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A NOVEL VIEW OF ROUGH SOFT SEMIGROUPS BASED ON FUZZY IDEALS

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Abstract. By using a special t-level relation $U(\mu,t) = \{(x,y) \in S \times S | (\mu(x) \wedge \mu(y)) \vee Id_S(x,y) \geq t\}$ based on a fuzzy ideal μ of a semigroup S, which is a congruence relation, we study the roughness of soft semigroups under this special ideal of S, such as rough soft subsemigroups, rough soft ideals and rough soft prime ideals.

Keywords: fuzzy ideal; rough soft set; rough soft (prime, bi) ideal.

1. Introduction

Pawlak firstly proposed rough set theory as a new way to solve imprecision, inconsistent and incomplete problems in [14]. The core of rough set theory is a pair of lower and upper approximations operators induced from an approximation space. Just because rough sets are widely used in many areas such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on. After rough sets were introduced, more and more scholars paid their attention to rough sets. Just like the study of rough sets in [2], [17] and [19]. With the studing of rough sets, some researchers turned their attention to the roughness of algebraic systems, such as in [9], Kuroki showed us rough ideals in semigroups. Followed by this, the roughness of τ-subsemigroups and ideals in τ-semigroups were discussed by Jun in [7]. In [3], making use of an ideal of a ring, the author introduced rough ideals and rough subrings and considered the ring as a universal set. Especially in [17], the roughness of semigroups were researched and Zhan et al. in [18] studied rough soft rings.

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Molodtsov firstly put forward soft sets in 1999 in [11], which were mainly used to solve the uncertainty and vagueness problems. Since Molodtsov in [11] proposed soft sets, the applications of soft sets were widely used, such as smoothness of functions, game theory, operations research, Rlemann-integration, Perron integration, probability, theory of measurement and so on. As a new mathematical branch, soft sets were researched by more and more researchers. With the help of rough sets of Pawlak [14], Maji et al. [13], defined a parameter set over soft sets, and gave the decision making problems of soft sets at the same time. Kong et al. [10] presented a new definition of the parameter reduction. Maji et al. in [12] gave a theoretical study on soft sets. Since then, the study of soft sets theories begun to build over algebraic structure increasingly in recent years. And then in [1], Aktaş and Çağman firstly put up the definitions of soft groups. They also showed us the relations between soft sets and fuzzy sets ang rough sets. Jun in [8], introduced the soft BCK/BCI-algebras and soft subalgebras. Jun in [7] put forward roughness of gamma-subsemigroups/ideals in gamma-subsemigroups. In [5], Feng et al. gave a study of soft semirings over soft sets theory, and then obtained some relevant conclusions. And in [4], Feng et al. introduced soft relations over semigroups, Shabir et al. researched soft ternary semigroups in [15]. As we all know, soft set is a new way to deal with decision-making problems.

In [11] and [14], as we all know, rough sets and soft sets are all the tools which can solve uncertainty and vagueness problems. And we know, a soft set (F, A) can be regarded as a parameterized family of subsets of universe U, which gives an approximation (soft) description of the objects in U. As pointed in [11], for any parameter $\epsilon \in A$, the subset $F(\epsilon) \subseteq U$ may be considered as the set of ϵ-approximation elements in the soft set (F, A). As a set, soft sets also have the rough approximations. So we realized that we can combine the rough sets and soft sets and research their properties. Just like in [6], Feng et al. told us a new tentative approach that applied fuzzy sets and rough sets to soft sets. And in the following researches, Zhan in [20] showed us that the roughness of soft hemirings and obtained the rough soft hemirings based on equivalence relation ρ, and got some related properties and conclusions. It is no doubt that an equivalence relation is essential in any rough sets, so in Zhan et al. in [21] made use of a strong h-ideal as the equivalence relation and got the rough soft hemirings, especially gave us the applications of rough soft hemirings in decision making. It is clearly that the t-level relation $U(\mu, t)$ of a fuzzy ideal μ is a congruence relation.

As a special algebraic structure, a semigroup has many good properties. Since 1950, the studies of finite semigroups are especially important in theoretical computer science, just because there is a natural connection between finite semigroups and finite automaton. Also we can combine semigroups with rough sets or soft sets, just like in [17], Xiao et al. gave us rough prime ideals and rough fuzzy prime ideals in semigroups. And in [22], Zhan et al. made use of the fuzzy ideal as the congruence relation and defined the rough n-ary semigroups, and then got some conclusions and properties. Recently, Wang and Zhan [16] defined a t-level relation $U(\mu, t)$ of a fuzzy ideal μ in a semigroups S and proved that it is a congruence relation.

In this paper, we study the rough soft semigroups in four parts. Firstly, we recall some basic notions which are needed in this paper in Section 2. And then we define the rough soft semigroups over fuzzy ideals in Section 3. Finally, we research the rough structure of soft semigroups over fuzzy ideals and study some relative properties in Section 4.

2. Preliminaries

In this section, some basic notions such as semigroups, rough sets, soft sets and so on, are briefly described.

2.1. Semigroups

A semigroup S is a nonempty set with a binary operation "·" such that

(i) $a \cdot b \in S$, for all $a, b \in S$;

(ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in S$.

A subset A of S is called a subsemigroup if A is closed under "·"; A is called a left (right) ideal if A is closed under "·" and $SA \subseteq A(AS \subseteq A)$; If A is not only a left ideal but also is a right ideal, then it is an ideal; A ideal A is called a prime ideal if $a \cdot b \in A$ implies $a \in A$ or $b \in A$, for all $a, b \in A$; A nonempty subsemigroup T of S is called a bi-ideal if $TST \subseteq T$.

A fuzzy set μ of S is called a fuzzy subsemigroup if it satisfies $\mu(x \cdot y) \geq \mu(x) \wedge \mu(y)$; μ is called a fuzzy left (right) ideal if it satisfies $(1)\mu(x \cdot y) \geq \mu(x) \wedge \mu(y)$ $(2)\mu(x \cdot y) \geq \mu(y)$ $(\mu(x \cdot y) \geq \mu(x))$, for all $x, y \in S$; μ is called a fuzzy ideal if it is not only a fuzzy left ideal but also is a fuzzy right ideal.

2.2. Soft sets

Definition 2.1 [11] A pair (F, A) is called a soft set over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping.

Definition 2.2 [21] Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two soft sets over a common semigroup. Then the "multiplication" of $\mathfrak{S}$ and $\mathfrak{T}$, denoted by $\mathfrak{S} \cdot \mathfrak{T} = (F, A) \cdot (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cdot G(y)$ for all $(x, y) \in A \times B$.

