A NOVEL STUDY OF SOFT SETS OVER n-ARY SEMIGROUPS

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Abstract. In this paper, we show that the regular n-ary semigroups can be described by using idealistic soft n-ary semigroups. The relationships between regular n-ary semigroups and soft regular n-ary semigroups are also discussed. Finally, we introduce quotient n-ary semigroups via soft congruence relations and establish some homomorphisms and related properties with respect to soft congruence relations.

Keyword: n-ary semigroups; soft n-ary semigroups; idealistic soft n-ary semigroups; soft congruence relations;soft homomorphisms.

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1. Introduction

The generalization of classical algebraic structures to n-ary structures was first initiated by Kasner [17] in 1904. In the following decades and nowadays, a number of different n-ary systems have been studied in depth in different contexts. Sioson [20] introduced regular n-ary semigroups and investigated their related properties. Since then, the nature of regular n-ary semigroups were discussed in detail by Dudek [11]. In [6], [7], [8], Dudek proved some results and presented many examples of n-ary groups. Earlier, Crombez et al. [2], [3] gave the generalized rings and named it as (m, n)-rings and introduced their quotient structure. Up till now, the theory of n-ary systems has many applications, for example, application in physics [21], [22] and in automata theory [15], fuzzy sets and rough set theory(see [4], [5], [9], [24]) and so on.

In dealing with uncertainties, many theories have been recently developed, including the theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets and theory of rough sets and so on. Although many new techniques have been developed as a result of theories, yet difficulties are still. The major difficulties posed by these theories are probably due to the inadequacy of parameters. In 1999, Molodtsov [19] initiated

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the concept of soft set theory, which was a completely new approach for modeling uncertainty and had a rich potential for applications in several directions. Later on, Maji et al. [18] introduced several operations in soft set theory and carried out a detailed theoretical study on soft sets. The algebraic structure of soft sets has been studied by several authors. For examples, Aktaş and Çağman [1] introduced the notion of soft groups and discussed their basic properties. Feng et al. [12] defined the notions of soft semirings, idealistic soft semirings, soft ideals and introduced the algebraic properties of semirings. Other applications of soft set theory in different algebraic structure can be found in [16], [23] and so on. Feng et al. [13] initiated the soft binary relations, some interesting properties of soft equivalence and soft congruence relations are discussed.

In this paper, we first recall some concepts and results on n-ary semigroups and soft sets. In Section 3, we define the notion of soft n-ary semigroups and idealistic soft n-ary semigroups over an n-ary semigroup. Some basic related properties with soft n-ary semigroups and idealistic soft n-ary semigroups are proposed. In Section 4, we show that the regular n-ary semigroups can be described by using idealistic soft n-ary semigroups. Moreover, we discuss relationships between regular n-ary semigroups and soft regular n-ary semigroups. In Section 5, we give the concept of soft congruence relations over an n-ary semigroup and introduce quotient n-ary semigroups via soft congruence relations. Some homomorphisms and related properties with respect to soft congruence relations are proposed.

2. Preliminaries

A non-empty set S together with one n-ary operation $f : S^n \to S$, where $n \geq 2$, is called an n-ary groupoid and is denoted by (S, f). According to the general convention used in the theory of n-ary groupoids, the sequence of elements $x_i, x_{i+1}, ..., x_j$ is denoted by x_i^j. In the case $j < i$, it is the empty symbol. If $x_{i+1} = x_{i+2} = \cdots = x_{i+t} = x$, then we write $\overset{(t)}{x}$ instead of x_{i+1}^{i+t}. In this convention,

$$f(x_1, x_2, ..., x_n) = f(x_1^n),$$

and

$$f(x_1, ..., x_i, \underbrace{x, ..., x}_{t}, x_{i+t+1}, ..., x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

An n-ary groupoid (S, f) is called (i, j)-associative if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

hold for all $x_1, x_2, ..., x_{2n-1} \in S$. If this identity holds for all $1 \leq i \leq j \leq n$, then we say that the operation f is associative, and (S, f) is called an n-ary semigroup. An n-ary semigroup (S, f) is called idempotent if $f(x, ..., x) = x$ for all $x \in S$.

A non-empty subset H of an n-ary semigroup (S, f) is an n-ary subsemigroup if (H, f) is an n-ary subsemigroup, i.e., if it is closed under the operation f. Throughout this paper, unless otherwise mentioned, S will denote an n-ary semigroup.

Definition 2.1 [11], [20] A non-empty subset I of S is called an i-ideal of S if for every $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n \in S$ with $a \in I$, then $f(x_1^{i-1}, a, x_{i+1}^n) \in I$. I is called an ideal of S if I is an i-ideal for every $1 \leq i \leq n$.

Definition 2.2 [4] Let R be an equivalence relation of S. R is called a congruence of S if $(x_i, y_i) \in R$ implies $(f(x_1^n), f(y_1^n)) \in R$ for all $1 \leq i \leq n$ and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in S$.

