APPROXIMATION OF CONTIONUOUS FUNCTIONS BY VALLEE-POUSSIN'S SUMS

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Abstract. Let $V_{n,m}^{(\alpha,\beta)}(f;x) = \frac{1}{m+1}\sum\limits_{k=n}^{n+m} S_k^{(\alpha,\beta)}(f;x)$ be the Vallee-Poussin's partial sums of Fourier-Jacobi series. In this paper, we study the deviations of $V_{n,m}^{(\alpha,\beta)}(f;x)$ on $[-1,1]$ for continuous function $f(x)$.

1. Introduction

Let $P_n^{(\alpha,\beta)}(x)$ $(n = 0, 1, 2, ...)$ denote the Jacobi orthonormal system of polynomials with weight function

$$(1-x)^\alpha(1+x)^\beta, \ (\alpha > -1, \beta > -1) \text{ on } [-1,1].$$

Furthermore, let

$$\sum_{k=0}^{\infty} C_k(f) P_k^{(\alpha,\beta)}(x), \tag{1.1}$$

be the Fourier-Jacobi series of the function $f(x)$, where

$$C_k(f) = \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta f(t) P_k^{(\alpha,\beta)}(t)dt. \tag{1.2}$$

Denote the Fourier-Jacobi series of partial sums, $S_n^{(\alpha,\beta)}(f;x)$ as

$$S_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} C_k(f) P_k^{(\alpha,\beta)}(x). \tag{1.3}$$

Denote the Fejer sum of $f(x)$, $\sigma_n^{(\alpha,\beta)}(f;x)$ as

$$\sigma_n^{(\alpha,\beta)}(f;x) = \frac{1}{(n+1)} \sum_{k=0}^{n} S_k(f;x). \tag{1.4}$$

Define the Vallee-Poussin's partial sums of Fourier-Jacobi series,$V_{n,m}^{(\alpha,\beta)}(f;x)$, as

$$V_{n,m}^{(\alpha,\beta)}(f;x) = \frac{1}{m+1} \sum_{k=n}^{n+m} S_k^{(\alpha,\beta)}(f;x). \tag{1.5}$$

An estimation for the deviation of the continuous function $f(x)$, with period 2π from its Fourier sum $S_n(f;x)$ is given in [1], when $f(x)$has a bounded variation and supreme modulus of continuity. This paper generalizes and improves some results in theory of approximation of continuous functions, as those reported in [1]–[10]. The efficient study for approximation by Vallee-Poussin sums has been carried out for several decades. Recently, several studies dealing with the Vallee-Poussin sums have been introduced, see [8]–[10]. The results presented in this paper generalize and improve many results of [1]–[10] and many others in theory of approximation of continuous functions.

Our problem, here, is to study the deviations of $S_n^{(\alpha,\beta)}(f;x)$ on $[-1,1]$ for continuous function of one variables $f(x)$.

In this regard, three theorems have been introduced.

2. Jackson's theorem

We state one of most important theorem in the approximation theory, namely Jackson's theorem, and which is used in our study.

Let $f(x)$ be continuous function on closed interval $[a,b]$. Denote by $E_n(f)$ the best uniform approximation element of a function $f(x)$ by algebraic polynomials of order not exceeded n on $[a,b]$, i.e.,

$$E_n(f) = \inf \left\{ \max_{x\in[a,b]} \left| f(x) - \sum_{k=0}^{\infty} C_k x^k \right| \right\}. \tag{2.1}$$

Jackson's Theorem. *If $\omega(f;\frac{1}{n})$ is a modulus of continuity of a function $f(x)$, then the inequality*

$$E_n(f) \le c\,\omega\left(f;\frac{b-a}{n}\right), \tag{2.2}$$

holds, where c an absolute, and $\omega(f;t)$ is defined as

$$\omega(f;t) = \sup_{\substack{|x_1-x_2| \\ x_1,x_2\in[a,b]}} |f(x_1)-f(x_2)|\,. \tag{2.3}$$

In case of $f(x) \in C[-1,1]$, then

$$E_n(f) \le c\,\omega\left(f;\frac{1}{n}\right). \tag{2.4}$$

3. The main result

Let $P_n^{(\alpha,\beta)}(x)$ $(n = 0,1,2,...)$ denote the Jacobi orthonormal system of polynomials with weight function

$$(1-x)^{\alpha}(1+x)^{\beta},\ (\alpha > -1, \beta > -1) \text{ on } [-1,1].$$

We start with the following special cases of the Jacobi polynomial:

1. $\alpha = -\frac{1}{2}$, $\beta = -\frac{1}{2}$, then

$$P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) = \sqrt{\frac{2}{\pi}}\cos(n\cos^{-1}x), \tag{3.1}$$

where

$$P_0^{(-\frac{1}{2},-\frac{1}{2})}(x) = \frac{1}{\sqrt{\pi}},$$

which are called the Tschebyscheff polynomials of the first kind.

