DYNAMICS OF ALMOST PERIODIC NICHOLSON'S BLOWFLIES MODEL WITH NONLINEAR DENSITY-DEPENDENT MORTALITY TERM

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Abstract. This paper deals with the dynamics of almost periodic Nicholson's blowflies model with nonlinear density-dependent mortality. Prior to the main results, we prove the boundedness and extinction of the solutions for the addressed model. By applying Shauder's fixed point theorem, we establish sufficient conditions for the existence of almost periodic positive solution. Under less restrictive assumptions, the exponential stability is derived by means of the Liapunov functional method. The reported results give an affirmative answer to the problem raised by L. Berezansky.
Keywords: Nicholson's blowflies model, density-dependent mortality, almost periodic solution, extinction, exponential stability.

1. Introduction

In [1], Gurney proposed the following delay differential equation

$$x'(t) = -ax(t) + px(t-\tau)e^{-\beta x(t-\tau)} \tag{1.1}$$

to describe the population of the Australian sheep-blowfly and to agree with the experimental data obtained by Nicholson in [2]. Equation (1.1) describes Nicholson's data of blowfly and thus it has been referred to as the Nicholson's blowflies model. The theory of this model has made a substantial progress during the last two decades [3-15]. Due to various seasonal effects of the environmental factors in real life situation (e.g., seasonal effects of weather, food supplies, mating habits, harvesting, etc.), researchers have found it rational and practical to study the population models under periodic assumptions. The recent years

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have witnessed the appearance of many papers that studied non autonomous differential equations with periodic coefficients of various versions of model (1.1); see for instance [4,5,7,9,12].

New studies indicated that the consideration of population models with density dependent mortality will be more accurate at low densities. In his remarkable paper [11], Berezansky has put forward an open problem about the dynamical behaviors of Nicholson's blowflies model with density-dependent mortality of the form

$$x'(t) = -M(x) + px(t-\tau)e^{-\beta x(t-\tau)}, \tag{1.2}$$

where M denotes the mortality term that might be expressed in the form $\frac{ax}{b+x}$ or $a - be^{-x}$. Although the papers [16-22] have dealt with the permanence and periodicity of solutions, they have provided insufficient outcomes to answer the problem raised by Berezansky for model (1.2). The almost periodicity which is a natural generalization of periodicity has been the object of many researchers in the last years. Indeed, it has been encompassed to Nicholson's model and thus several results have been recently reported; see for instance the papers [23,24] and the monograph [28] for more details.

Motivated by the above discussions, we consider the non-autonomous almost periodic Nicholson's blowflies model with density-dependent mortality term of the form

$$x'(t) = -\frac{a(t)x(t)}{b(t)+x(t)} + p(t)x(t-\tau(t))e^{-\beta(t)x(t-\tau(t))}, \tag{1.3}$$

where $a(t), b(t), \beta(t), p(t), \tau(t) \in C(R, R^+)$ and $a(t), b(t), \beta(t), p(t), \tau(t)$ are bounded almost periodic functions. Due to biological significance, we restrict our attention to positive solutions of equation (1.3). The initial condition of equation (1.2) is $x(t) = \phi(t) > 0$ for $t \in [-\overline{\tau}, 0]$, $\overline{\tau} = \sup_{t\in R} \tau(t)$, $\phi \in BC([-\overline{\tau}, 0], R^+)$, where $BC([-\overline{\tau}, 0], R^+) = \{\phi | \phi : [-\overline{\tau}, 0] \to R^+$ is bounded continuous function $\}$.

In this paper, we provide sufficient conditions for the existence and exponential stability of almost periodic solution for model (1.3). Prior to the main results, we prove the boundedness and extinction for the addressed model. Unlike previously obtained results such as those given in [23,24], we utilize Shauder's fixed point theorem to prove the existence result. In addition to this, the exponential stability has been proved under less restrictive assumptions. To the best of our observation, no published paper has dealt with model (1.3) by the implementation of these two distinctive features.

2. Preliminaries

For any bounded function $f(t)$, we denote $\overline{f} = \sup_{t\in R} f(t)$ and $\underline{f} = \inf_{t\in\mathbb{R}} f(t)$. Therefore, in the remaining part of the paper, we assume that the bounded almost periodic functions $a(t), b(t), \beta(t), p(t), \tau(t)$ satisfy $0 \le a \le a(t) \le \overline{a}$,

$0 \le b \le b(t) \le \bar{b}$, $0 \le \underline{\beta} \le \beta(t) \le \overline{\beta}$, $0 < \underline{p} \le p(t) \le \overline{p}$ and $0 < \underline{\tau} \le \tau(t) \le \overline{\tau}$. In what follows, we set forth some assertions that will be used throughout the rest of the paper.

Definition 2.1 ([25])**.** Let $u(t) : \mathbb{R} \to \mathbb{R}^n$ be continuous in $t, u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$,the set $T(u, \varepsilon) = \{\delta : |u(t + \delta) - u(t)| < \varepsilon, t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, for which there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t + \delta) - u(t)| < \varepsilon$, for all $t \in \mathbb{R}$.

Definition 2.2 ([25])**.** Let $x \in \mathbb{R}$ and $Q(t)$ be $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system

$$x'(t) = Q(t)x(t) \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (2.1) satisfying

$$\begin{aligned} &\|X(t)PX^{-1}(s)\| \le ke^{-\alpha(t-s)} \text{ for } t \ge s \\ &\|X(t)(I - P)X^{-1}(s)\| \le ke^{-\alpha(s-t)} \text{ for } t \le s. \end{aligned}$$

Definition 2.3 ([27])**.** Let $\widetilde{x}(t)$ be an almost periodic solution of Eq. (1.3), $x(t)$ be another solution of Eq. (1.3). The solution $x(t)$ is said to be exponentially convergent to $\widetilde{x}(t)$ as $t \to +\infty$ if there exist constants $\lambda > 0, K > 0$ such that $|x(t) - \widetilde{x}(t)| \le Ke^{-\lambda t}$, for all $t > 0$.