Definition 2.3 Let (F, A) be a soft set over S. Then

(1) (F, A) is called a soft semigroup over S if $F(x)$ is a semigroup of S for all $x \in \text{Supp}(F, A)$.

(2) (F, A) is called an idealistic soft semigroup if $F(x)$ is an ideal of S for all $x \in \text{Supp}(F, A)$.

(3) (F, A) is called a prime idealistic soft semigroup if $F(x)$ is a prime ideal of S for all $x \in \text{Supp}(F, A)$.

(4) (F, A) is called a bi-idealistic soft semigroup if $F(x)$ is a bi-ideal of S for all $x \in \text{Supp}(F, A)$.

2.3. Rough sets

Definition 2.4 [14] Let ρ be an equivalence relation on the universe U, (U, ρ) be a Pawlak approximation space. A subset $A \subseteq U$ is called definable if $\underline{\rho}(A) = \overline{\rho}(A)$, otherwise, U is a rough set, where

$$\underline{\rho}(A) = \{x \in U : [x]_\rho \subseteq A\},$$

and

$$\overline{\rho}(A) = \{x \in U : [x]_\rho \cap A \neq \emptyset\}.$$

Definition 2.5 [6] Let (U, ρ) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft set over U. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ w.r.t. (U, ρ) are denoted by $\underline{\rho}(\mathfrak{S}) = (\underline{F}, A)$, and $\overline{\rho}(\mathfrak{S}) = (\overline{F}, A)$, which are soft sets over U with

$$\underline{F}(x) = \underline{\rho}(F(x)) = \{y \in U | [y]_\rho \subseteq F(x)\},$$

and

$$\overline{F}(x) = \overline{\rho}(F(x)) = \{y \in U | [y]_\rho \cap F(x) \neq \emptyset\},$$

for all $x \in A$.

If $\underline{\rho}(\mathfrak{S}) = \overline{\rho}(\mathfrak{S})$, $\mathfrak{S}$ is called definable; otherwise $\mathfrak{S}$ is called a rough soft set.

3. Rough soft semigroups based on fuzzy ideals

In [16], we proved that $U(\mu, t)$ is a congruence relation in a semigroup S if μ is a fuzzy ideal of S. By means of this point, we study rough soft semigroups in this section. And in this paper, let S be a semigroup in the following.

Definition 3.1 [16] Let μ be a fuzzy ideal of S. For each $t \in [0, 1]$, the set $U(\mu, t) = \{(x, y) \in S \times S | (\mu(x) \wedge \mu(y)) \vee Id_S(x, y) \geq t\}$ is called a t-level relation of μ.

Lemma 3.2 [16] *Let μ and ν be two fuzzy ideals of S such that $\mu \subseteq \nu$ and $t \in [0, 1]$. Then $[x]_{(\mu,t)} \subseteq [x]_{(\nu,t)}$ for all $x \in S$.*

For any fuzzy ideal μ, $\mu(0) \geq \mu(x)$ and $\mu(0) \leq 1$, so when $t \in [0, \mu(0)]$, $U(\mu, t)$ is a congruence relation on S (see [16]). We say x is congruent to y model μ, written $x \equiv_t y(mod\mu)$. If for elements $x, y \in S$, $t \in [0, 1]$, $(\mu(x) \wedge \mu(y)) \vee Ids(x, y) \geq t$, let $[x]_{(\mu,t)}$ as the equivalence class of x. However, $U(\mu, t)$ is not a complete congruence relation. We can only obtain a conclusion as follows.

Lemma 3.3 [16] *Let μ be a fuzzy ideal of S, and $t \in [0,1]$, then $[x]_{(\mu,t)} \cdot [y]_{(\mu,t)} \subseteq [xy]_{(\mu,t)}$.*

Example 3.4 Let $S = \{a, b, c, d\}$ be a semigroup with the following "·" table.

·	a	b	c	d
a	a	b	b	d
b	b	b	b	d
c	b	b	b	d
d	d	d	d	d

Let $\mu = \frac{0.3}{a} + \frac{0.5}{b} + \frac{0.1}{c} + \frac{0.8}{d}$ be the fuzzy set of S and $t = 0.4$, it is easy to prove μ is a fuzzy ideal of S. Thus $U(\mu; 0.4) = \{(a,a), (b,b), (c,c), (d,d), (b,d)\}$, so $[a]_{(\mu,0.4)} = \{a\}$, $[b]_{(\mu,0.4)} = \{b, d\}$, $[c]_{(\mu,0.4)} = \{c\}$, $[d]_{(\mu,0.4)} = \{b, d\}$. $[a]_{(\mu,0.4)} \cdot [c]_{(\mu,0.4)} = \{b\}$. Since $a \cdot c = b$, so $[ac]_{(\mu,0.4)} = \{b, d\}$. Obviously, $[a]_{(\mu,0.4)} \cdot [c]_{(\mu,0.4)} \subseteq [a \cdot c]_{(\mu,0.4)}$.

Definition 3.5 $U(\mu; t)$ is called a complete congruence relation if it satisfies: For any elements $x, y \in S$, $[x]_{(\mu,t)} \cdot [y]_{(\mu,t)} = [xy]_{(\mu,t)}$.

Example 3.6 Let $S = \{0, a, b, c\}$ be a semigroup with the following "·" table.

·	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	b

Assume that $\mu = \frac{0.1}{0} + \frac{0.4}{a} + \frac{0.7}{b} + \frac{0.7}{c}$ is a fuzzy set of S and $t = 0.7$, obviously, μ is a fuzzy ideal of S. Thus $U(\mu; 0.7) = \{(0,0), (a,a), (b,b), (c,c), (b,c)\}$, so $[0]_{(\mu,0.7)} = \{0\}$, $[a]_{(\mu,0.7)} = \{a\}$, $[b]_{(\mu,0.7)} = \{b, c\}$, $[c]_{(\mu,0.7)} = \{b, c\}$. Obviously, we can easily check $U(\mu; t)$ is a complete congruence relation.

Let μ be a fuzzy ideal of S, $t \in [0, 1]$. Thus $U(\mu; t)$ is a congruence relation. Therefore, when $U = S$ and ρ is the above equivalence relation, then we use (S, μ, t) instead of approximation space (U, ρ).

