, +, \cdot)$, then the product of I and J is the set*

$$IJ = \{\sum_{finite} a_i b_i : a_i \in I, b_i \in J\}.$$

It follows from [10] that if I and J are ideals then both $I + J$ and IJ are ideals.

3. Basic properties of ideals

This section is concerned with the fundamental properties of ideals of a near left almost ring. The idea of these properties has come from the source [8] in which the author has done similar calculations for an LA-ring. We start with the result given below which is true for a left almost ring. Here we prove it for a near left almost ring.

Proposition 3.1. *Let $(T, +, \cdot)$ be a near left almost ring and "e" the left identity of T with respect to "$\cdot$". If I is a right ideal of T, then I is a left ideal of T.*

Proof. Suppose I is a right ideal, then I is a near left almost subring of T. Now let $t \in T$ and $c \in I$, then

$$tc = (et)c = (ct)e \in I \quad \because by\ left\ invertive\ law$$

It follows that I is a left ideal. $\square$

Lemma 3.2. *Let $(T, +, \cdot)$ be a near left almost ring and "e" the left identity of T with respect to "$\cdot$". Let $a,\ c,\ x,\ y \in T$ be such that $ax = cy$, then $xa = yc$.*

Proof. Let $ax = cy$ for all $a,\ c,\ x,\ y \in T$. Now $xa = (ex)a = (ax)e = (cy)e = (ey)c = yc$. $\square$

Proposition 3.3. *Let T be an LA-ring with left identity "e" and let N be a right ideal of T. Then the following results are equivalent:*

(i) $N = T$,

(*ii*) $e \in N$,

(*iii*) *N contain a unit,*

(*iv*) *N contains an element which is left invertible or right invertible.*

Proof. (*i*) $\implies$ (*ii*) Let us suppose that $N = T$, then $e \in T \implies e \in N$.

(*ii*) $\implies$ (*iii*) Let $e \in N$. Then obviously N contains a unit.

(*iii*) $\implies$ (*iv*) Suppose N contains a unit. Then N must contain an element that must be left as well as right invertible.

(*iv*) $\implies$ (*i*) Let N contain an element r such that r is left invertible or right invertible. But by the above lemma in nLA-rings left invertibility implies right invertibility and right invertibility implies left invertibility. Thus r is a unit. □

Proposition 3.4. *Let M be a proper ideal of an nLA-ring T with left identity "e", then $e \notin M$.*

Proof. Suppose that $e \in M$. Now let $t \in T$, then $t = et \in MT \subseteq M \implies T \subseteq M$ but $M \subseteq T \implies M = T$. This is a contradiction. Hence $e \notin M$. □

As we know that if T is an nLA-ring and N_1, N_2 are two left (right) ideals of T, then $N_1 \cap N_2$ is a left (right) ideal of T. The following example shows that if T is an nLA-ring and N_1, N_2 are two left (right) ideals of T, then $N_1 \cup N_2$ is not necessary to be a left (right) ideal of T.

Example 3.5. Let $T = \{r, s, t, u, v, w\}$. Define "+" and "·" in T as follows:

$+$	r	s	t	u	v	w	$\cdot$	r	s	t	u	v	w
r	r	s	t	u	v	w	r	r	r	r	r	r	r
s	w	r	s	t	u	v	s	r	s	t	u	v	w
t	v	w	r	s	t	u	t	r	t	v	r	t	v
u	u	v	w	r	s	t	u	r	u	r	u	r	u
v	t	u	v	w	r	s	v	r	v	t	r	v	t
w	s	t	u	v	w	r	w	r	w	v	u	t	s

Then according to [11], $(T, +, \cdot)$ is an nLA-ring. Now let $N_1 = \{r, t, v\}$ and $N_2 = \{r, u\}$, then it is easy to see that N_1 and N_2 are ideals but $N_1 \cup N_2$ is not an ideal because $t, u \in N_1 \cup N_2$ but $t + u = s$ and $s \notin N_1 \cup N_2$.

The following results shows that when the union of left (right) ideals of an nLA-ring becomes a left (right) ideal.