Lemma 2.4 ([25])**.** If the linear system (2.1) admits an exponential dichotomy, then the almost periodic system

$$x'(t) = Q(t)x(t) + g(t) \tag{2.2}$$

has a unique almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)ds - \int_{t}^{+\infty} X(t)(I - P)X^{-1}(s)g(s)ds.$$

Lemma 2.5. Let $c_i(t)$ be almost periodic function on $\mathbb{R}$ and

$$M[c_i] = \lim_{T \to +\infty} \frac{1}{T} \int_{t}^{t+T} c_i(s)ds > 0, \quad i = 1, 2, \dots, n.$$

Then, the linear system

$$x'(t) = \mathrm{diag}(-c_1(t), -c_2(t), \dots, -c_n(t))x(t)$$

admits an exponential dichotomy on $\mathbb{R}$.

Lemma 2.6. Every solution of equation (1.3) is positive.

Proof. Let $x(t)$ be any solution of equation (1.3) with initial condition $x(t) = \phi(t) > 0$ for $t \in [-\overline{\tau}, 0]$. We claim that

$$x(t) > 0, \quad \text{for all } t > 0. \tag{2.3}$$

Suppose the claim (2.3) is not true, then there must exist $t_1 \in (0, +\infty)$ such that $x(t_1) = 0$, $x'(t_1) \leq 0$ and $x(t) > 0$ for $t \in [-\overline{\tau}, t_1)$. From (1.3), we have

$$\begin{aligned} x'(t_1) &= -\frac{a(t_1)x(t_1)}{b(t_1) + x(t_1)} + p(t_1)x(t_1 - \tau(t_1))e^{-\beta(t_1)x(t_1-\tau(t_1))} \\ &= p(t_1)x(t_1 - \tau(t_1))e^{-\beta(t_1)x(t_1-\tau(t_1))}. \end{aligned} \tag{2.4}$$

Since $x(t_1 - \tau(t_1)) > 0$, then it follows from (2.4) that $x'(t_1) > 0$, which contradicts $x'(t_1) \leq 0$. Therefore, the claim (2.3) is true. The proof is complete. □

Schauder fixed point theorem is an important tool in our proof.

Lemma 2.7 ([26])**.** (Shauder's fixed point theorem) Let Ω be a closed convex subset of Banach space X, $A : \Omega \to \Omega$ be a continuous operator such that $A\Omega$ is relatively compact. Then the operator A has at least one fixed point in Ω.

3. Boundedness and extinction of solutions

Let $m = \frac{\overline{p}}{\underline{\beta}e}$ and $H = \frac{m\overline{b}}{\underline{a}-m}$. We make the assumption:

$$(C_1) \qquad \underline{a} > m.$$

Theorem 3.1. *Let (C_1) hold. Then, every solution of equation (1.3) is bounded.*

Proof. Let $x(t)$ be any solution of equation (1.3) with initial condition $x(t) = \phi(t) > 0$ for $t \in -[\overline{\tau}, 0]$. By Lemma 2.6, we know that $x(t) > 0$, for all $t > 0$.

Now, we prove that $x(t)$ is bounded.

Suppose $x(t)$ is unbounded, then there exists $t^* > 0$ such that $x(t') > H$, and there also exists $\overline{t} > 0$ satisfying $0 < \overline{t} < t^*$, $x(t) < x(t^*)$. From (1.3), we have

$$\begin{aligned} x'(t^*) &= -\frac{a(t^*)x(t^*)}{{}^{b}(t^*)} + p(t^*)x(t^* - \tau(t^*))e^{-\beta(t^*)x(t^*-\tau(t^*))} \\ &\leq -\frac{\underline{a}x(t^*)}{\overline{b} + x(t^*)} + \overline{p}x(t^* - \tau(t^*))e^{-\underline{\beta}x(t^*-\tau(t^*))}. \end{aligned} \tag{3.1}$$

It is clear that the function $f(u) = ue^{-\underline{\beta}u}$, $u \in [0, +\infty)$ reaches its maximum $\frac{1}{\underline{\beta}e}$ at $u = \frac{1}{\underline{\beta}}$. Then, we get $x(t^* - \tau(t^*))e^{-\underline{\beta}x(t^*-\tau(t^*))} \leq \frac{1}{\underline{\beta}e}$. Thus, (3.1) implies that

$$x'(t^*) \leq -\frac{\underline{a}x(t^*)}{\overline{b} + x(t^*)} + \overline{p}\frac{1}{\underline{\beta}e}. \tag{3.2}$$

Note that the function $g(u) = \frac{\underline{a}u}{\overline{b}+u}$ is strictly increasing on $u \in (0, +\infty)$. Since $x(t^*) > H$, then we have $g(x(t^*)) > g(H)$, that is

$$\frac{\underline{a}x(t^*)}{\overline{b} + x(t^*)} > \frac{\underline{a}H}{\overline{b} + H}. \tag{3.3}$$

By (3.2) and (3.3), we get

$$x'(t^*) < -\frac{\underline{a}H}{\overline{b} + H} + \overline{p}\frac{1}{\underline{\beta}e} = 0.$$

Let $x(\widehat{t}) = \max_{\widetilde{t} \le t \le t^*}$. Since $x(\widetilde{t}) < x(t^*)$ and $x'(t^*) < 0$, then it follows that $\widetilde{t} < \widehat{t} < t^*$ and $x(\widehat{t}) > x(t^*)$. Hence we have