Definition 3.7 Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft set over S. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ with respect to (S, μ, t) are denoted by: $\underline{U}(\mu, t, \mathfrak{S}) = (\underline{F}_\mu, A)$ and $\overline{U}(\mu, t, \mathfrak{S}) = (\overline{F}_\mu, A)$, which are soft sets over S with $\underline{F}_\mu(x) = \underline{U}(\mu, t, F(x)) = \{y \in S | [y]_{(\mu,t)} \subseteq F(x)\}$ and $\overline{F}_\mu(x) = \overline{U}(\mu, t, F(x)) = \{y \in S | [y]_{(\mu,t)} \cap F(x) \neq \emptyset\}$, for all $x \in A$.

(i) $\underline{U}(\mu, t, \mathfrak{S}) = \overline{U}(\mu, t, \mathfrak{S})$, $\mathfrak{S}$ is called definable.

(ii) $\underline{U}(\mu, t, \mathfrak{S}) \neq \overline{U}(\mu, t, \mathfrak{S})$, $\underline{U}(\mu, t, \mathfrak{S})(\overline{U}(\mu, t, \mathfrak{S}))$ is called a lower (upper) rough soft set. Moreover, $\mathfrak{S}$ is called a rough soft set.

Example 3.8 Based on Example 3.4, then we have $[a]_{(\mu,0.4)} = \{a\}$, $[b]_{(\mu,0.4)} = \{b,d\}$, $[c]_{(\mu,0.4)} = \{c\}$, $[d]_{(\mu,0.4)} = \{b,d\}$. Define a soft set $\mathfrak{S} = (F,A)$ over S. Let $A = \{x_1, x_2\}$, where $F(x_1) = \{a,b\}$, $F(x_2) = \{c,d\}$, obviously, $\underline{F}_\mu(x_1) = \{a\}$, $\underline{F}_\mu(x_2) = \{c\}$, $\overline{F}_\mu(x_1) = \{a,b,d\}$, $\overline{F}_\mu(x_2) = \{a,b,d\}$. Thus, $\mathfrak{S}$ is a rough soft set.

The following two theorems are straightforward and we omit the proofs.

Theorem 3.9 *Let μ be a fuzzy ideal of S and $\mathfrak{S} = (F,A)$ be a soft set over S. Then we have:*

(1) $\underline{U}(\mu,t,\mathfrak{S}) \subseteq \mathfrak{S} \subseteq \overline{U}(\mu,t,\mathfrak{S})$.

(2) $\underline{U}(\mu,t,\underline{U}(\mu,t,\mathfrak{S})) = \underline{U}(\mu,t,\mathfrak{S})$.

(3) $\overline{U}(\mu,t,\overline{U}(\mu,t,\mathfrak{S})) = \overline{U}(\mu,t,\mathfrak{S})$.

(4) $\overline{U}(\mu,t,\underline{U}(\mu,t,\mathfrak{S})) = \underline{U}(\mu,t,\mathfrak{S})$.

(5) $\underline{U}(\mu,t,\overline{U}(\mu,t,\mathfrak{S})) = \overline{U}(\mu,t,\mathfrak{S})$.

(6) $\underline{U}(\mu,t,\mathfrak{S}) = (\overline{U}(\mu,t,\mathfrak{S})^r)^r$.

(7) $\overline{U}(\mu,t,\mathfrak{S}) = (\underline{U}(\mu,t,\mathfrak{S})^r)^r$.

Theorem 3.10 *Let μ be a fuzzy ideal of S and $\mathfrak{S} = (F,A)$ and $\mathfrak{T} = (G,B)$ be soft sets over S. Then we have:*

(1) $\underline{U}(\mu,t,\mathfrak{S} \sqcap \mathfrak{T}) = \underline{U}(\mu,t,\mathfrak{S}) \sqcap \underline{U}(\mu,t,\mathfrak{T})$.

(2) $\underline{U}(\mu,t,\mathfrak{S} \sqcap_\varepsilon \mathfrak{T}) = \underline{U}(\mu,t,\mathfrak{S}) \sqcap_\varepsilon \underline{U}(\mu,t,\mathfrak{T})$.

(3) $\overline{U}(\mu,t,\mathfrak{S} \sqcap \mathfrak{T}) \subseteq \overline{U}(\mu,t,\mathfrak{S}) \sqcap \overline{U}(\mu,t,\mathfrak{T})$.

(4) $\overline{U}(\mu,t,\mathfrak{S} \sqcap_\varepsilon \mathfrak{T}) \subseteq \overline{U}(\mu,t,\mathfrak{S}) \sqcap_\varepsilon \overline{U}(\mu,t,\mathfrak{T})$.

(5) $\underline{U}(\mu,t,\mathfrak{S} \cup_R \mathfrak{T}) \supseteq \underline{U}(\mu,t,\mathfrak{S}) \cup_R \underline{U}(\mu,t,\mathfrak{T})$.

(6) $\underline{U}(\mu,t,\mathfrak{S} \tilde{\cup} \mathfrak{T}) \supseteq \underline{U}(\mu,t,\mathfrak{S}) \tilde{\cup} \underline{U}(\mu,t,\mathfrak{T})$.

(7) $\overline{U}(\mu,t,\mathfrak{S} \cup_R \mathfrak{T}) = \overline{U}(\mu,t,\mathfrak{S}) \cup_R \overline{U}(\mu,t,\mathfrak{T})$.

(8) $\overline{U}(\mu,t,\mathfrak{S} \tilde{\cup} \mathfrak{T}) = \overline{U}(\mu,t,\mathfrak{S}) \tilde{\cup} \overline{U}(\mu,t,\mathfrak{T})$.

(9) $\mathfrak{S} \subseteq \mathfrak{T} \Rightarrow \underline{U}(\mu,t,\mathfrak{S}) \subseteq \underline{U}(\mu,t,\mathfrak{T})$, $\overline{U}(\mu,t,\mathfrak{S}) \subseteq \overline{U}(\mu,t,\mathfrak{T})$.