Definition 2.3 [4] A mapping $\varphi : S \to T$ from S into T is called a homomorphism if $\varphi(f(x_1^n)) = g(\varphi(x_1), \varphi(x_2), ..., \varphi(x_n))$ for all $x_1, x_2, ..., x_n \in S$.

Definition 2.4 [19] A pair (F, A) is called a soft set over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping.

For a soft set (F, A), the set $\mathrm{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of (F, A). Thus a null soft set is indeed a soft set with an empty support, and we say that a soft set (F, A) is non-null if $\mathrm{Supp}(F, A) \neq \emptyset$.

Definition 2.5 [19] A soft set (F, A) over S is called an absolute soft set if $F(a) = S$ for all $a \in A$.

Definition 2.6 [14] A soft set (F, A) over S is called a full soft set if $\bigcup\limits_{x \in A} F(x) = S$.

Definition 2.7 [12] Let (F, A) and (G, B) be two soft sets over a common universe U. The inclusion symbol "$\widetilde{\subseteq}$" of (F, A) and (G, B), denoted by $(F, A) \widetilde{\subseteq} (G, B)$, is defined as

(1) $A \subseteq B$;

(2) $F(x) \subseteq G(x)$ for all $x \in A$.

If $(F, A) \widetilde{\subseteq} (G, B)$ and $(G, B) \widetilde{\subseteq} (F, A)$, then we denote $(F, A) = (G, B)$.

3. Soft n-ary semigroups and idealistic soft n-ary semigroups

In this section, we define the notion of soft n-ary semigroups and idealistic soft n-ary semigroups over S. Some basic related properties with soft n-ary semigroups and idealistic soft n-ary semigroups are proposed.

Definition 3.1 Let $(F_1, A_1), (F_2, A_2), ..., (F_n, A_n)$ be soft sets over S. Then the $\widetilde{f}$-product of them, denoted by $\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$, is defined as a soft set $(G, B) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$, where $B = \bigcap\{A_i | i = 1, 2, ..., n\} \neq \emptyset$ and $G : B \to P(S)$ defined by $G(a) = f(F_1(a), F_2(a), ..., F_n(a))$ for all $a \in B$.

Definition 3.2 Let (F, A) be a non-null soft set over S. Then (F, A) is called a soft n-ary semigroup over S if $F(a)$ is an n-ary subsemigroup of S for all $a \in \mathrm{Supp}(F, A)$.

Example 3.3 Let $S = \{-i, 0, i\}$ be a set with a ternary operation f as the usual multiplication of complex numbers. Then (S, f) is a ternary semigroup. Let (F, A) be a soft set over S, where $A = \{a, b, c\}$ and $F : A \to P(S)$ be a set-valued function defined by $F(x) = \{y \in S | (x, y) \in R\}$ for all $x \in A$, where $R = \{(a, 0), (c, -i), (c, 0), (c, i)\}$. Then $F(a) = \{0\}, F(b) = \emptyset, F(c) = \{-i, 0, i\}$. Therefore (F, A) is a soft ternary semigroup over S.

Proposition 3.4 *A non-null soft set (F, A) over S is a soft n-ary semigroup if and only if $\widetilde{f}((F, A), ..., (F, A)) \widetilde{\subseteq} (F, A)$.*

Proof. Let (F, A) be a soft n-ary semigroup over S, then for all $a \in \mathrm{Supp}(F, A)$, $F(a)$ is an n-ary subsemigroup of S. By Definition 3.1, we denote $\widetilde{f}((F, A), ..., (F, A)) = (G, A)$, where $G : A \to P(S)$ defined by $G(a) = f(F(a), ..., F(a))$ for all $a \in \mathrm{Supp}(F, A)$. $F(a)$ is an n-ary subsemigroup of S, it follows that $f(F(a), ..., F(a)) \subseteq F(a)$, that is $G(a) \subseteq F(a)$ for all $a \in A$. Hence $\widetilde{f}((F, A), \cdots, (F, A)) \widetilde{\subseteq} (F, A)$.

Conversely, if $\widetilde{f}((F, A), ..., (F, A)) \widetilde{\subseteq} (F, A)$, it follows that $f(F(a), ..., F(a)) \subseteq F(a)$ for all $a \in \mathrm{Supp}(F, A)$. This means $F(a)$ is an n-ary subsemigroup of S. By Definition 3.2, (F, A) is a soft n-ary semigroup over S. ∎

Definition 3.5 Let (F, A) be a non-null soft set over S. Then (F, A) is called a j-idealistic soft n-ary semigroup over S, if $F(x)$ is a j-ideal of S for all $x \in \mathrm{Supp}(F, A)$. Moreover, if (F, A) is a j-idealistic soft n-ary semigroup of S for each $j = 1, 2, ..., n$, then (F, A) is called an idealistic soft n-ary semigroup.