$$x'(\widehat{t}) = 0. \tag{3.4}$$

On the other hand, from (1.3), we get

$$\begin{aligned}
x'(\widehat{t}) &= -\frac{a(\widehat{t})x(\widehat{t})}{b(\widehat{t}) + x(\widehat{t})} + p(\widehat{t})x(\widehat{t} - \tau(\widehat{t}))e^{-\beta(\widehat{t})x(\widehat{t})x(\widehat{t}-\tau(\widehat{t}))} \\
&\le -\frac{\underline{a}x(\widehat{t})(\widehat{t})}{\overline{b} + x(\widehat{t})} + \overline{p}x(\widehat{t} - \tau(\widehat{t}))e^{-\underline{\beta}x(\widehat{t}-\tau(\widehat{t}))} \\
&\le -\frac{\underline{a}x(\widehat{t})}{\overline{b} + x(\widehat{t})} + \overline{p}\frac{1}{\underline{\beta}e} \\
&< -\frac{\underline{a}x(t^*)}{\overline{b} + x(t^*)} + \overline{p}\frac{1}{\underline{\beta}e} \\
&- \frac{\underline{a}H}{\overline{b} + H} + \overline{p}\frac{1}{\underline{\beta}e} = 0,
\end{aligned}$$

which contradicts (3.4). Therefore, $x(t)$ is bounded. The proof is complete. □

Let $L > H$, we define $U_L = \{\phi | \phi \in BC([-\overline{\tau}, 0], \mathbb{R}^+), 0 < \phi(t) < L, t \in [-\overline{\tau}, 0]\}$.

Theorem 3.2. *Let* (C_1) *hold. Then every solution* $x(t)$ *of equation* (1.3) *with the initial function* $\phi \in U_L$ *satisfies*

$$0 < x(t) < L, \quad \textit{for all}\, t > 0.$$

Proof. Let $x(t)$ be any solution of equation (1.3) with initial function $\phi \in U_L$. For $t \in [-\overline{\tau}, 0]$, we have $0 < x(t) = \phi(t) < L$. By Lemma 2.6, we have $x(t) > 0$, for all $t > 0$. Now, we claim that

$$x(t) < L, \quad \text{for all}\, t > 0. \tag{3.5}$$

Suppose that (3.5) is not true. Then there must exist a $t_1 \in (0, +\infty)$ such that $x(t_1) = L$, $x'(t) \geq 0$ and $0 < x(t) < L$ for $t \in (0, t_1)$. From (1.3), we have

$$\begin{aligned}
x'(t_1) &= -\frac{a(t_1)x(t_1)}{b(t_1) + x(t_1)} + p(t_1)x(t_1 - \tau(t_1))e^{\beta(t_1)x(t_1 - \tau(t_1))} \\
&= -\frac{a(t_1)L}{b(t_1) + L} + p(t_1)x(t_1)x(t_1 - \tau(t_1))e^{-\beta(t_1)x(t_1 - \tau(t_1))} \\
&\leq -\frac{\underline{a}L}{\bar{b} + L} + \bar{p}x(t_1 - \tau(t_1))e^{-\underline{\beta}x(t_1 - \tau(t_1))} \\
&\leq -\frac{\underline{a}L}{\bar{b} + L} + \bar{p}\frac{1}{\underline{\beta}e} \& < -\frac{\underline{a}H}{\bar{b} + H} + \bar{p}\frac{1}{\underline{\beta}e} = 0,
\end{aligned}$$

which contradicts $x'(t_1) \geq 0$. So the claim (3.5) is true. The proof is complete. $\square$

Furthermore, we assume that

$$(C_2) \qquad \frac{\underline{a} - \bar{p}\bar{b}}{\bar{p}} > H.$$

Theorem 3.3. *Assume that (C_1) and (C_2) hold. Let L be a positive constant satisfying $H < L < \frac{\underline{a} - \bar{p}\bar{b}}{\bar{p}}$. Then every solution $x(t)$ of equation (1.3) with initial function $\phi \in U_L$ satisfies*

$$x(t) \to 0 \qquad as \qquad t \to +\infty.$$

That is , every solution $x(t)$ with initial function $\phi \in U_L$ tends to extinction.

Proof. Let $x(t)$ be any solution of equation (1.3) with the initial function $\phi \in U_L$. For $t \in [-\bar{\tau}, 0]$, we have $0 < x(t) = \phi(t) < L$. By Theorem 2, we know that $0 < x(t) < L$ for all $t > 0$. From $H < L < \frac{\underline{a} - \bar{p}\bar{b}}{\bar{p}}$, it follows that $\underline{a} > \bar{p}\bar{b} + \bar{p}L$. Consider the function $F(x) = \bar{p}\bar{b}e^{\tau x} + \bar{b}x + Lx + \bar{p}L - \underline{a}$, $x \in [0, 1]$. Since $F(0) = \bar{p}\bar{b} + \bar{p}L - \underline{a} < 0$, then there exists a constant $\lambda \in (0, 1)$ such that $F(\lambda) < 0$. That is

$$\bar{p}\bar{b}e^{\overline{\lambda\tau}} + \lambda\bar{b} + \lambda L + \bar{p}L - \underline{a} < 0 \tag{3.6}$$

Let $\delta(t) = x(t)e^{\lambda t}$, then we have

$$\begin{aligned}
\delta'(t) &= x'(t)e^{\lambda t} + \lambda x(t)e^{\lambda t} \\
&= [-\frac{a(t)x(t)}{b(t) + x(t)} + p(t)x(t - \tau(t))e^{-\beta(t)x(t - \tau(t))} + \lambda x(t)]e^{\lambda t}.
\end{aligned} \tag{3.7}$$

Let $M = L + \sup_{-\bar{\tau} \leq t \leq 0} \phi(t)$. For all $t \in [-\bar{\tau}, 0]$, we have

$$0 < \delta(t) = x(t)e^{\lambda t} = \phi(t)e^{\lambda t} \leq \phi(t) \leq - \sup_{-\bar{\tau} \leq t \leq 0} \phi(t) < L + \sup_{-\bar{\tau} \leq t \leq 0} \phi(t) = M.$$