Proposition 3.11 *Let* $\mathfrak{S} = (F, A)$ *and* $\mathfrak{T} = (G, B)$ *be two non-null soft sets over* S, μ *be a fuzzy ideal of* S. *Then*

$$\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \subseteq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T}).$$

Proof. For any $a \in Supp(F, A)$, $b \in Supp(G, B)$. Let $m \in \overline{U}(\mu, t, F(x)) \cdot \overline{U}(\mu, t, G(y))$. There exists $a_i \in \overline{U}(\mu, t, F(x))$, $b_i \in \overline{U}(\mu, t, G(y))$ such that $m = a_i b_i$, so $[a_i]_{(\mu, t)} \cap F(x) \neq \emptyset$ and $[b_i]_{(\mu, t)} \cap G(y) \neq \emptyset$. Now let $x_i \in [a_i]_{(\mu, t)} \cap F(x)$, $y_i \in [b_i]_{(\mu, t)} \cap G(y)$, then $(x_i, a_i) \in U(\mu, t)$ and $(y_i, b_i) \in U(\mu, t)$. As $U(\mu, t)$ is a congruence relation, so $(x_i y_i, a_i b_i) \in U(\mu, t)$. Since $x_i y_i \in F(x) \cdot G(y)$, Therefore $[a_i b_i]_{(\mu, t)} \cap F(x) \cdot G(y) \neq \emptyset$, that is $[m]_{(\mu, t)} \cap F(x) \cdot G(y) \neq \emptyset$. This means $m \in \overline{U}(\mu, t, F(x)G(y))$ and $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \subseteq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. ∎

The following example shows that the converse of Proposition 3.11 is not true.

Example 3.12 Let $S = \{a, b, c, d\}$ be a semigroup with the following "$\cdot$" table.

$\cdot$	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

Let $\mu = \frac{0.5}{a} + \frac{0.3}{b} + \frac{0.1}{c} + \frac{0.8}{d}$ be a fuzzy set of S, μ is a fuzzy ideal of S. Let $t = 0.3$, then $[a]_{(\mu, 0.3)} = [b]_{(\mu, 0.3)} = [d]_{(\mu, 0.3)} = \{a, b, d\}$, $[c]_{(\mu, 0.3)} = \{c\}$. Obviously, $U(\mu; t)$ is a congruence relation. Then we define two soft sets $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ over S, where $F(e_1) = \{a, c\}$, $F(e_2) = \{b, c\}$ and $G(e_3) = \{a, b, d\}$. Then $\overline{U}(\mu, t, F(e_1)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, F(e_2)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, G(e_3)) = \{a, b, d\}$. Thus $\overline{U}(\mu, t, F(e_1)) \cdot \overline{U}(\mu, t, G(e_3)) = \{a, b, d\}$, $\overline{U}(\mu, t, F(e_2)) \cdot \overline{U}(\mu, t, G(e_3)) = \{a, b, d\}$. On the other hand, $\overline{U}(\mu, t, F(e_1) \cdot G(e_3)) = \{a, b, d\}$, $\overline{U}(\mu, t, F(e_2) \cdot G(e_3)) = \{a, b, c, d\}$. Therefore, obviously, $\overline{U}(\mu, t, F(e_2)) \cdot \overline{U}(\mu, t, G(e_3)) \subseteq \overline{U}(\mu, t, F(e_2) \cdot G(e_3))$. Hence, $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \subsetneqq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$.

Proposition 3.13 *Let* $\mathfrak{S} = (F, A)$ *and* $\mathfrak{T} = (G, B)$ *be two non-null soft sets over* S, μ *be a fuzzy ideal of* S, $U(\mu, t)$ *be a complete congruence relation. Then*

$$\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) = \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T}).$$

Proof. Firstly, from Proposition 3.11, we have $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \subseteq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. So we only prove $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \supseteq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. For any $a \in Supp(F, A)$, $b \in Supp(G, B)$. Let $m \in \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$, then $[m]_{(\mu, t)} \cap F(x)G(y) \neq \emptyset$, that is, there exists $n \in [m]_{(\mu, t)}$ and $n \in F(x) \cdot G(y)$ such that $n = a_i b_i$, for $a_i \in F(x), b_i \in G(y)$ and $m \in [n]_{(\mu, t)} = [a_i b_i]_{(\mu, t)} = [a_i]_{(\mu, t)} \cdot [b_i]_{(\mu, t)}$. Since $U(\mu, t)$ is a complete congruence relation, then there exists $x_i \in [a_i]_{(\mu, t)}$ and $y_i \in [b_i]_{(\mu, t)}$ such that $m = x_i y_i$, hence $a_i \in [x_i]_{(\mu, t)} \cap F(x)$ and $b_i \in [x_i]_{(\mu, t)} \cap G(y)$, therefore $m \in \overline{U}(F(x)) \cdot \overline{U}(G(y))$, so $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) = \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. ∎

Example 3.14 Let $S = \{a, b, c, d\}$ be a semigroup with the following "·" table:

·	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

Assume that $\mu = \frac{0.6}{a} + \frac{0.4}{b} + \frac{0.1}{c} + \frac{0.9}{d}$ is a fuzzy set of S and $t = 0.5$, μ is a fuzzy ideal of S, then we have $U(\mu; 0.5) = \{(a, a), (b, b), (c, c), (d, d), (a, d)\}$, so we have $[a]_{(\mu, 0.5)} = [d]_{(\mu, 0.5)} = \{a, d\}$, $[b]_{(\mu, 0.5)} = \{b\}$, $[c]_{(\mu, 0.5)} = \{c\}$. Obviously, $U(\mu; t)$ is a complete congruence relation.

Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two soft sets over S, where $A = \{e_1, e_2\}$, $B = \{e_3\}$, $G(e_3) = \{b, c, d\}$. And $F(e_1) = \{a, b, c\}$, $F(e_2) = \{b, d\}$ and $G(e_3) = \{b, c, d\}$. Then $\overline{U}(\mu, t, F(e_1)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, F(e_2)) = \{a, b, d\}$, $\overline{U}(\mu, t, G(e_3)) = \{a, b, c, d\}$. Thus $\overline{U}(\mu, t, F(e_1)) \cdot \overline{U}(\mu, t, G(e_3)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, F(e_2)) \cdot \overline{U}(\mu, t, G(e_3)) = \{a, b, d\}$. $\overline{U}(\mu, t, F(e_1) \cdot G(e_3)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, F(e_2) \cdot G(e_3)) = \{a, b, d\}$. So $\overline{U}(\mu, t, F(e_1)) \cdot \overline{U}(\mu, t, G(e_3)) = \overline{U}(\mu, t, F(e_1) \cdot G(e_3))$, $\overline{U}(\mu, t, F(e_2)) \cdot \overline{U}(\mu, t, G(e_3)) = \overline{U}(\mu, t, F(e_2) \cdot G(e_3))$. Therefore, we know Proposition 3.13 is proper.