Example 3.6 Consider the natural numbers $\mathbb{N}$ with usual multiplication. Let $S = 2\mathbb{N}$. We define the 4-ary operation f, $f(a, b, c, d) = \frac{abcd}{2}$ for all $a, b, c, d \in S$. Then (S, f) is a 4-ary semigroup. Let (F, A) be a soft set over S, where $A = \mathbb{N}$ and $F : A \to P(S)$ is a set-valued function defined by $F(x) = \{4xn | n \in \mathbb{N}\}$ for all $x \in \mathbb{N}$. Since for all $x_1, x_2, x_3 \in S$ and $r \in F(x)$, $f(x_1^{i-1}, r, x_{i+1}^4) \in F(x)$ for $i = 1, 2, 3, 4$. Hence $F(x)$ is an ideal of (S, f). Therefore (F, A) is an idealistic soft 4-ary semigroup of S.

Remark 3.7 If an idealistic soft n-ary semigroup (F, A) satisfying $\widetilde{f}((F, A), ..., (F, A)) = (F, A)$, then (F, A) is called a soft idempotent idealistic soft n-ary semigroup.

Proposition 3.8 *Let (H, E) be an absolute soft set over S. Then, a non-null soft set (F, A) over S is a j-idealistic soft n-ary semigroup if and only if* $\widetilde{f}((\overset{(j-1)}{H, E}), (F, A), (\overset{(n-j)}{H, E})) \widetilde{\subseteq} (F, A)$, *where $j = 1, 2, ..., n$.*

Proof. Let (F, A) be a j-idealistic soft n-ary semigroup, then $F(a)$ is a j-ideal of S for all $a \in \mathrm{Supp}(F, A)$, it follows that $f(x_1^{k-1}, r, x_{k+1}^n) \in F(a)$ for all $x_1, x_2, ..., x_n \in S$ and all $r \in F(a)$. By Definition 3.1, we denote $\widetilde{f}((\overset{(k-1)}{H, E}), (F, A), (\overset{(n-k)}{H, E})) = (G, A)$, where $G : A \to P(S)$ defined by $G(a) = f(\overset{j-1}{H(a)}, F(a), \overset{n-j}{H(a)})$ for all $a \in \mathrm{Supp}(F, A)$. Since (H, E) is an absolute soft set over S, so $H(a) = S$ for all $a \in \mathrm{Supp}(F, A)$. Hence $G(a) = f(\overset{j-1}{S}, F(a), \overset{n-j}{S}) \subseteq F(a)$. Therefore, $\widetilde{f}((\overset{(j-1)}{H, E}), (F, A), (\overset{(n-j)}{H, E})) \widetilde{\subseteq} (F, A)$.

Conversely, if $\widetilde{f}((\overset{(j-1)}{H, E}), (F, A), (\overset{(n-j)}{H, E})) \widetilde{\subseteq} (F, A)$, then $f(\overset{j-1}{H(a)}, F(a), \overset{n-j}{H(a)}) \subseteq F(a)$ for all $a \in \mathrm{Supp}(F, A)$. Since (H, E) is an absolute soft set over S, so $H(a) = S$ for all $a \in \mathrm{Supp}(F, A)$. Hence $f(\overset{j-1}{S}, F(a), \overset{n-j}{S}) \subseteq F(a)$. This means $F(a)$ is a j-ideal of S. By Definition 3.5, (F, A) is a j-idealistic soft n-ary semigroup over S. ∎

Proposition 3.9 *Let (H, E) be an absolute soft set over S. Then, a non-null soft set (F, A) over S is an idealistic soft n-ary semigroup if (F, A) is a j-idealistic soft n-ary semigroup for all $1 \leq j \leq n$.*

Proof. It is straightforward. ∎

4. The characterizations of regular n-ary semigroups

In this section, we show that the regular n-ary semigroups can be described by using idealistic soft n-ary semigroups. Moreover, we discuss relationships between regular n-ary semigroups and soft regular n-ary semigroups.

Definition 4.1 [11] An element $a \in S$ is called regular if there exist $x_2, x_3, ..., x_{n-1} \in S$ such that $f(a, x_2^{n-1}, a) = a$. S is called regular if every element of S is regular.

Definition 4.2 [20] S is called regular if for all $a \in S$, there exist $x_{ij} \in S$ $(i, j = 1, 2, ..., n)$ such that

$$a = f(f(a, x_{12}^{1n}), f(x_{21}, a, x_{23}^{2n}), ..., f(x_{n1}^{nn-1}, a)).$$

Remark 4.3 In [11], Dudek proved that Definitions 4.1 and 4.2 are equivalent.

Example 4.4 Let $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, where the ternary operation f is the usual matrix multiplication. One can easily show that (S, f) is a regular ternary semigroup.