For all $t \in (0, +\infty)$, it is obvious that $\delta(t) = x(t)e^{\lambda t} > 0$. Now, we claim that

$$\delta(t) < M \text{ for all } t > 0. \tag{3.8}$$

Suppose that (3.8) is not true. Then, there must exist a $t^* > 0$, such that $\delta(t^*) = M$, $\delta'(t^*) \geq 0$ and $\delta(t) < M$ for $t < t^*$. It follows from (3.7) that

$$\begin{aligned}
0 \leq \delta'(t^*) &= [-\frac{a(t^*)x(t^*)}{b(t^*) + x(t^*)} \\
&\quad + p(t^*)x(t^* - \tau(t^*))e^{-\beta(t^*)x(t^*-\tau(t^*))} + \lambda x(t^*)]e^{\lambda t^*} \\
&\leq [-\frac{\underline{a}x(t^*)}{\overline{b} + x(t^*)} + \overline{p}x(t^* - \tau(t^*)) + \lambda x(t^*)]e^{\lambda t^*} \\
&= \frac{\begin{array}{c}-\underline{a}x(t^*)e^{\lambda t^*} + \overline{p}\overline{b}x(t^* - \tau(t^*))e^{\lambda t^*} + \overline{p}x(t^* - \tau(t^*))x(t^*)e^{\lambda t^*} \\ +\lambda\overline{b}x(t^*)e^{\lambda t^*} + \lambda x^2(t^*)e^{\lambda t^*}\end{array}}{\overline{b} + x(t^*)} \\
&= \frac{\begin{array}{c}-\underline{a}\delta(t^*) + \overline{p}\overline{b}x(t^* - \tau(t^*))e^{\lambda(t^*-\tau(t^*))} + \overline{p}x(t^* \\ -\tau(t^*))\delta(t^*) + \lambda\overline{b}\delta(t^*) + \lambda x(t^*)\delta(t^*)\end{array}}{\overline{b} + x(t^*)} \qquad (3.9) \\
&= \frac{-\underline{a}M + \overline{p}\overline{b}\delta(t^* - \tau(t^*))e^{\lambda\tau(t^*)} + \overline{p}x(t^* - \tau(t^*))M + \lambda\overline{b}M + \lambda x(t^*)M}{\underline{b} + x(t^*)} \\
&< \frac{-\underline{a}M + \overline{p}\overline{b}Me^{\lambda\overline{\tau}} + \overline{p}LM + \lambda\overline{b}M + \lambda LM}{\overline{b} + x(t^*)} \\
&= \frac{M}{\overline{b} + x(t^*)}(-\underline{a} + \overline{p}\overline{b}e^{\lambda\overline{\tau}} + \overline{p}L + \lambda\overline{b} + \lambda L)
\end{aligned}$$

Thus, (3.9) implies $-\underline{a} + \overline{p}\overline{b}e^{\lambda\overline{\tau}} + \overline{p}L + \lambda\overline{b} + \lambda L > 0$, which contradicts (3.6). Therefore (3.8) is true. Hence, $\delta(t) = x(t)e^{\lambda t} < M$, for all $t > 0$. That is

$$0 < x(t) < Me^{-\lambda t}, \quad \text{for all } t > 0.$$

which implies that $x(t) \to 0$ as $t \to +\infty$. The proof is complete. □

4. Existence of almost periodic positive solution

It is assumed that (C_3) there exist two positive constants $L_2 > L_1 \geq \frac{1}{\underline{\beta}}$, and a bounded positive almost periodic function $\gamma(t) \in C(R, R^+)$ with $\underline{\gamma} > 0$, satisfying the following inequalities

$$\sup_{t \in R}\{-\frac{a(t)L_1}{b(t) + L_1} + \gamma(t)L_2 + \frac{p(t)}{\beta(t)}\frac{l}{e}\} \leq \underline{\gamma}L_2$$

and

$$\inf_{t \in R, v \in [L_1, L_2]}\{-\frac{a(t)L_2}{b(t) + L_2} + \gamma(t)L_1 + p(t)ve^{-\overline{\beta}v}\} \geq \overline{\gamma}L_1.$$

Let $X = \{w(t) \in C(\mathbb{R}, \mathbb{R}), w(t)$ is almost periodic function $\}$. For $w \in X$, we define $\|w\| = \sup_{t\in\mathbb{R}} |w(t)|$, then X is a Banach space.

We note that equation (1.3) is equivalent to

$$(4.1)\quad x'(t) = -\gamma(t)x(t) - \frac{a(t)x(t)}{b(t)+x(t)} + \gamma(t)x(t) + p(t)x(t-\tau(t))e^{-\beta(t)x(t-\tau(t))}.$$

For $w(t) \in X$, we consider the equation

$$(4.2)\quad \omega' = -\gamma(t)\omega(t) - \frac{a(t)w(t)}{b(t)+w(t)} + \gamma(t)w(t) + p(t)w(t-\tau(t))e^{-\beta(t)w(t-\tau(t))}.$$

Since $M[\gamma] > 0$, then from Lemma 2.5 we know that the linear equation $x' = -\gamma(t)x(t)$ admits exponential dichotomy on $\mathbb{R}$.

Hence, by Lemma 1, we know that equation (4.2) has exactly one almost periodic solution:

$$\begin{aligned} x_w(t) = \int_{-\infty}^{t} & e^{-\int_s^t \gamma(u)du}[-\frac{a(s)w(s)}{b(s)+w(s)} + \gamma(s)w(s) \\ & + p(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))}]ds. \end{aligned}$$

We define the operator $A : X \to X$,

$$\begin{aligned} (Aw)(t) = \int_{-\infty}^{t} & e^{-\int_s^t \gamma(u)du}[-\frac{a(s)w(s)}{b(s)+w(s)} \\ & + \gamma(s)w(s) + p(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))}]ds,\ w \in X. \end{aligned}$$

Obviously, $w(t) \in C(\mathbb{R}, \mathbb{R})$ is the almost periodic solution of equation (4.1) if and only if w is the fixed point of operator A.