Proposition 3.15 *Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two non-null soft sets over S, μ be a fuzzy ideal of S and $U(\mu, t)$ be a complete congruence relation. Then*

$$\underline{U}(\mu, t, \mathfrak{S}) \cdot \underline{U}(\mu, t, \mathfrak{T}) \subseteq \underline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T}).$$

Proof. For any $a \in Supp(F, A)$, $b \in Supp(G, B)$. Let $m \in \underline{U}(\mu, t, \mathfrak{S}) \cdot \underline{U}(\mu, t, \mathfrak{T})$, then $m = a_i b_i$ where $a_i \in \underline{U}(\mu, t, F(x))$ and $b_i \in \underline{U}(\mu, t, G(y))$, that is, $[a_i]_{(\mu, t)} \subseteq F(x)$ and $[b_i]_{(\mu, t)} \subseteq G(y)$. So $[m]_{(\mu, t)} = [a_i b_i]_{(\mu, t)} = [a_i]_{(\mu, t)} \cdot [b_i]_{(\mu, t)} \subseteq F(x) \cdot G(y)$, therefore $m \in \underline{U}(\mu, t, F(x) \cdot G(y))$, that implies, $\underline{U}(\mu, t, \mathfrak{S}) \cdot \underline{U}(\mu, t, \mathfrak{T}) \subseteq \underline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. ∎

Example 3.16 In Example 3.4, let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two soft sets over S, where $F(e_1) = \{a, b, c\}$, $F(e_2) = \{b, d\}$, $G(e_3) = \{b, c, d\}$, thus $\underline{U}(\mu, t, F(e_1)) = \{b, c\}$, $\underline{U}(\mu, t, F(e_2)) = \{b\}$, $\underline{U}(\mu, t, G(e_3)) = \{b, c\}$, so $\underline{U}(\mu, t, F(e_1)) \cdot \underline{U}(\mu, t, G(e_3)) = \{a, b, c\}$, $\underline{U}(\mu, t, F(e_2)) \cdot \underline{U}(\mu, t, G(e_3)) = \{a, b\}$, $\underline{U}(\mu, t, F(e_1) \cdot G(e_3)) = \{a, b, c, d\}$, $\underline{U}(\mu, t, F(e_2) \cdot G(e_3)) = \{a, b, d\}$, so $\underline{U}(\mu, t, \mathfrak{S}) \cdot \underline{U}(\mu, t, \mathfrak{T}) \subsetneq \underline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$.

4. Characterizations of rough soft semigroups based on $U(\mu, t)$

In this section, we study the characterizations of rough soft semigroups based on $U(\mu, t)$.

Proposition 4.1 *Let μ and ν be two fuzzy ideals of S and $t \in [0, 1]$. If $\mathfrak{S} = (F, A)$ is a non-null soft set over S, then*

$$\overline{U}(\mu \cap \nu, t, \mathfrak{S}) \subseteq \overline{U}(\mu, t, \mathfrak{S}) \cap \overline{U}(\nu, t, \mathfrak{S}).$$

Proof. For all $x \in \text{Supp}(F, A)$, let $a \in \overline{F}_{\mu \cap \nu}(x)$, then we have $[a]_{(\mu \cap \nu, t)} \cap F(x) \neq \emptyset$. So there exists $m \in [a]_{(\mu \cap \nu, t)} \cap F(x)$, it follows that $(a, m) \in U(\mu \cap \nu, t)$ and $m \in F(x)$. This means $((\mu \cap \nu)(a) \wedge (\mu \cap \nu)(m)) \vee Id_S(a, m) \geq t$ and $m \in F(x)$. If $a = m$, then $a \in F(x)$ and $a \in [a]_{(\mu, t)}$. So $a \in [a]_{(\mu, t)} \cap F(x)$, that is $a \in \overline{F}_\mu(x)$. In similar way, we have $a \in \overline{F}_\nu(x)$. Hence $a \in \overline{F}_\mu(x) \cap \overline{F}_\nu(x)$. If $a \neq m$, then $(\mu \cap \nu)(a) \wedge (\mu \cap \nu)(m) \geq t$ and $m \in F(x)$, it implies $\mu(a) \wedge \mu(m) \geq t$ and $\nu(a) \wedge \nu(m) \geq t$. Thus, $(a, m) \in U(\mu, t), (a, m) \in U(\nu, t)$ and $m \in F(x)$. Hence $m \in [a]_{(\mu, t)} \cap F(x)$ and $m \in [a]_{(\nu, t)} \cap F(x)$. This means $[a]_{(\mu, t)} \cap F(x) \neq \emptyset$ and $[a]_{(\nu, t)} \cap F(x) \neq \emptyset$, it implies $a \in \overline{F}_\mu(x)$ and $a \in \overline{F}_\nu(x)$. So $a \in \overline{F}_\mu(x) \cap \overline{F}_\nu(x)$. Hence $\overline{F}_{(\mu \cap \nu)}(x) \subseteq \overline{F}_\mu(x) \cap \overline{F}_\nu(x)$. Therefore $\overline{U}(\mu \cap \nu, t, \mathfrak{S}) \subseteq \overline{U}(\mu, t, \mathfrak{S}) \cap \overline{U}(\nu, t, \mathfrak{S})$. This completes the proof. ∎

Proposition 4.2 *Let μ and ν be two fuzzy ideals of S and $t \in [0, 1]$. If $\mathfrak{S} = (F, A)$ is a non-null soft set over S, then*

$$\underline{U}(\mu, t, \mathfrak{S}) \cap \underline{U}(\nu, t, \mathfrak{S}) \subseteq \underline{U}(\mu \cap \nu, t, \mathfrak{S}).$$

Proof. For all $x \in \text{Supp}(F, A)$, let $a \in \underline{F}_\mu(x) \cap \underline{F}_\nu(x)$, we have $a \in \underline{F}_\mu(x)$ and $a \in \underline{F}_\nu(x)$, it implies $[a]_{(\mu, t)} \subseteq F(x)$ and $[a]_{(\nu, t)} \subseteq F(x)$. By Lemma 3.2, $[a]_{(\mu \cap \nu, t)} \subseteq [a]_{(\mu, t)} \subseteq F(x)$, this means $a \in \underline{F}_{\mu \cap \nu}(x)$. Hence $\underline{U}(\mu, t, \mathfrak{S}) \cap \underline{U}(\nu, t, \mathfrak{S}) \subseteq \underline{U}(\mu \cap \nu, t, \mathfrak{S})$. ∎