Theorem 3.6. *Let $\{Y_k : k \in \Lambda\}$ be a family of left (right) ideals of an nLA-ring T. If $Y_k \subseteq Y_{k+1}$ for all $k \in \Lambda$. Then $\cup_{k\in\Lambda} Y_k$ is an ideal of T.*

Proof. For left ideal, let $x, y \in \cup_{k\in\Lambda} Y_k$, then there exist $i, j \in \Lambda$ such that $x \in Y_i, y \in Y_j$. Let $i > j$, then $x, y \in Y_i$. Now since Y_i is a left ideal, so it follows that $x - y \in Y_i$ for some $i \in \Lambda \implies x - y \in \cup_{k\in\Lambda} Y_k$. Now let $a \in T$, then $a \cdot x \in Y_i$ for some $i \in \Lambda$ because each Y_i is a left ideal. This implies that $a \cdot x \in \cup_{k\in\Lambda} Y_k \implies \cup_{k\in\Lambda} Y_k$ is a left ideal.

For right ideal, let $x, y \in \cup_{k\in\Lambda} Y_k$, then there exist $i, j \in \Lambda$ such that $x \in Y_i, y \in Y_j$. Let $i > j$, then $x, y \in Y_i$. Now since Y_i is a right ideal, so it follows that $x - y \in Y_i$ for some $i \in \Lambda \implies x - y \in \cup_{k\in\Lambda} Y_k$. Now let $a, b \in T$, then $(x + a)b - ab \in Y_i$ for some $j \in \Lambda$ because each Y_i is a right ideal. This implies that $(x + a)b - ab \in \cup_{k\in\Lambda} Y_k \implies \cup_{k\in\Lambda} Y_k$ is a right ideal of T.

This completes the proof. □

4. Fully idempotent near left almost rings

This section is concerned with the fully idempotent near left almost ring. We start with the following definition and then explore some properties of the fully idempotent near left almost ring.

Definition 4.1. *An ideal M of an nLA-ring T is said to be idempotent if $M^2 = M$. If every ideal of the nLA-ring T is idempotent, then T is said to be fully idempotent.*

Let us describe some properties. The following result gives us equivalent conditions for a fully idempotent nLA-ring. The result is true in case of a ring. Here we prove it for an nLA-ring.

Proposition 4.2. *If T is an nLA-ring, then the following conditions are equivalent:*

(i) *T is fully idempotent.*

(ii) *For each pair of ideals M, N of $T, M \cap N = MN$.*

Proof. $(i) \implies (ii)$ As $MN \subseteq MT \subseteq M$ and also $MN \subseteq TN \subseteq N$, so it follows that $MN \subseteq M \cap N$. Conversely $M \cap N = (M \cap N)^2 \subseteq MN$. Hence $M \cap N = MN$.

$(ii) \implies (i)$ By (ii), we have $M \cap M = MM$ for each ideal M of T. Hence $M = MM = M^2$. This shows that T is fully idempotent. □

5. Prime, fully prime and irreducible ideals

In this section, we discuss prime, fully prime and irreducible ideals in an nLA-ring. We define the said terms and then discuss some elegant and intersecting results which give relations among the said terms. Firstly we are going to define prime and semiprime ideals.

Definition 5.1. *If T is an nLA-ring and M a two sided ideal of T, then M is known to be a prime ideal if for ideals I and J of T such that $IJ \subseteq M$ implies either $I \subseteq M$ or $J \subseteq M$. If each ideal of an nLA-ring T is prime, then it is called fully prime. M is said to be semiprime if for each ideal I of an nLA-ring T such that $I^2 \subseteq M \implies I \subseteq M$. If each ideal of an nLA-ring T is semiprime, then it is called fully semiprime.*

It is clear from the above definition that every prime ideal is semiprime. We are now going to define totally ordered set of an nLA-ring under inclusion.

Definition 5.2. *The set of two sided ideals of a near left almost ring T is known as totally ordered under inclusion if $\forall$ ideals M, N of T, either $M \subseteq N$ or $N \subseteq M$. The set of two sided ideals of T is shortly denoted by ideal(T).*

Let us state and prove some properties. The idea of these properties has come from the source [8] in which the author has done similar calculations for an LA-ring. Here we generalize his idea to an nLA-ring.