Theorem 4.1. *Let (C_3) hold. Then, equation (1.3) has at least one almost periodic positive solution.*

Proof. Define a closed convex subset Ω of X as follows

$$\Omega = \{w | w \in X, L_1 \le w(t) \le L_2, t \in \mathbb{R}\}.$$

Firstly, we prove that $A\Omega \subset \Omega$. For all $w \in \Omega$, we have

$$\begin{aligned} (Aw)(t) = \int_{-\infty}^{t} & e^{-\int_s^t \gamma(u)du}[-\frac{a(s)w(s)}{b(s)+w(s)} + \gamma(s)w(s) \\ & + p(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))}]ds \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{t} e^{-\int_s^t \gamma(u)du}[-\frac{a(s)L_1}{b(s)+L_1} + g(s)L_2 \\
&+ \frac{p(s)}{\beta(s)}\beta(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))}]ds \\
&\leq \int_{-\infty}^{t} e^{-\int_s^t \underline{\gamma}(u)du}[-\frac{a(s)L_1}{b(s)+L_1} + \gamma(s)L_2 + \frac{p(s)}{\beta(s)}\frac{1}{e}]ds \qquad (4.3)\\
&\sup_{t\in\mathbb{R}}\{-\frac{a(t)L_1}{b(t)+L_1} + \gamma(t)L_2 + \frac{p(t)}{\beta(t)}\frac{1}{e}\}\int_{-\infty}^{t} e^{-\underline{\gamma}(t-s)}ds \\
&= \sup_{t\in\mathbb{R}}\{-\frac{a(t)L_1}{b(t)+L_1} + \gamma(t)L_2 + \frac{p(t)}{\beta(t)}\frac{1}{e}\}\frac{1}{\underline{\gamma}} \leq L_2.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(Aw)(t) &= \int_{-\infty}^{t} e^{-\int_s^t \gamma(u)du}[\frac{a(s)w(s)}{b(s)+w(s)} + \gamma(s)w(s) \\
&+ p(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))}]ds \\
&\geq \int_{-\infty}^{t} e^{-\int_s^t \gamma(u)du}[\frac{a(s)L_2}{b(s)+L_2} + g(s)L_1 \\
&+ \frac{p(s)}{\beta(s)}\beta(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))}]ds \\
&\geq \int_{-\infty}^{t} e^{-\int_s^t \underline{\gamma}(u)du}[-\frac{a(s)L_2}{b(s)+L_2} + \gamma(s)L_1 \\
&+ p(s)w(s-\tau(s))e^{-\overline{\beta}w(s-\tau(s))}]ds \qquad (4.4)\\
&\geq \inf_{t\in\mathbb{R}, v\in[L_1,L_2]}\{-\frac{a(t)L_2}{b(t)+L_2} + \gamma(t)L_1 + p(t)ve^{-\overline{\beta}v}\}\int_{-\infty}^{t} e^{-\int_s^t \overline{\gamma}du}ds \\
&= \int_{t\in\mathbb{R}, v\in[L_1,L_2]}\{-\frac{a(t)L_2}{b(t)+L_2} + \gamma(t)L_1 + p(t)ve^{-\overline{\beta}v}\}\frac{1}{\overline{\gamma}} \geq L_1.
\end{aligned}
$$

Hence, (4.3) and (4.4) imply

$$
L_1 \leq (Aw)(t) \leq L_2. \qquad (4.5)
$$

In addition, for all $w \in \Omega$, then $w(t)$ is almost periodic. By Lemma 1, we know that equation (4.2) has exactly one almost periodic solution:

$$
x_w(t) = \int_{-\infty}^{t} e^{-\int_s^t \gamma(u)du}[-\frac{a(s)w(s)}{b(s)+w(s)} + \gamma(s)w(s) + p(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))}]ds.
$$

Since $x_w(t)$ is almost periodic, then $(Aw)(t)$ is almost periodic. This, together with (4.5), imply that $Aw \in \Omega$. So we have $A\Omega \subset \Omega$.

Next, we prove that the operator $A:\Omega\to\Omega$ is continuous. Let $x_n=x_n(t)\in\Omega$ be such that $x_n\to x\in\Omega$ as $n\to+\infty$. Then, we have

$$
\begin{aligned}
&\|Ax_n-Ax\|=\sup_{t\in\mathbb{R}}|(Ax_n)(t)-(Ax)(t)|\\
&=\sup_{t\in\mathbb{R}}\bigg|\int_{-\infty}^{t}e^{-\int_s^t\gamma(u)du}[-\frac{a(s)x_n(s)}{b(s)+x_n(s)}+\gamma(s)x_n(s)\\
&+p(s)x_n(s-\tau(s))e^{-\beta(s)x_n(s-\tau(s))}]ds\\
&-\int_{-\infty}^{t}e^{-\int_s^t\gamma(u)du}[-\frac{a(s)x(s)}{b(s)+x(s)}+\gamma(s)x(s)\\
&+p(s)x(s-\tau(s))e^{-\beta(s)x(s-\tau(s))}]ds\bigg|\\
&=\sup_{t\in\mathbb{R}}\bigg|\int_{-\infty}^{t}e^{-\int_s^t\gamma(u)du}[(-\frac{a(s)x_n(s)}{b(s)+x_n(s)}\\
&-\frac{a(s)x(s)}{b(s)+x(s)})+(\gamma(s)x_n(s)-\gamma(s)x(s))\\
&+(p(s)x_n(s-\tau(s))e^{-\beta(s)x_n(s-\tau(s))}-p(s)x(s-\tau(s))e^{-\beta(s)x(s-\tau(s))})]ds\bigg|\\
&\le\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\int_s^t\gamma(u)du}[\bigg|\frac{a(s)x_n(s)}{b(s)+x_n(s)}-\frac{a(s)x(s)}{b(s)+x(s)}\bigg|\\
&+|\gamma(s)x_n(s)-\gamma(s)x(s)|\\
&+|p(s)x_n(s-\tau(s))e^{-\beta(s)x_n(s-\tau(s))}-p(s)x(s-\tau(s))e^{-\beta(s)x(s-\tau(s))}|]ds\\
&\le\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\int_s^t\gamma(u)du}[\overline{a}|\frac{x_n(s)}{b(s)+x_n(s)}-\frac{x(s)}{b(s)+x(s)}|+\overline{\gamma}|x_n(s)-x(s)|\\
&+\overline{p}|x_n(s-\tau(s))e^{-\beta(s)x_n(s-\tau(s))}-x(\tau(s))e^{-\beta(s)x(s-\tau(s))}|]ds.
\end{aligned}
\tag{4.6}
$$