Example 4.3 Let $S = \{a, b, c, d\}$ be a semigroup in Example 3.12. Let $\mathfrak{S} = (F, A)$ be a soft set and μ and ν be fuzzy ideals over S, where $F(e_1) = \{b, c\}$, $F(e_2) = \{a, b\}$, $F(e_3) = \{c, d\}$, $\mu = \frac{0.7}{a} + \frac{0.3}{b} + \frac{0.5}{c} + \frac{0.8}{d}$, $\nu = \frac{0.5}{a} + \frac{0.5}{b} + \frac{0.4}{c} + \frac{0.9}{d}$, $t = 0.5$, $\mu \cap \nu = \frac{0.5}{a} + \frac{0.3}{b} + \frac{0.4}{c} + \frac{0.8}{d}$, then we have $U(\mu, 0.5) = \{\{a, a\}, \{b, b\}, \{c, c\}, \{d, d\}, \{a, c\}, \{a, d\}, \{c, d\}\}$, so $[a]_{(\mu, 0.5)} = [c]_{(\mu, 0.5)} = [d]_{(\mu, 0.5)} = \{a, c, d\}, [b]_{(\mu, 0.5)} = \{b\}$. $U(\nu, 0.5) = \{\{a, a\}, \{b, b\}, \{c, c\}, \{d, d\}, \{a, b\}, \{a, d\}, \{b, d\}\}$ and $[a]_{(\mu, 0.5)} = [b]_{(\mu, 0.5)} = [d]_{(\mu, 0.5)} = \{a, b, d\}$, $[c]_{(\mu, 0.5)} = \{c\}$, $U(\mu \cap \nu, 0.5) = \{\{a, a\}, \{b, b\}, \{c, c\}, \{d, d\}, \{a, d\}\}$ and $[a]_{(\mu, 0.5)} = [d]_{(\mu, 0.5)} = \{a, d\}$, $[b]_{(\mu, 0.5)} = \{b\}$, $[c]_{(\mu, 0.5)} = \{c\}$, thus $\overline{U}(\mu, t, F(e_1)) = \{a, b, c, d\}, \overline{U}(\mu, t, F(e_2)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, F(e_3)) = \{a, c, d\}$ and $\underline{U}(\mu, t, F(e_1)) = \{b\}, \underline{U}(\mu, t, F(e_2)) = \{b\}$, $\underline{U}(\mu, t, F(e_3)) = \emptyset$, $\underline{U}(\nu, t, F(e_1)) = \{c\}, \underline{U}(\nu, t, F(e_2)) = \emptyset, \underline{U}(\nu, t, F(e_3)) = \{c\}$ and $\overline{U}(\nu, t, F(e_1)) = \{a, b, c, d\}, \overline{U}(\nu, t, F(e_2)) = \{a, b, d\}, \overline{U}(\nu, t, F(e_3)) = \{a, b, c, d\}$, $\overline{U}(\mu \cap \nu, t, F(e_1)) = \{b, c\}, \overline{U}(\mu \cap \nu, t, F(e_2)) = \{a, b, d\}, \overline{U}(\mu \cap \nu, t, F(e_3)) = \{a, c, d\}$ and $\underline{U}(\mu \cap \nu, t, F(e_1)) = \{b, c\}, \underline{U}(\mu \cap \nu, t, F(e_2)) = \{b\}, \underline{U}(\mu \cap \nu, t, F(e_3)) = \{c\}$, so, obviously, we obtain $\overline{U}(\mu \cap \nu, t, F(e_1)) \subsetneq \cap \underline{U}(\mu, t, F(e_1)) \cap \overline{U}(\nu, t, F(e_1))$ and $\underline{U}(\mu, t, F(e_3)) \cap \underline{U}(\nu, t, F(e_3)) \subsetneq \underline{U}(\mu \cap \nu, t, F(e_3))$. Therefore, we cannot use " $='$ " to replace " $\subseteq$ " in Propositions 4.1 and 4.2.

Definition 4.4 Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft set over S.

(1) Then $\underline{U}(\mu, t, \mathfrak{S})(\overline{U}(\mu, t, \mathfrak{S}))$ is called a lower (upper) rough soft semigroup (resp., ideal, prime ideal, bi-ideal) of S, if $\underline{F}_\mu(x)(\overline{F}_\mu(x))$ is a subsemigroup (resp., ideal, prime ideal, bi-ideal) of S, for all $x \in A$.

(2) Moreover, $\mathfrak{S}$ is called a rough soft semigroup (resp., rough soft ideal, rough soft prime ideal, bi-ideal) of S, if $\underline{F}_\mu(x)$ and $\overline{F}_\mu(x)$ are subsemigroups (resp., ideals, prime ideals, bi-ideal) of S for all $x \in A$.

Theorem 4.5 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft semigroup over S. Then $\mathfrak{S} = (F, A)$ is an upper rough soft semigroup of S.*

Proof. For any $a \in A$, let $x_1 \in \overline{F}_\mu(a), x_2 \in \overline{F}_\mu(a)$, then $[x_1]_{(\mu,t)} \cap F(a) \neq \emptyset$, $[x_2]_{(\mu,t)} \cap F(a) \neq \emptyset$. This means there exist $b_1 \in [x_1]_{(\mu,t)} \cap F(a)$, $b_2 \in [x_2]_{(\mu,t)} \cap F(a)$. Thus $b_1 \in [x_1]_{(\mu,t)}$, $b_1 \in F(a), b_2 \in [x_2]_{(\mu,t)}$, $b_2 \in F(a)$ and $(x_1, b_1) \in U(\mu, t)$, $(x_2, b_2) \in U(\mu, t)$. Since $U(\mu, t)$ is a congruence relation on S, we have $(x_1 x_2, b_1 b_2) \in U(\mu, t)$, it implies $[x_1 x_2]_{(\mu,t)} = [b_1 b_2]_{(\mu,t)}$. Since $\mathfrak{S} = (F, A)$ is a soft semigroup over S, so $F(a)$ is an subsemigroup of S, so $b_1 b_2 \in F(a)$. That is $[b_1 b_2]_{(\mu,t)} \cap F(a) \neq \emptyset$. Hence $[x_1 x_2]_{(\mu,t)} \cap F(a) \neq \emptyset$ and $x_1 x_2 \in \overline{F}_\mu(a)$. Then $\overline{F}_\mu(a)$ is a subsemigroup of S. Therefore $\mathfrak{S} = (F, A)$ is a upper rough soft semigroup over S. ∎