Lemma 4.5 [20] *The following conditions are equivalent:*

(1) *S is regular,*
(2) $\bigcap_{i=1}^{n} B_i = f(B_1, B_2, ..., B_n)$ *for all i-deals B_i,*
(3) *every ideal is idempotent.*

Theorem 4.6 *S is regular if and only if*

$$(F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$$

for every j-idealistic soft n-ary semigroup $(F_j, A_j)(j = 1, 2, ..., n)$.

Proof. Let S be a regular n-ary semigroup and (F_j, A_j) be an j-idealistic soft n-ary semigroup over S, respectively. Then $(G, B) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$, where $B = \bigcap_{j=1}^{n} A_j$ and G is defined as $G(a) = f(F_1(a), F_2(a), ..., F_n(a))$ for all $a \in B$. Also, $(K, C) = (F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n)$, where $C = \bigcap_{j=1}^{n} A_j$ and K is defined as $K(a) = F_1(a) \cap F_2(a) ... \cap F_n(a)$ for all $a \in C$. By Proposition 3.8, we have

$$\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)) \widetilde{\subseteq} \widetilde{f}((F_1, A_1), (H, E), ..., (H, E)) \widetilde{\subseteq} (F_1, A_1),$$

$$\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)) \widetilde{\subseteq} \widetilde{f}((H, E), (F_2, A_2), ..., (H, E)) \widetilde{\subseteq} (F_2, A_2),$$

....

$$\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)) \widetilde{\subseteq} \widetilde{f}((H, E), (H, E), ..., (F_n, A_n)) \widetilde{\subseteq} (F_n, A_n),$$

where (H, E) is an absolute soft set over S.

Thus, $\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)) \widetilde{\subseteq} (F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n)$.

Let $b \in F_1(a) \cap F_2(a) ... \cap F_n(a)$, then $b \in F_j(a)$. Since S is regular, there exist $x_{ij} \in S(i, j = 1, 2, ..., n)$ such that $b = f(f(b, x_{12}^{1n}), f(x_{21}, b, x_{23}^{2n}), ..., f(x_{n1}^{nn-1}, b))$. By (F_j, A_j) is an j-idealistic soft n-ary semigroup over S, we have $F_j(a)$ is an j-ideal of S, so $f(b, x_{12}^{1n}) \in F_1(a), f(x_{21}, b, x_{23}^{2n}) \in F_2(a), ..., f(x_{n1}^{nn-1}, b) \in F_n(a)$. Hence $b \in f(F_1(a), F_2(a), ..., F_n(a))$. This means $F_1(a) \cap F_2(a) ... \cap F_n(a) \subseteq f(F_1(a), F_2(a), ..., F_n(a))$, for all $a \in \bigcap_{j=1}^{n} A_j$. It implies $(F_1,A_1) \Cap (F_2,A_2) ... \Cap (F_n,A_n) \widetilde{\subseteq} \widetilde{f}((F_1,A_1), (F_2,A_2), ..., (F_n,A_n))$.

Hence $(F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$ for every j-idealistic soft n-ary semigroup$(F_j, A_j)(j = 1, 2, ..., n)$.

Conversely, suppose that $A_1 = A_2 = \cdots = A_n = S$ and F_j is a function from A_j to $P(S)$. For all $a \in S$, we define $F_1(a) = \{a\} \cup f(a, S, ..., S)$, $F_2(a) = \{a\} \cup f(S, a, S, ..., S)$, ..., $F_n(a) = \{a\} \cup f(S, S, ..., S, a)$. Since

$$\begin{aligned} f(F_1(a), S, ..., S) &= f(\{a\} \cup f(a, S, ..., S), S, ..., S) \\ &\subseteq f(a, S, ..., S) \cup f(f(a, S, ..., S), S..., S) \\ &= f(a, S, ..., S) \cup f(a, S, ..., S, f(S, ..., S)) \\ &\subseteq f(a, S, ..., S) \cup f(a, S, ..., S) \\ &= f(a, S, ..., S). \end{aligned}$$

Hence $a \in F_1(a) \cap F_2(a) ... \cap F_n(a) = f(F_1(a), F_2(a), ..., F_n(a)) \subseteq f(F_1(a), S, ..., S) \subseteq f(a, S, ..., S)$. In similar way, we have $a \in f(S, a, S, ..., S), ..., a \in f(S, ..., S, a)$. So

$$\begin{aligned} & f(f(a, S, ..., S), f(S, a, S, ..., S), ..., f(S, ..., S, a)) \\ &= f(a, f(S, ..., S), f(a, S, ..., S), ..., f(S, ..., S), a) \\ &\subseteq f(a, S, ..., S, a). \end{aligned}$$

Therefore,

$$\begin{aligned} & f(F_1(a), F_2(a), ..., F_n(a)) \\ &= f(\{a\} \cup f(a, S, ..., S), ..., \{a\} \cup f(S, S, ..., S, a)) \\ &= f(a, ..., a), ..., \cup f(f(a, S, ..., S), f(S, a, S, ..., S), ..., f(S, ..., S, a)) \\ &= f(f(a, S, ..., S), f(S, a, S, ..., S), ..., f(S, ..., S, a)) \\ &\subseteq f(a, S, ..., S, a). \end{aligned}$$