Define the function $\Phi(x)=\frac{x}{x+1}$, $x\in(0,+\infty)$, then $\Phi'(x)=\frac{1}{(x+1)^2}$. By the mean value theorem, we then have

$$
\begin{aligned}
&\left|\frac{x_n(s)}{b(s)+x_n(s)}-\frac{x(s)}{b(s)+x(s)}\right|=\left|\frac{\frac{x_n(s)}{b(s)}}{1+\frac{x_n(s)}{b(s)}}-\frac{\frac{x(s)}{b(s)}}{1+\frac{x(s)}{b(s)}}\right|\\
&=\left|\Phi\left(\frac{x_n(s)}{b(s)}\right)-\Phi\left(\frac{x(s)}{b(s)}\right)\right|\\
&\le\left|\Phi'(\xi_1)\left(\frac{x_n(s)}{b(s)}-\frac{x(s)}{b(s)}\right)\right|=\frac{1}{(1+\xi_1)^2}\left|\frac{x_n(s)}{b(s)}-\frac{x(s)}{b(s)}\right|\\
&\le\left|\frac{x_n(s)}{b(s)}-\frac{x(s)}{b(s)}\right|=\frac{1}{b(s)}|x_n(s)-x(s)|\le\frac{1}{\underline{b}}|x_n(s)-x(s)|,
\end{aligned}
\tag{4.7}
$$

in which ξ_1 lies between $\frac{x_n(s)}{b(s)}$ and $\frac{x(s)}{b(s)}$.

We define the function $\Psi(x) = xe^{-x}$, then $\Psi'(x) = (1-x)e^{-x}$. Again by the mean value theorem, we get

$$\begin{aligned}
&|x_n(s-\tau(s))e^{-\beta(s)x_n(s-\tau(s))} - x(s-\tau(s))e^{-\beta(s)x(s-\tau(s))}|\\
&= \frac{1}{\beta(s)}|\beta(s)x_n(s-\tau(s))e^{-\beta(s)x_n(s-\tau(s))} - \beta(s)x(s-\tau(s))e^{-\beta(s)x(s-\tau(s))}|\\
&= \frac{1}{\beta(s)}|\Psi(\beta(s)x_n(s-\tau(s))) - \Psi(\beta(s)x(s-\tau(s)))| \qquad (4.8)\\
&\le \frac{1}{\beta(s)}|\Psi'(\xi_2)(\beta(s)x_n(s-\tau(s)) - \beta(s)x(s-\tau(s)))|\\
&= \frac{1}{\beta(s)}|(1-\xi_2)e^{-\xi_2}||\beta(s)x_n(s-\tau(s)) - \beta(s)x(s-\tau(s))|\\
&= |(1-\xi_2)e^{-\xi_2}||x_n(s-\tau(s)) - x(s-\tau(s))|,
\end{aligned}$$

in which ξ_2 lies between $\beta(s)x_n(s-\tau(s))$ and $\beta(s)x(s-\tau(s))$. Since $x_n, x \in \Omega$, $L_1 \le x_n(t) \le L_2$ and $L_1 \le x(t) \le L_2$ for $t \in \mathbb{R}$, then we have

$$1 \le \underline{\beta}L_1 \le \beta(s)x_n(s-\tau(s)) \le \overline{\beta}L_2 \text{ and } 1 \le \underline{\beta}L_1 \le \beta(s)x(s-\tau(s)) \le \overline{\beta}L_2.$$

This implies

$$1 \le \underline{\beta}L_1 < \xi_2 < \overline{\beta}L_2.$$

Note that the function $h(x) = |(1-x)e^{-x}|$, $x \in [1, +\infty)$ has maximum $h_{\max} = \frac{1}{e^2}$. Thus we have $h(\xi_2) = |(1-\xi_2)e^{-\xi_2}| \le \frac{1}{e^2}$. It follows from (4.8) that

$$\begin{aligned}
&|x_n(s-\tau(s))e^{-\beta(s)x_n(s-\tau(s))} - x(s-\tau(s))e^{-\beta(s)x((s-\tau(s)))}|\\
&\le \frac{1}{e^2}|x_n(s-\tau(s)) - x(s-\tau(s))|. \qquad (4.9)
\end{aligned}$$

From (4.6) , (4.7) and (4.9), we get

$$\begin{aligned}
\|Ax_n - Ax\| &\le \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t \gamma(u)du}(\frac{\overline{a}}{\underline{b}}|x_n(s) - x(s)| + \overline{\gamma}|x_n(s) - x_s|\\
&\quad + \overline{p}\frac{1}{e^2}|x_n(s-\tau(s)) - x(s-\tau(s))|)ds\\
&= \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t \underline{\gamma}du}(\frac{\overline{a}}{\underline{b}}\|x_n - x\| + \overline{\gamma}\|x_n - x\| + \overline{p}\frac{1}{e^2}\|x_n - x\|)ds\}\\
&= \sup_{t\in\mathbb{R}}\{(\frac{\overline{a}}{\underline{b}}\|x_n - x\| + \overline{g}\|x_n - x\| + \overline{p}\frac{1}{e^2}\|x_n - x\|)\frac{1}{\underline{\gamma}}\} \qquad (4.10)\\
&= \frac{1}{\underline{\gamma}}(\frac{\overline{a}}{\underline{b}} + \overline{\gamma} + \frac{\overline{p}}{e^2})\|x_n - x\|.
\end{aligned}$$