Theorem 4.6 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft semigroup over S. If $U(\mu, t)$ is a complete congruence relation on S and $\underline{U}(\mu, t, \mathfrak{S})$ is non-null, then $\mathfrak{S} = (F, A)$ is a lower rough soft semigroup of S.*

Proof. For any $a \in A$, since $\underline{U}(\mu, t, \mathfrak{S})$ is non-null, then $\underline{F}_\mu(a) \neq \emptyset$. Let $x_1 \in \underline{F}_\mu(a), x_2 \in \underline{F}_\mu(a)$, then $[x_1]_{(\mu,t)} \subseteq F(a), [x_2]_{(\mu,t)} \subseteq F(a)$. Since $U(\mu, t)$ is a complete congruence relation and $\mathfrak{S} = (F, A)$ is a soft semigroup over S, we have $[x_1 \cdot x_2]_{(\mu,t)} = [x_1]_{(\mu,t)} \cdot [x_2]_{(\mu,t)} \subseteq F(a) \cdot F(a) \subseteq F(a)$, so $x_1 \cdot x_2 \in \underline{F}_\mu(a)$, Hence $\underline{F}_\mu(a)$ is an subsemigroup of S. Therefore $\underline{U}(\mu, t, \mathfrak{S})$ is a soft semigroup of S. ∎

Theorem 4.7 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ an idealistic soft semigroup over S. Then $\mathfrak{S} = (F, A)$ is an upper rough soft ideal of S.*

Proof. For any $a \in A, x \in \overline{F}_\mu(a), m \in S$, since $\mathfrak{S} = (F, A)$ an idealistic soft semigroup over S, according to Theorem 4.5, we can obtain that $\overline{U}(\mu, t, \mathfrak{S})$ is a soft semigroup of S. In the following, we only prove the properties of left (right). $x \in \overline{F}_\mu(a)$, so we have $[x]_{(\mu,t)} \cap F(a) \neq \emptyset$. Now let $n \in [x]_{(\mu,t)} \cap F(a)$, then we have $(n, x) \in U(\mu, t)$, and $n \in F(a)$, Since $U(\mu, t)$ is a congruence relation on S, so $(mn, mx) \in U(\mu, t)$, we have $[mn]_{(\mu,t)} = [mx]_{(\mu,t)}$. For $n \in F(a), m \in S$, and $F(a)$ is a ideal of S, then we have $mn \in F(a)$, so $[mx]_{(\mu,t)} \cap F(a) \neq \emptyset$, so $mx \in \overline{F}_\mu(a)$. Therefore, $\overline{F}_\mu(a)$ is a left ideal, we can prove $\overline{F}_\mu(a)$ in the similar way. On balance, $\overline{F}_\mu(a)$ is an ideal of S. Then $\mathfrak{S} = (F, A)$ is an upper rough soft ideal of S. ∎

Theorem 4.8 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ an idealistic soft semigroup over S. If $U(\mu, t)$ is a complete congruence relation on S and $\underline{U}(\mu, t, \mathfrak{S})$ is non-null, then $\mathfrak{S} = (F, A)$ is a lower rough soft ideal of S.*

Proof. For any $a \in A$, since $\mathfrak{S} = (F, A)$ an idealistic soft semigroup over S, then $F(a)$ is an ideal of S. Since $\underline{U}(\mu, t, \mathfrak{S})$ is non-null, then $\underline{F}_\mu(a) \neq \emptyset$. Let

$m \in \underline{F}_\mu(a), x \in S$, then $[m]_{(\mu,t)} \subseteq F(a)$. Since $U(\mu, t)$ is a complete congruence relation on S, then $[mx]_{(\mu,t)} = [m]_{(\mu,t)} \cdot [n]_{(\mu,t)} \subseteq F(a) \cdot S \subseteq F(a)$, and so $mx \in \underline{F}_\mu(a)$. Similarly, we can prove that $xm \in \underline{F}_\mu(a)$. This means $\underline{F}_\mu(a)$ is an ideal of S. Hence $\underline{U}(\mu, t, \mathfrak{S})$ is a soft ideal of S. ∎

Theorem 4.9 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a prime idealistic soft semigroup over S. If $U(\mu, t)$ is a complete congruence relation over S, then $\mathfrak{S} = (F, A)$ is an upper rough soft prime ideal of S.*

Proof. Since $\mathfrak{S} = (F, A)$ is prime idealistic soft semigroup over S, then for all $a \in A$, $F(a)$ is a prime ideal of S. By Theorem 4.7, we know $\overline{F}_\mu(a)$ is an ideal of S. Let $ab \in \overline{F}_\mu(a)$ for some $a, b \in S$, since $U(\mu, t)$ is a complete congruence relation on S, then we have $[a]_{(\mu,t)} \cdot [b]_{(\mu,t)} \cap F(a) = [ab]_{(\mu,t)}) \cap F(a) \neq \emptyset$. So there exist $x_1 \in [a]_{(\mu,t)}, x_2 \in [b]_{(\mu,t)}$ such that $x_1 \cdot x_2 \in F(a)$. Since $F(a)$ is a prime ideal of S, then we have $x_1 \in F(a)$ or $x_2 \in F(a)$. Thus $[a]_{(\mu,t)} \cap F(a) \neq \emptyset$ or $[b]_{(\mu,t)} \cap F(a) \neq \emptyset$, and so $a \in \overline{F}_\mu(a)$ or $b \in \overline{F}_\mu(a)$. Hence $\overline{F}_\mu(a)$ is a prime ideal of S. Thus $\mathfrak{S} = (F, A)$ is an upper rough soft prime ideal of S. ∎

Example 4.10 Let $S = \{a, b, c, d, e\}$ be a semigroup and "·" a operation defined by the table:

·	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

Let $\mu = \frac{0.4}{a} + \frac{0.7}{b} + \frac{0.5}{c} + \frac{0.9}{d} + \frac{0.1}{e}$ be a fuzzy set of S, μ is a fuzzy ideal of S. Let $t = 0.5$, then $U(\mu; 0.5) = \{(a, a), (b, b), (c, c), (d, d), (e, e), (b, c), (b, d)\}$, so $[a]_{(\mu,0.5)} = \{a\}$, $[b]_{(\mu,0.5)} = [c]_{(\mu,0.5)} = [d]_{(\mu,0.5)} = \{b, c, d\}$, $[e]_{(\mu,0.5)} = \{e\}$. And let $\mathfrak{S} = (F, A)$ be a soft set over S, $A = \{x_1, x_2\}$, $F(x_1) = \{b, e\}$, $F(x_2) = \{a, b, e\}$. Then we have $\underline{F}_\mu(x_1) = \{e\}, \underline{F}_\mu(x_2) = \{a, e\}, \overline{F}_\mu(x_1) = \{b, c, d, e\}, \overline{F}_\mu(x_2) = \{a, b, c, d, e\}$. Obviously, we have $\underline{F}_\mu(x_2), \overline{F}_\mu(x_1), \overline{F}_\mu(x_2)$ are ideals of S, but $\underline{F}_\mu(x_1)$ is not a ideal of S. Therefore, we say $\mathfrak{S}$ is a upper rough ideal of S, but it is not a lower rough ideal of S.