This means $a \in f(a, S, ..., S, a)$. Hence S is regular. ∎

Theorem 4.7 *S is regular if and only if every idealistic soft n-ary semigroup over S is soft idempotent.*

Proof. Let S be a regular n-ary semigroup and (F, A) an idealistic soft n-ary semigroup. Putting $(F, A) = (F_1, A_1) = (F_2, A_2) = ... = (F_n, A_n)$, then by Theorem 4.1, $(F, A) = (F, A) \Cap (F, A) ... \Cap (F, A) = \widetilde{f}((F, A), ..., (F, A))$. Thus (F, A) is soft idempotent.

Conversely, if all (F_j, A_j) are j-idealistic soft n-ary semigroups over S, where $j = 1, 2, ..., n$. Then $(F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n)$ is an idealistic soft n-ary semigroup over S and $(F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n) \widetilde{\subseteq} (F_j, A_j)$ for each $j = 1, 2, \cdots, n$. This implies that

$$\begin{aligned} & (F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n) \\ &= \widetilde{f}((F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n), ..., (F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n)) \\ &\widetilde{\subseteq} \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)). \end{aligned}$$

But $\widetilde{f}((F_1,A_1),(F_2,A_2),...,(F_n,A_n))\widetilde{\subseteq}(F_1,A_1)\Cap(F_2,A_2)...\Cap(F_n,A_n)$ always holds. Thus, $(F_1,A_1)\Cap(F_2,A_2)...\Cap(F_n,A_n)=\widetilde{f}((F_1,A_1),(F_2,A_2),...,(F_n,A_n))$. Hence, by Theorem 4.6, S is regular. ■

By Theorems 4.6 and 4.7, we have the following Corollary.

Corollary 4.8 *Then the following conditions are equivalent:*

(1) *S is regular,*

(2) $(F_1,A_1)\Cap(F_2,A_2)...\Cap(F_n,A_n)=\widetilde{f}((F_1,A_1),(F_2,A_2),...,(F_n,A_n))$ *for every j-idealistic soft n-ary semigroup $(F_j,A_j)(j=1,2,...,n)$,*

(3) *every idealistic soft n-ary semigroup is soft idempotent*

Definition 4.9 A soft n-ary semigroup (F,A) over S is called a soft regular n-ary semigroup if for each $a\in A, F(a)$ is regular.

The following examples show that if S is a regular n-ary semigroup then soft n-ary semigroup (F,A) over S may not be soft regular and if the soft n-ary semigroup (F,A) over S is soft regular then S may not be regular.

Example 4.10 Consider the regular ternary semigroup in Example 4.4. Let (F,A) be a soft set over S, where $A=S$ and $F:A\to P(S)$ be a set-valued function defined by $F(\begin{pmatrix}0&0\\0&0\end{pmatrix})=S, F(\begin{pmatrix}1&0\\0&0\end{pmatrix})=F(\begin{pmatrix}0&1\\0&0\end{pmatrix})=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$, $F(\begin{pmatrix}0&0\\1&0\end{pmatrix})=F(\begin{pmatrix}0&0\\0&1\end{pmatrix})=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$. Then (F,A) is soft ternary semigroup over S. But it is not soft regular, because $F(\begin{pmatrix}0&0\\1&0\end{pmatrix})=F(\begin{pmatrix}0&0\\0&1\end{pmatrix})=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is not regular ternary subsemigroup of S.

Example 4.11 Let (S,f) be a 4-ary semigroup derived from the semigroup $(S,\cdot)$, where the 4-ary operation defined $f(d,d,d,d)=d$ and $f(x,y,z,u)=a$ where $x,y,z,u\in\{a,b,c\}$. Clearly, (S,f) is not a regular 4-ary semigroup. Let $A=\{\alpha,\beta\}$ be a set of parameters such that $F(\alpha)=\{a\}, F(\beta)=\{a,d\}$. Then (F,A) is a soft regular 4-ary semigroup over (S,f) because $F(\alpha)$ and $F(\beta)$ are regular 4-ary subsemigroups of (S,f).

Remark 4.12 In the above examples, we have demonstrated that the regularity of a n-ary semigroup S does not imply the regularity of a soft n-ary semigroup over S. Also, the regularity of a soft n-ary semigroup over a given n-ary semigroup S does not imply the regularity of the n-ary semigroup. However, we still have the following proposition.