However, since $\|x_n - x\| \to$ as $n \to \infty$, then it follows from (4.10) that

$$\|Ax_n - Ax\| \to 0 \text{ as } n \to +\infty.$$

which means that the operator A is continuous. Finally, we show that $A\Omega$ is relatively compact. For all $w \in \Omega$, we have

$$\begin{aligned}
|(Ax)(t)| &= |\int_{-\infty}^{t} e^{-\int_s^t \gamma(u)du}[-\frac{a(s)x(s)}{b(s)+x(s)} + \gamma(s)x(s) \\
&+ p(s)x(s-\tau(s))e^{-\beta(s)x(s-\tau(s))}]ds| \\
&\le \int_{-\infty}^{t} e^{-\int_s^t \gamma(u)du}[\frac{a(s)x(s)}{b(s)+x(s)} + \gamma(s)x(s) \\
&+ p(s)x(s-\tau(s))e^{-\beta(s)x(s-\tau(s))}]ds \\
&\le \int_{-\infty}^{t} e^{-\int_s^t \underline{\gamma}du}[\frac{\overline{a}x(s)}{\underline{b}+x(s)} + \overline{\gamma}L_2 + \overline{p}x(s-\tau(s))e^{-\underline{\beta}x(s-\tau(s))}]ds \\
&\le \int_{-\infty}^{t} e^{-\int_s^t \underline{\gamma}du}(\frac{\overline{a}L_2}{\underline{b}+L_2} + \overline{\gamma}L_2 + \overline{p}\frac{1}{\underline{\beta}e})ds \qquad (4.11) \\
&= (\frac{\overline{a}L_2}{\underline{b}+L_2} + \overline{\gamma}L_2 + \overline{p}\frac{1}{\underline{\beta}e}) \int_{-\infty}^{t} e^{-\underline{\gamma}(t-s)ds} \\
&= \frac{1}{\underline{\gamma}}(\frac{\overline{a}L_2}{\underline{b}+L_2} + \overline{\gamma}L_2 + \overline{p}\frac{1}{\underline{\beta}e}),
\end{aligned}$$

which implies that $A\Omega \to \Omega$ is uniformly bounded. By calculating the derivative of operator A, we get

$$\frac{d}{dt}(Ax)(t) = -\gamma(t)(Ax)(t) - \frac{a(t)x(t)}{b(t)+x(t)} + \gamma(t)x(t) + p(t)x(t-\tau(t))e^{-\beta(t)x(t-\tau(t))}.$$

Hence we have

$$\begin{aligned}
&|\frac{d}{dt}(Ax)(t)| = |-\gamma(t)(Ax)(t) - \frac{a(t)x(t)}{b(t)+x(t)} + \gamma(t)x(t) \\
&+ p(t)x(t-\tau(t))e^{-\beta(t)x(t-\tau(t))}| \\
&\le \gamma(t)|(Ax)(t)| + \frac{a(t)x(t)}{b(t)+x(t)} + \gamma(t)x(t) + p(t)x(t-\tau(t))e^{-\beta(t)x(t-\tau(t))} \\
&\le \overline{\gamma}|(Ax)(t)| + \frac{\overline{a}x(t)}{\underline{b}+x(t)} + \overline{\gamma}x(t) + \overline{p}x(t-\tau(t))e^{-\underline{\beta}x(t-\tau(t))} \\
&\le \overline{\gamma}\frac{1}{\underline{\gamma}}(\frac{\overline{a}L_2}{\underline{b}+L_2} + \overline{\gamma}L_2 + \overline{p}\frac{1}{\underline{\beta}e}) + \frac{\overline{a}L_2}{\underline{b}+L_2} + \overline{\gamma}L_2 + \overline{p}\frac{1}{\underline{\beta}e} \\
&= (1+\frac{\overline{\gamma}}{\underline{\gamma}})(\frac{\overline{a}L_2}{\underline{b}+L_2} + \overline{\gamma}L_2 + \overline{p}\frac{1}{\underline{\beta}e}),
\end{aligned}$$

which implies that $A : \Omega \to \Omega$ is equicontinuous. Since $A : \Omega \to \Omega$ is uniformly bounded and equicontinuous, by the Ascoli-Arzela theorem, we conclude that $A\Omega$ is relatively compact.

Thus, by Shauder's fixed point theorem, the operator A has at least one fixed point in Ω. This means that equation (1.3) has at least one almost periodic positive solution $w^*(t)$ satisfying $L_1 \le w^*(t) \le L_2$. The proof of Theorem 4 is complete. □

5. Exponential stability

To prove the main result of this section, we make the following assumptions:

(C_4) $L_2 > H$,

(C_5) $\overline{p} < \frac{\underline{ab}}{(\overline{b}+L_2)^2}$.

Let $U_{L_2} = \{\phi | \phi \in BC([-\overline{\tau}, 0], \mathbb{R}^*), < \phi(t) < L_2, t \in [-\overline{\tau}, 0]\}$.

Theorem 5.1. *Assume that the conditions $(C_1), (C_3), (C_4)$ and (C_5) hold. Then, every solution $x(t)$ of equation (1.3) with initial function $\phi \in U_{L_2}$ converges exponentially to $w^*(t)$ as $t \to +\infty$, where $w^*(t)$ is the almost periodic positive solution of equation (1.3) satisfying $L_1 \le w^*(t) \le L_2$.*

Proof. By Theorem 4 we know equation (1.3) has an almost periodic positive solution $w^*(t)$, and $L_1 \le w^*(t) \le L_2$. Assume the initial function of the almost periodic positive solution $w^*(t)$ is $w^*(t) = \psi(t) > 0$ for $-\overline{\tau} \le t \le 0$. Suppose $x(t)$ is arbitrary solution of equation (1.3) with initial function $phi \in U_{L_2}$, here $0 < \phi(t) < L_2$ and $x(t) = \phi(t)$ for $-\overline{\tau} \le t \le 0$. By Theorem 3.2 we know $0 < x(t) < L_2$ for all $t > 0$.

Consider function $G(x) = x - \frac{\underline{ab}}{(\overline{b}+L_2)^2} + \overline{p}e^{\overline{\tau}x}$, $x \in [0, 1]$.