Theorem 5.3. *Suppose "e' is the left identity of an nLA-ring T, then it is fully prime if and only if each ideal of T is idempotent and the set ideal(T) is totally ordered under inclusion.*

Proof. Suppose T is fully prime and let M be an ideal of T. Now clearly $M^2 \subseteq M$. Also $MM \subseteq M^2 \implies M \subseteq M^2$. Therefore $M^2 = M$ and hence M is idempotent. Now let M, N be ideals of T, then $MN \subseteq M$ and $MN \subseteq N \implies MN \subseteq M \cap N$. As M and N are prime ideals, so clearly $M \cap N$ is also a prime ideal of T. Therefore either $M \subseteq M \cap N$ or $N \subseteq M \cap N \implies$ either $M \subseteq N$ or $N \subseteq M$. Therefore the set ideal(T) is totally ordered under inclusion.

On the other hand suppose each ideal of T is idempotent and ideal(T) is totally ordered under inclusion. Assume I, J and M are some ideals of T with $IJ \subseteq M$ such that $I \subseteq J$. As I is idempotent, then $I = I^2 = II \subseteq IJ \subseteq M$. This implies that $I \subseteq M$. Hence T is fully prime. □

Let us define strongly irreducible and irreducible ideals.

Definition 5.4. *An ideal M of an nLA-ring T is strongly irreducible if for ideals I and J of T such that $I \cap J \subseteq M \implies$ either $I \subseteq M$ or $J \subseteq M$. M is called irreducible if for ideals I and $J, M = I \cap J \implies$ either $M = I$ or $M = J$.*

It is clear from the above definition that every strongly irreducible ideal is irreducible. We now state and prove some results. The following results are true in case of a ring. Here we prove them for an nLA-ring.

Proposition 5.5. *An ideal M of an nLA-ring T is a prime ideal if and only if M is semiprime and strongly irreducible.*

Proof. If M is a prime ideal, then it must be semiprime. Let I and J be ideals of T and $I \cap J \subseteq M$. Now as $IJ \subseteq I \cap J \subseteq M$, so by primeness it follows that either $I \subseteq M$ or $J \subseteq M$. Hence I is strongly irreducible.

Conversely, suppose that M is an ideal of T which is both semiprime and strongly irreducible. If I and J are ideals of T, $IJ \subseteq M$, then $(I \cap J)^2 \subseteq IJ \subseteq M$. By semiprimeness, it follows that $I \cap J \subseteq M$. Thus by strongly irreducible it follows that either $I \subseteq M$ or $J \subseteq M$. Thus M is a prime ideal. □

Before starting the next result firstly we are going to state a result which is called Zorn's Lemma and will be used in the next result. The result has been taken from [1].

Lemma 5.6. *If A is a partially ordered set such that every chain in A has an upper bound in A, then A has at least one maximal element.*

Proposition 5.7. *Let a be an element of a near left almost ring T and I a two sided ideal of T such that $a \notin I$. Then there exist an irreducible ideal M of T such that $I \subseteq M$ and $a \notin M$.*

Proof. If $\{H_i : i \in \Lambda\}$ is a chain of ideals in T containing I and $a \notin I$, then $\cup H_{i\in\Lambda}$ is an ideal of T. Therefore by the above lemma, the set of all ideals of T not containing a has a maximal element M. Assume $M = J \cap K$ where J and K are ideals of T properly containing M. Then by choice of $M, a \in J$ and $a \in K \implies a \in J \cap K = M$, which is a contradiction. Hence M must be irreducible. □

Proposition 5.8. *Let T be an nLA-ring with left identity "e". Then an ideal M of T such that $M \neq T$ is the intersection of all irreducible ideals containing it.*

Proof. As $e \notin M$, by the above proposition, there exists an irreducible ideal of T containing M. Suppose N is the intersection of all irreducible ideals of T containing M, then $M \subseteq N$. We now show that $N \subseteq M$. For this let $a \notin M$, then again by the above proposition, there exists an irreducible ideals H of T containing M but not containing a. Thus $a \notin N$ and hence $M = N$. □

6. M-system, I-system and P-system in near left almost rings

In this section, we study M-system, I-system and P-system in an nLA-ring containing a left identity element. We define the said terms and relate these concepts through some results. We start with the following definitions.