Proposition 4.13 *Let (F,A) be a full soft regular n-ary semigroup over S. Then S is a regular n-ary semigroup.*

Proof. Let (F,A) be a soft regular n-ary semigroup over S. Then $F(\alpha)$ is regular for each $\alpha\in A$. Now let $a\in S$, because (F,A) is a full soft set, we have $S=\bigcup_{\alpha\in A}F(\alpha)$, then there exists $\beta\in A$ such that $a\in F(\beta)$. Since $F(\beta)$ is regular, then exist $x_2,x_3,...,x_{n-1}\in F(\beta)$ such that $f(a,x_2^{n-1},a)=a$. Since $x_2,x_3,...,x_{n-1}\in F(\beta)\subseteq\bigcup_{\alpha\in A}F(\alpha)=S$, hence S is regular. ■

Proposition 4.14 *If S is an idempotent n-ary semigroup, then every soft n-ary semigroup (F,A) over S is soft regular.*

Proof. It is straightforward. ∎

5. Soft congruence relations over n-ary semigroups and homomorphisms

In this section, we give the concept of soft congruence relations over S and introduce quotient n-ary semigroups via soft congruence relations. Some homomorphisms and related properties with respect to soft congruence relations are proposed.

Definition 5.1 A non-null soft set (ρ, A) over $S\times S$ is called a soft congruence relation over S if $\rho(\alpha)$ is a congruence relation on S for all $\alpha \in \mathrm{Supp}(\rho, A)$.

If $\mathrm{Supp}(\rho, A) = \emptyset$, then (ρ, A) is called a null soft congruence relation over S, denoted $\emptyset^2_A$.

Example 5.2 Let (S,f) be an n-ary semigroup in Example 3.6 and $A=\mathbb{N}^+$. Consider the set-valued function $\rho : A \to P(S\times S)$ given by $\rho(\alpha) = \{(x,y)\in S\times S | x \equiv y (mod\ \alpha)\}$ for all $\alpha \in A$. Then $\rho(\alpha)$ is a congruence relation on (S,f). Hence (ρ, A) is a soft congruence relation on (S,f).

Theorem 5.3 *Let (ρ, A) be a soft congruence relation over an n-ary semigroup (S,f) and $S/(\rho,A) = \{[x]_{\rho(\alpha)} | x\in S\}$ where $[x]_{\rho(\alpha)} = \{y \in S | (x,y)\in \rho(\alpha), \alpha\in A\}$. Then for any $\alpha \in A$, $S/(\rho, A)$ is an n-ary semigroup under the n-ary operation defined by*

$$F([x_1]_{\rho(\alpha)}, [x_2]_{\rho(\alpha)}, ..., [x_n]_{\rho(\alpha)}) = [f(x_1^n)]_{\rho(\alpha)}$$

for all $x_1, x_2, ..., x_n \in S$.

Proof. We shall first show that F is well defined. Let $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in S$ be such that

$$[x_1]_{\rho(\alpha)} = [y_1]_{\rho(\alpha)}, [x_2]_{\rho(\alpha)} = [y_2]_{\rho(\alpha)}, ..., [x_n]_{\rho(\alpha)} = [y_n]_{\rho(\alpha)}$$

for all $\alpha\in A$. It follows that $(x_1,y_1)\in\rho(\alpha), (x_2,y_2)\in\rho(\alpha), ..., (x_n,y_n)\in\rho(\alpha)$. Since (ρ, A) is a soft congruence relation over (S,f), by Definition 4.1, for any $\alpha\in A$, $\rho(\alpha)$ is a congruence relation on (S,f). Hence we have $(f(x_1^n), f(y_1^n))\in\rho(\alpha)$. This means $[f(x_1^n)]_{\rho(\alpha)} = [f(y_1^n)]_{\rho(\alpha)}$. Hence F is well defined. $S/(\rho,A)$ is closed under the operation F and F is (i,j)-associative is obvious. Therefore $(S/(\rho,A),F)$ is an n-ary semigroup for all $\alpha\in A$. ∎

Theorem 5.4 *Let (ρ, A) and (σ, B) be two soft congruence relations over an n-ary semigroup S with $(\rho, A)\tilde{\subseteq}(\sigma, B)$. Then the soft binary relation $(\sigma,B)/(\rho,A)$ over $S/(\rho,A)$, defined by $(\delta, C) = (\sigma,B)/(\rho,A)$ where $C = A\cap B$ and*

$$\delta(\alpha) = \sigma(\alpha)/\rho(\alpha) = \{([x]_{\rho(\alpha)}, [y]_{\rho(\alpha)}) \in S/(\rho,A)\times S/(\rho,A), (x,y)\in\sigma(\alpha)\}$$

for all $\alpha \in C$, is a soft congruence relation over $S/(\rho,A)$ and

$$(S/(\rho,A))/((\sigma,B)/(\rho,A)) \cong S/(\sigma,B).$$

Proof. Since $C = A \cap B = A$, then for any $\alpha \in C$, $\rho(\alpha)$ and $\sigma(\alpha)$ are congruence relations on S, so $\delta(\alpha)$ is an equivalence relation on $S/(\rho, A)$.

Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \in S$, if $(a_1, b_1) \in \sigma(\alpha), ..., (a_n, b_n) \in \sigma(\alpha)$, then we have $(f(a_1^n), f(b_1^n)) \in \sigma(\alpha)$ and $([a_1]_{\rho(\alpha)}, [b_1]_{\rho(\alpha)}) \in \delta(\alpha), ..., ([a_n]_{\rho(\alpha)}, [b_n]_{\rho(\alpha)}) \in \delta(\alpha)$. According to Theorem 5.3,

$$(F([a_1]_{\rho(\alpha)}, ..., [a_n]_{\rho(\alpha)}), F([b_1]_{\rho(\alpha)}, ..., [b_n]_{\rho(\alpha)}) = ([f(a_1^n)]_{\rho(\alpha)}, [f(b_1^n)]_{\rho(\alpha)}) \in \delta(\alpha).$$

This means $\delta(\alpha)$ is a congruence relation on $S/(\rho, A)$ for all $\alpha \in C$. Hence $(\sigma, B)/(\rho, A)$ is a soft congruence relation over $S/(\rho, A)$.

From Theorem 5.3, we know $(S/(\rho, A))/((\sigma, B)/(\rho, A))$ and $S/(\sigma, B)$ are two n-ary semigroups. Define a mapping:

$$h : (S/(\rho, A))/((\sigma, B)/(\rho, A)) \to S/(\sigma, B)$$

by $h([[x]_{\rho(\alpha)}]_{\delta(\alpha)}) = [x]_{\sigma(\alpha)}$ for all $x \in S$ and $\alpha \in A$. If $[[x]_{\rho(\alpha)}]_{\delta(\alpha)} = [[y]_{\rho(\alpha)}]_{\delta(\alpha)}$, then $([x]_{\rho(\alpha)}, [x]_{\rho(\alpha)}) \in \delta(\alpha)$, it follows that $(x, y) \in \sigma(\alpha)$, this means $[x]_{\sigma(\alpha)} = [y]_{\sigma(\alpha)}$. Hence h is well defined.

Let F^* be an n-ary operation of $(S/(\rho, A))/((\sigma, B)/(\rho, A))$. Then we have

$$\begin{aligned} h(F^*([[x_1]_{\rho(\alpha)}]_{\delta(\alpha)}, ..., [[x_n]_{\rho(\alpha)}]_{\delta(\alpha)}) &= h([F([x_1]_{\rho(\alpha)}, ..., [x_n]_{\rho(\alpha)})]_{\delta(\alpha)}) \\ &= h([[f(x_1^n)]_{\rho(\alpha)}]_{\delta(\alpha)}) \\ &= [f(x_1^n)]_{\sigma(\alpha)} \\ &= F([x_1]_{\sigma(\alpha)}, ..., [x_n]_{\sigma(\alpha)}) \\ &= F(h([[x_1]_{\rho(\alpha)}]_{\delta(\alpha)}), ..., h([[x_n]_{\rho(\alpha)}]_{\delta(\alpha)})). \end{aligned}$$

This means h is a homomorphism. If $[x]_{\sigma(\alpha)} = [y]_{\sigma(\alpha)}$, then $(x, y) \in \sigma(\alpha)$, so $([x]_{\rho(\alpha)}, [y]_{\rho(\alpha)}) \in \delta(\alpha)$. It follows that $[[x]_{\rho(\alpha)}]_{\delta(\alpha)} = [[x]_{\rho(\alpha)}]_{\delta(\alpha)}$, and h is injective.

Furthermore, for any $[y]_{\sigma(\alpha)} \in S/(\sigma, B)$, there exists $\rho(\alpha) = \sigma(\alpha)$ such that $h([[y]_{\sigma(\alpha)}]_{\delta(\alpha)}) = h([[y]_{\rho(\alpha)}]_{\delta(\alpha)}) = [y]_{\sigma(\alpha)}$ for all $\alpha \in A$. Hence h is surjective. This completes the proof. ∎

Lemma 5.5 *Let $\varphi : (S_1, f) \to (S_2, g)$ be an n-ary semigroup epimorphism.*

(1) *If γ is a congruence relation on S_1, define $\varphi(\gamma)=\{(\varphi(x), \varphi(y)) \in S_2 \times S_2 | (x, y) \in \gamma\}$, then $\varphi(\gamma)$ is a congruence relation on S_2.*

(2) *If θ is a congruence relation on S_2 such that $\varphi^{-1}(\theta) \neq \emptyset$, where $\varphi^{-1}(\theta) = \{(x, y) \in S_1 \times S_1 | (\varphi(x), \varphi(y)) \in \theta\}$, then $\varphi^{-1}(\theta)$ is a congruence relation on S_1.*

Proposition 5.6 *Let $\varphi : (S_1, f) \to (S_2, g)$ be an n-ary semigroup epimorphism.*

(1) *If (ρ, A) is a soft congruence relation over S_1, the image $\varphi(\rho, A)$ of (ρ, A) is denoted by $(\varphi(\rho), A)$, then $(\varphi(\rho), A)$ is a soft congruence relation over S_2, where*

$$\varphi(\rho)(\alpha) = \{(\varphi(x), \varphi(y)) \in S_2 \times S_2 | (x, y) \in \rho(\alpha)\}$$

for all $\alpha \in A$.