Since $G(0) = x - \frac{\underline{ab}}{(\overline{b}+L_2)^2} + \overline{p} < 0$, then there exists a constant $\lambda \in (0, 1)$ such that $G(\lambda) < 0$. That is

$$\lambda - \frac{\underline{ab}}{(\overline{b}+L_2)^2} + \overline{p}e^{\lambda\overline{\tau}} < 0. \tag{5.1}$$

We define $V(t) = |x(t) - w^*(t)|e^{\lambda t}$, then it follows that

$$\begin{aligned}
D^*V(t) \le & [-a(t)|\frac{x(t)}{b(t)+x(t)} - \frac{w^*(t)}{b(t)+w^*(t)}| \\
& + |p(t)x(t-\tau(t))e^{-\beta(t)x(t-\tau(t))} - p(t)w^*(t-\tau(t))e^{-\beta(t)w^*(t-\tau(t))}|]e^{e^{\lambda t}} \\
& + \lambda|x(t) - w^*(t)|e^{\lambda t} \\
= & [-a(t)b(t)\frac{|x(t)-w^*(t)|}{(b(t)+x(t))(b(t)+w^*(t))} \\
& + p(t)|x(t-\tau(t))e^{-\beta(t)x(t-\tau(t))} - w^*(t-\tau(t))e^{-\beta(t)w^*(t-\tau(t))}|]e^{\lambda t} \\
& + \lambda|x(t) - w^*(t)|e^{\lambda t} \\
\le & [-\underline{ab}\frac{|x(t)-w^*(t)|}{(\overline{b}+L_2)^2}] + \overline{p}|x(t-\tau(t))e^{-\beta(t)x(t-\tau(t))} \\
& - w^*(t-\tau(t))e^{-\beta(t)w^*(t-\tau(t))}| \\
& + \lambda|x(t) - w^*(t)|e^{\lambda t}.
\end{aligned} \tag{5.2}$$

Using the inequality $|xe^{-x} - ye^{-y}| \leq |x - y|$, for $x > 0, y > 0$, we get

$$
\begin{aligned}
&|x(t-\tau(t))|e^{-\beta(t)x(t-\tau(t))} - w^*(t-\tau(t))e^{-\beta(t)w^*(t-\tau(t))}| \\
&= \frac{1}{\beta(t)}|\beta(t)x(t-\tau(t))e^{-\beta(t)x(t-\tau(t))} - \beta(t)|w^*(t-\tau(t))e^{-\beta(t)w^*(t-\tau(t))}| \\
&\leq \frac{1}{\beta(t)}|\beta(t)x(t-\tau(t)) - \beta(t)w^*(t-\tau(t))| \\
&= |x(t-\tau(t)) - w^*(t-\tau(t))|.
\end{aligned} \tag{5.3}
$$

Hence, (5.2) and (5.3) imply that

$$
\begin{aligned}
D^+V(t) &\leq [-\frac{ab}{(\bar{b}+L_2)^2}|x(t) - w^*(t)| + \overline{p}|x(t-\tau(t)) \\
&\quad - w^*(t-\tau(t))|]e^{\lambda t} + \lambda|x(t) - w^*(t)|e^{\lambda t} \\
&= \lambda V(t) - \frac{ab}{(\bar{b}+L_2)^2}V(t) + \overline{p}|x(t-\tau(t)) - w^*(t-\tau(t))|e^{\lambda t}.
\end{aligned} \tag{5.4}
$$

Let $h = L_2 + \sup_{-\overline{\tau}\leq t\leq 0}|\phi(t) - \psi(t)|$. For all $t \in [-\overline{\tau}, 0]$, we get

$$
\begin{aligned}
V(t) &= |x(t) - w^*(t)|e^{\lambda t} \leq |x(t) - w^*(t)| = |\phi(t) - \psi(t)| \\
&\leq \sup_{-\overline{\tau}\leq t\leq 0}|\phi(y) - \psi(y)| < L_2 + \sup_{-\overline{\tau}\leq t\leq 0}|\phi(t) - \psi(t)| = h.
\end{aligned}
$$

We claim that

$$
V(t) < h, \quad \text{for all} \quad t > 0. \tag{5.5}
$$

Suppose the claim (5.5) is not true, then there must exist a $t^* > 0$, such that $V(t^*) = h$, $D^+V(t)|_{t=t^*} \geq 0$ and $V(t) < h$ for $t < t^*$. It follows from (5.4) that

$$
\begin{aligned}
0 &\leq D^+V(t)|_{t=t^*} \leq \lambda V(t^*) - \frac{ab}{(\bar{b}+L_2)^2}V(t^*) \\
&\quad + \overline{p}|x(t^*-\tau(t^*)) - w^*(t^*-\tau(t^*))|e^{\lambda t^*} \\
&= \lambda h - \frac{ab}{(\bar{b}+L_2)^2}h + \overline{p}|x(t^*-\tau(t^*)) - w^*(t^*-\tau(t^*))|e^{\lambda(t^*-\tau(t^*))}e^{\lambda t(t^*)} \\
&= \lambda h - \frac{ab}{(\bar{b}+L_2)^2}h + \overline{p}V(t^*-\tau(t^*))e^{\lambda\tau(t^*)} \\
&< \lambda h - \frac{\overline{a}\overline{b}}{(\bar{b}+L_2)^2}h + \overline{p}he^{\lambda\tau(t^*)} \\
&\leq \lambda h - \frac{ab}{(\bar{b}+L_2)^2}h + \overline{p}he^{\lambda\overline{\tau}} \\
&= (\lambda - \frac{ab}{(\bar{b}+L_2)^2} + \overline{p}e^{\lambda\overline{\tau}}).
\end{aligned} \tag{5.6}
$$

Thus, from (5.6) we have $\lambda - \frac{ab}{(\bar{b}+L_2)^2} + \overline{p}e^{\lambda\overline{\tau}} > 0$, which contradicts (5.1).

Therefore, the claim (5.5) is true. Hence $V(t) = |x(t) - w^*(t)|e^{\lambda t} < h$, for all $t > 0$.

That is $|x(t) - w^*(t)|$, for all $t > 0$, which means $x(t)$ converges exponentially to $w^*(t)$ as $t \to +\infty$. The proof of Theorem 5 is complete. $\square$

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