Definition 6.1. *Let T be an nLA-ring with left identity "e" and $\emptyset \neq S \subseteq T$. Then S is called an M-system if $\forall\, x, y \in S$ there exist t in T such that $x(ty) \in S$.*

Definition 6.2. *Suppose "e" is a left identity element of an nLA-ring T, then the principal left ideal generated by an element x of T is defined by*

$$\langle x \rangle = Tx = \{tx : t \in T\}.$$

Definition 6.3. *Let T be an nLA-ring and N a left ideal of T. Then N is called a quasi-prime ideal if for left ideals I and J of T such that $IJ \subseteq N \implies$ either $I \subseteq N$ or $J \subseteq N$ and N is called quasi-semiprime if $I^2 \subseteq N \implies I \subseteq N$, where I is a left ideal of T. A left ideal N of an nLA-ring T is known to be quasi-strongly irreducible if for left ideals I and J of T such that $I \cap J \subseteq N \implies$ either $I \subseteq N$ or $J \subseteq N$. N is known to be quasi-irreducible if for left ideals I and $J, N = I \cap J \implies$ either $N = I$ or $N = J$.*

Let us state and prove some properties. The idea of these properties has come from the source [8] in which the author has done similar calculations for an LA-ring. We generalize his idea to an nLA-ring. The result given below gives us equivalent conditions for a quasi-prime ideal.

Lemma 6.4. *Let T be an nLA-ring having left identity "e" and N a left ideal of T. Then the under below conditions are equivalent:*

(i) *N is quasi-prime.*

(ii) *If I and J are any left ideals of T such that $IJ = \langle IJ \rangle \subseteq N \implies$ either $I \subseteq N$ or $J \subseteq N$.*

(iii) *Let $I \nsubseteq N$ and $J \nsubseteq N$, then $IJ \nsubseteq N$, where I and J are any left ideals of T.*

(iv) *If m_1, m_2 are element of T such that $m_1 \notin N$ and $m_2 \notin N$, then $\langle m_1 \rangle \langle m_2 \rangle \nsubseteq N$.*

(v) *If m_1, m_2 are elements of T satisfying $m_1(Tm_2) \subseteq N$, then either $m_1 \in N$ or $m_2 \in N$.*

Proof. $(i) \iff (ii)$ Let N be quasi-prime. By definition if $IJ = \langle IJ \rangle \subseteq N$, then it implies that either $I \subseteq N$ or $J \subseteq N$ for all left ideal I and J of T. Conversely suppose that $IJ = \langle IJ \rangle \subseteq N$ implies that $I \subseteq N$ or $J \subseteq N$. Then this implies that N is quasi-prime.

$(ii) \iff (iii)$ is trivial.

$(i) \implies (iv)$ Let $\langle m_1 \rangle \langle m_2 \rangle \subseteq N$ then either $\langle m_1 \rangle \subseteq N$ or $\langle m_2 \rangle \subseteq N \implies$ either $m_1 \in N$ or $m_2 \in N$. So, by contrapositive statement $m_1 \notin N$ and $m_2 \notin N$, then $\langle m_1 \rangle \langle m_2 \rangle \nsubseteq N$.

$(i) \iff (v)$ Let $m_1(Tm_2) \subseteq N$, then $T(m_1(Tm_2)) \subseteq TN \subseteq N$. Now consider

$T(m_1(Tm_2)) = (TT)(m_1(Tm_2))$

$= (Tm_1)(T(Tm_2)) \qquad \because$ *by medial law*

$= (Tm_1)((TT)(Tm_2))$
$= (Tm_1)((m_2T)(TT) \quad \because$ *by paramedial law*
$= (Tm_1)((TT)m_2) \quad \because$ *by left invertive law*
$= (Tm_1)(Tm_2) \subseteq N.$
Since Tm_1 and Tm_2 are left ideals, so either $m_1 \in N$ or $m_2 \in N$.

Conversely suppose $IJ \subseteq N$ where I and J are any left ideals of T. Let $I \nsubseteq N$, then there exist $a \in I$ such that $a \notin N$. Now $\forall\ b \in J$, we have $a(Tb) \subseteq I(TJ) \subseteq IJ \subseteq N \implies J \subseteq N$ and hence N is a quasi-prime ideal of T. □

Further we have the following result which gives equivalent condition for quasi-prime ideal and for M-system.

Theorem 6.5. *Let "e" be a left identity element of an nLA-ring T and N a left ideal of T, then N is quasi-prime if and only if $T \setminus N$ is an M-system.*

Proof. Suppose that N is a quasi-prime ideal. Assume $x, y \in T\setminus N \implies x \notin N$ and $y \notin N$. So by the above lemma, $x(Ty) \nsubseteq N \implies$ there exist $t \in T$ such that $x(ty) \notin N \implies x(ty) \in T \setminus N \implies T \setminus N$ is an M-system.