Theorem 4.11 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a prime idealistic soft semigroup over S. If $U(\mu, t)$ is a complete congruence relation on S and $\underline{U}(\mu, t, \mathfrak{S})$ is non-null, then $\mathfrak{S} = (F, A)$ is a lower rough soft prime ideal of S.*

Proof. Since $\mathfrak{S} = (F, A)$ is prime idealistic soft semigroup over S, then for all $a \in A$, $F(a)$ is a prime ideal of S. By Theorem 4.6, we know $\underline{F}_\mu(a)$ is an ideal of S. Let $x_1 \cdot x_2 \in \underline{F}_\mu(a)$ for some $x_1, x_2 \in S$, since $U(\mu, t)$ is a congruence relation over S, then we have $[x_1]_{(\mu,t)} \cdot [x_2]_{(\mu,t)} = [x_1 \cdot x_2]_{(\mu,t)} \subseteq F(a)$. We suppose that $\underline{F}_\mu(a)$ is

not a prime ideal of S, then there exist $x_1, x_2 \in S$ such that $x_1 \cdot x_2 \in \underline{F}_\mu(a)$ but $x_1 \notin \underline{F}_\mu(a), x_2 \notin \underline{F}_\mu(a)$. Thus $[x_1]_{(\mu,t)} \nsubseteq F(a), [x_2]_{(\mu,t)} \nsubseteq F(a)$, then exist $x_1' \in [x_1]_{(\mu,t)}, x_1' \notin F(a), x_2' \in [x_2]_{(\mu,t)}, x_2' \notin F(a)$. Thus $x_1' \cdot x_2' \in [x_1]_{(\mu,t)} \cdot [x_2]_{(\mu,t)} \subseteq F(a)$. Since $F(a)$ is a prime ideal of S, we have $x_1' \in F(a)$ or $x_2' \in F(a)$. It contradicts the supposition. Hence $\mathfrak{S} = (F, A)$ is a lower rough soft prime ideal of S. ∎

Theorem 4.12 *Let* (S, μ, t) *be a Pawlak approximation space and* $\mathfrak{S} = (F, A)$ *a bi-idealistic soft semigroup over* S. *If* $U(\mu, t)$ *is a congruence relation over* S, *then* $\overline{U}(\mu, t, \mathfrak{S})$ *is a soft bi-ideal of* S.

Proof. Since $\mathfrak{S} = (F, A)$ is bi-idealistic soft semigroup over S, then for all $a \in A$, $F(a)$ is a bi-ideal of S, that is $F(a)$ is a subsemigroup of S and $F(a) \cdot S \cdot F(a) \subseteq F(a)$. By Theorem 4.5, we know $\overline{F}_\mu(a)$ is a subsemigroup of S. Let $y \in \overline{F}_\mu(a) \cdot S \cdot \overline{F}_\mu(a)$, there exists $m \in \overline{F}_\mu(a), s \in S, n \in \overline{F}_\mu(a)$ such that $y = msn$, let $k \in [m]_{(\mu,t)} \cap F(a) \neq \emptyset$ and $l \in [n]_{(\mu,t)} \cap F(a) \neq \emptyset$, then we have $[k, m] \in U(\mu, t), [l, n] \in U(\mu, t), [ksl, msn] \in U(\mu, t)$, so $ksl \in [msn]_{(\mu,t)}$ and $ksl \in F(a) \cdot S \cdot F(a) \subseteq F(a)$, therefore $[y]_{(\mu,t)} = [msn]_{(\mu,t)} \cap F(a) \neq \emptyset$, $y \in \overline{F}_\mu(a)$. We have $\overline{F}_\mu(a) \cdot S \cdot \overline{F}_\mu(a) \subseteq \overline{F}_\mu(a)$. Then $\overline{U}(\mu, t, \mathfrak{S})$ is a soft bi-ideal of S. ∎

Theorem 4.13 *Let* (S, μ, t) *be a Pawlak approximation space and* $\mathfrak{S} = (F, A)$ *a bi-idealistic soft semigroup over* S. *If* $U(\mu, t)$ *is a complete congruence relation over* S *and* $\underline{U}(\mu, t, \mathfrak{S}) \neq \emptyset$. $\mathfrak{S} = (F, A)$ *is a lower rough soft bi-ideal of* S.

Proof. Since $\mathfrak{S} = (F, A)$ is bi-idealistic soft semigroup over S, then for all $a \in A$, $F(a)$ is a bi-ideal of S, that is $F(a)$ is a subsemigroup of S and $F(a) \cdot S \cdot F(a) \subseteq F(a)$. By Theorem 4.6, we know $\underline{F}_\mu(a)$ is a subsemigroup of S. Let $y \in \underline{F}_\mu(a) \cdot S \cdot \underline{F}_\mu(a)$, there exists $m \in \underline{F}_\mu(a), s \in S, n \in \underline{F}_\mu(a)$ such that $y = msn$, let $[m]_{(\mu,t)} \subseteq F(a)$ and $[n]_{(\mu,t)} \subseteq F(a)$, since $\mathfrak{S} = (F, A)$ a bi-idealistic soft semigroup over S and $U(\mu, t)$ is a complete congruence relation on S, then we have $[m]_{(\mu,t)} \cdot s \cdot [n]_{(\mu,t)} \subseteq [m]_{(\mu,t)} \cdot [s]_{(\mu,t)} \cdot [n]_{(\mu,t)} = [msn]_{(\mu,t)} \subseteq F(a) \cdot S \cdot F(a) \subseteq F(a)$, so $[y]_{(\mu,t)} \subseteq F(a)$, therefore $y \in \underline{F}_\mu(a)$. Therefore, $\mathfrak{S} = (F, A)$ is a lower rough soft bi-ideal of S. ∎

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