(2) *If (σ, B) is a soft congruence relation over S_2 such that $\varphi^{-1}(\sigma, B) \neq \emptyset_B^2$, where $\varphi^{-1}(\sigma, B)$ is the inverse image of (σ, B) is denoted by $(\varphi^{-1}(\sigma), B)$ and*

$$\varphi^{-1}(\sigma)(\beta) = \{(x, y) \in S_1 \times S_1 | (\varphi(x), \varphi(y)) \in \sigma(\beta)\}$$

for all $\beta \in B$. Then $\varphi^{-1}(\sigma, B)$ is a soft congruence relation over S_1.

Theorem 5.7 *Let $\varphi : (S_1, f) \to (S_2, g)$ be an n-ary semigroup epimorphism. If (ρ, A) is a soft congruence relation over S_1, then $S_1/(\rho, A) \cong S_2/\varphi(\rho, A)$.*

Proof. By Theorem 5.3 and Proposition 5.6, $S_1/(\rho, A)$ and $S_2/\varphi(\rho, A)$ are n-ary semigroups. Define a mapping

$$\psi : S_1/(\rho, A) \to S_2/\varphi(\rho, A) \text{ by } \psi([x]_{\rho(\alpha)}) = [\varphi(x)]_{\varphi(\rho)(\alpha)}$$

for all $x \in S_1$ and $\alpha \in A$.

We first show that ψ is well defined. In fact, let $x, x' \in S_1$, if $[x]_{\rho(\alpha)} = [x']_{\rho(\alpha)}$, then $(x.x') \in \rho(\alpha), \alpha \in A$. From Proposition 5.6, we have $(\varphi(x), \varphi(x')) \in \varphi(\rho)(\alpha)$ for all $\alpha \in A$, which implies that $[\varphi(x)]_{\varphi(\rho)(\alpha)} = [\varphi(x')]_{\varphi(\rho)(\alpha)}$. Hence ψ is well defined.

Moreover, ψ is a homomorphism. Let $x_1, x_2, ..., x_n \in S_1, \alpha \in A$, F and F' be two n-ary operations of $S_1/(\rho, A)$ and $S_2/\varphi(\rho, A)$, respectively. Then we have

$$\begin{aligned}
\psi(F([x_1]_{\rho(\alpha)}, ..., [x_n]_{\rho(\alpha)}) &= \psi([f(x_1^n)]_{\rho(\alpha)}) \\
&= \varphi(f(x_1^n))_{\varphi(\rho)(\alpha)} \\
&= [g(\varphi(x_1), \varphi(x_2), ..., \varphi(x_n))]_{\varphi(\rho)(\alpha)} \\
&= F'([\varphi(x_1)]_{\varphi(\rho)(\alpha)}, [\varphi(x_2)]_{\varphi(\rho)(\alpha)}, ..., [\varphi(x_n)]_{\varphi(\rho)(\alpha)}).
\end{aligned}$$

Hence ψ is a homomorphism.

For any $x, x' \in S_1, \alpha \in A$, if $[\varphi(x)]_{\varphi(\rho)(\alpha)} = [\varphi(x')]_{\varphi(\rho)(\alpha)}$, then $(\varphi(x), \varphi(x')) \in \varphi(\rho)(\alpha)$. By Proposition 5.1, we have $(x, x') \in \rho(\alpha)$, which implies $[x]_{\rho(\alpha)} = [y]_{\rho(\alpha)}$. This means ψ is injective.

Since φ is surjective, then for any $[y]_{\varphi(\rho)(\alpha)} \in S_2/\varphi(\rho, A), y \in S_2, \alpha \in A$, there exists $x_1 \in S_1$ such that $\varphi(x) = y$. So $\psi([x]_{\rho(\alpha)}) = [\varphi(x)]_{\varphi(\rho)(\alpha)} = [y]_{\varphi(\rho)(\alpha)}$, this means that ψ is surjective. Hence ψ is a isomorphism and $S_1/(\rho, A) \cong S_2/\varphi(\rho, A)$. ∎

Theorem 5.8 *Let $\varphi : (S_1, f) \to (S_2, g)$ be an n-ary semigroup epimorphism. If (σ, B) is a soft congruence relation over S_2 and $\varphi^{-1}(\sigma, B) \neq \emptyset_B^2$, then*

$$S_1/\varphi^{-1}(\sigma, B) \cong S_2/(\sigma, B).$$

Proof. It is similar to the proof of Theorem 5.7 and we omit it. ∎

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