Conversely suppose $T \setminus N$ is an M-system. Let $x, y \in T$ be such that $x(Ty) \subseteq N$. Suppose that $x \notin N$ and $y \notin N \implies x, y \in T \setminus N$. Since $T \setminus N$ is an M-system, so there exist $t \in T$ such that $x(ty) \in T \setminus N \implies x(Ty) \nsubseteq N$, which is impossible. Hence $x \in N$ or $y \in N \implies N$ is a quasi-prime ideal. □

We are now going to define a P-system.

Definition 6.6. *Let T be an nLA-ring with left identity "e" and $\emptyset \neq S \subseteq T$. Then S is said to be a P-system if $\forall\ x \in S$, there exist $r \in T$ such that $x(rx) \in S$.*

It is clear from the definition that every M-system is an P-system. Let us state and prove some properties. Again the idea of these properties has come from the source [8]. The following result gives us equivalent conditions for a quasi-semiprime ideal.

Lemma 6.7. *Let T be an nLA-ring having left identity "e" and N a left ideal of T, then the under below conditions are equivalent:*

(i) *N is quasi-semiprime.*

(ii) *$I^2 = \langle I^2 \rangle \subseteq N \implies I \subseteq N$, where I is any left ideal of T.*

(iii) *For every left ideal I of T such that $I \nsubseteq N$ implies that $I^2 \nsubseteq N$.*

(iv) *Let $x \in T$ such that $\langle x \rangle^2 \subseteq N$, then this implies that $x \in N$.*

(v) *$\forall\ y \in T$ such that $y(Ty) \subseteq N$ implies that $y \in N$.*

Proof. $(i) \iff (ii)$ Let N be quasi-semiprime. Let $I^2 = \langle I^2 \rangle \subseteq N$, then by definition $I \subseteq N$, where I is any left ideal of T. Conversely, as $I^2 = \langle I^2 \rangle \subseteq N \implies I \subseteq N \implies N$ is quasi-semiprime.

$(ii) \iff (iii)$ is trivial.

$(i) \implies (iv)$ Let $\langle x \rangle^2 \subseteq N$. But by hypothesis N is quasi-semiprime $\implies \langle x \rangle \subseteq N \implies x \in N$.

$(iv) \implies (ii)$ For all left ideal I of T, let $I^2 = \langle I^2 \rangle \subseteq N$. If $x \in I$, then by $(iv), \langle x \rangle^2 \subseteq N \implies x \in N \implies I \subseteq N$.

$(i) \iff (v)$ clear. □

Further the following result gives us equivalent conditions for quasi-semiprime ideal and for P-system.

Proposition 6.8. *Let "e" be a left identity element of an nLA-ring T and N a left ideal of T, then N is quasi-semiprime if and only if $T \setminus N$ is a P-system.*

Proof. Suppose N is quasi-semiprime and assume $u \in T \setminus N$. Let us suppose that there does not exist an element $t \in T$ such that $u(tu) \in T \setminus N \implies u(tu) \in N$. As N is quasi-semiprime, so by the above lemma, $u \in N$. This is a contradiction. Thus there exists $t \in T$ such that $u(tu) \in T \setminus N$. Hence $T \setminus N$ is a P-system.

Conversely, suppose that $T \setminus N$ is a P-system. Let $u \in T$ be such that $u(Tu) \subseteq N$. Suppose that $u \notin N \implies u \in T \setminus N$. Since $T \setminus N$ is a P-system, so there exist $t \in T$ such that $u(tu) \in T \setminus N \implies u(Tu) \nsubseteq N$ which is a contradiction. Hence $u \in N$ and so by the above lemma, N is quasi-semiprime □

We are now going to define an I-system.

Definition 6.9. *Let T be an nLA-ring with left identity "e' and $\emptyset \neq S \subseteq T$. Then S is said to be an I-system if $(\langle x \rangle \cap \langle y \rangle) \cap S \neq \emptyset \ \forall \ x, y \in S$.*

Let us state and prove some properties. Again the idea of these properties has come from the source [8]. The following result gives us equivalent condition for quasi-strongly irreducible ideal and for I-system.

Proposition 6.10. *Let "e" be a left identity element of an nLA-ring T. Let N be a left ideal of T, then the following conditions are equivalent:*

(i) *N is quasi-strongly irreducible.*

(ii) *For $x, y \in T$ such that $\langle x \rangle \cap \langle y \rangle \subseteq N \implies x \in N$ or $y \in N$.*

(iii) *$T \setminus N$ is an I-system.*

Proof. $(i) \implies (ii)$ Let $x, y \in T$ such that $\langle x \rangle \cap \langle y \rangle \subseteq N$, then by definition of a quasi-strongly irreducible ideal either $\langle x \rangle \subseteq N$ or $\langle y \rangle \subseteq N$. Thus it follows that either $x \in N$ or $y \in N$.

$(ii) \implies (iii)$ Let $x, y \in T \setminus N$ and let us suppose that $(\langle x \rangle \cap \langle y \rangle) \cap T \setminus N = \emptyset \implies \langle x \rangle \cap \langle y \rangle \subseteq N \implies$ either $x \in N$ or $y \in N$. It follows that our supposition is wrong and so $(\langle x \rangle \cap \langle y \rangle) \cap T \setminus N \neq \emptyset$.

$(iii) \implies (i)$ Let I and J be ideals of T such that $I \cap J \subseteq N$. Suppose $I \nsubseteq N$ and $J \nsubseteq N$, then there exists an element $x \in I$ and $y \in J$ such that $x \in I \setminus N$ and $y \in I \setminus N \implies x, y \in T \setminus N$, so by hypothesis it follows that $(\langle x \rangle \cap \langle y \rangle) \cap T \setminus N \neq \emptyset \implies$ there exist an element $z \in \langle x \rangle \cap \langle y \rangle$ such that $z \in T \setminus N \implies z \in \langle x \rangle \cap \langle y \rangle \subseteq I \cap J$ such that $z \notin N \implies I \cap J \nsubseteq N$, which is a contradiction. Hence either $I \subseteq N$ or $J \subseteq N$ and so N is quasi-strongly irreducible. $\square$

References

[1] Fraleigh, J. B., *A first course in abstract algebra,* Pearson Education Inc, 2003.

[2] Hussain, F. and Khan, W., *Congruences on Left Almost Rings*, International Journal of Algebra and Statistics, 4(1) (2015),1-6.

[3] Hussain, F. and Firdous, S., *Direct Product of Left Almost Rings*, International Journal of Sciences: Basic and Applied Research, 5(2) (2016), 113-127.

[4] Hussain, F., Firdous, S. and Sadiq, N., *Direct Product of two Near Left Almost Rings*, International Journal of Algebra and Statistics, 5(2) (2016), 77-89.

[5] Kazim, M. A. and Naseeruddin, M., *On left almost semigroups*, The Alig. Bull. Math, 2 (1972), 1-7.

[6] Kamran, M. S., *Conditions for LA-semigroups to resemble associative* structures, Ph.D. Thesis, Quaid-i-Azam University, Islamabad, 1993. Available at http://eprints.hec.gov.pk/2370/1/2225.htm.

[7] Rehman, I. Shah, M. Shah, T. and Razzaque, A., *On existence of nonassociative LA-ring*, An. St. Univ. Ovidius Constanta, 21(3) (2013), 223-228.

[8] Rehman, I., *On generalized commutative ring and related structures*, PhD. Thesis, Quaid-i-Azam University, Islamabad, 2012.

[9] Shah, M. and Shah, T., *Some basic properties of LA-rings*, International Mathematical Forum, 6(44) (2011), 2195-2199.

[10] Shah, T. Ali, G. and Rehman, F., *Direct sum of ideals in a generalized LA-ring*, International Mathematical Forum, 6(22) (2011), 1095-1101.

[11] Shah, T. Rehman, F and Raees, M., *On near left almost rings*, International Mathematical Forum, 6(23) (2011), 1103-1111.

[12] Shah, T. and Rehman, I., *On LA-rings of finitely non-zero functions*, Int. J. Contemp. Math. Sciences, 5(5) (2010), 209-222.

[13] Yusuf, S. M., *On left almost rings*, Proc. of 7th International Pure Math. Conference, 2006.

Accepted: 21.04.2017