2. $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, then

$$P_n^{(\frac{1}{2},\frac{1}{2})}(x) = \sqrt{\frac{2}{\pi}}\frac{\sin\left((n+1)\cos^{-1}x\right)}{\sin\left(\cos^{-1}x\right)}, \tag{3.2}$$

which are called the Tschebyscheff polynomials of the second kind.

3. $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, then

$$P_n^{(\frac{1}{2},-\frac{1}{2})}(x) = \sqrt{\frac{2}{\pi}}\frac{\sin\frac{2n+1}{2}\cos^{-1}x}{\sin\frac{1}{2}\left(\cos^{-1}x\right)}, \tag{3.3}$$

4. $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, then

$$P_n^{(-\frac{1}{2},\frac{1}{2})}(x) = \sqrt{\frac{2}{\pi}}\frac{\cos\frac{2n+1}{2}\cos^{-1}x}{\cos\frac{1}{2}\left(\cos^{-1}x\right)}, \tag{3.4}$$

In this work, we will prove the following three theorems:

Theorem 1. *If* $M = \max_{-1 \le x \le 1} |f(x)|$, *then the following inequalities hold*

$$\left| V_{n,m}^{(-\frac{1}{2},-\frac{1}{2})}(f;x) \right| \le CM \left(1 + \ln \left(\frac{n+m+1}{m+t1} \right) \right). \tag{3.5}$$

$$\left| V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x) \right| \le \frac{CM}{\sqrt{1-x^2}} \left(1 + \ln \left(\frac{n+m+1}{m+1} \right) \right). \tag{3.6}$$

$$\left| V_{n,m}^{(\frac{1}{2},-\frac{1}{2})}(f;x) \right| \le \frac{CM}{\sqrt{1-x}} \left(1 + \ln \left(\frac{n+m+1}{m+1} \right) \right). \tag{3.7}$$

$$\left| V_{n,m}^{(-\frac{1}{2},\frac{1}{2})}(f;x) \right| \le \frac{CM}{\sqrt{1+x}} \left(1 + \ln \left(\frac{n+m+1}{m+1} \right) \right). \tag{3.8}$$

Proof. First, we consider the case $\alpha = \beta = \frac{1}{2}$ (equation (3.6)). Define

$$P_n(x) = P_n^{(\frac{1}{2},\frac{1}{2})}(x),\ S_n(f;x) = S_n^{(\frac{1}{2},\frac{1}{2})}(f;x), \sigma_n(f;x) = \sigma_n^{(\frac{1}{2},\frac{1}{2})}(f;x),$$
$$V_{n,m}(f;x) = V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x), S_n(f;x) = \sum_{k=0}^{n} C_k(f) P_k(x),$$

where

$$C_k(f) = \int_{-1}^{1} \sqrt{1-t^2} f(t) P_k(t) dt.$$

Using the definition of $P_k(x)$, given in (3.2), we have

$$\begin{aligned} C_k(f) &= \sqrt{\frac{2}{\pi}} \int_{-1}^{1} \sqrt{1-t^2} f(t) \frac{\sin\left((k+1)\cos^{-1} t\right)}{\sin(\cos^{-1} t)} dt \\ &= \sqrt{\frac{2}{\pi}} \int_{-1}^{1} f(t) \sin\left((k+1)\cos^{-1} t\right) dt \\ &= \sqrt{\frac{2}{\pi}} \int_{0}^{\pi} f(\cos t) \sin(k+1)t \sin t dt. \end{aligned} \tag{3.9}$$

Then

$$\begin{aligned} S_n(f;x) &= \sum_{k=0}^{n} C_k(f) P_k(x) \\ &= \frac{2}{\pi \sin y} \int_{0}^{\pi} f(\cos t) \sin t \sum_{k=0}^{n} \sin(k+1)t \sin(k+1) y dt, \end{aligned}$$

where $y=\cos^{-1}x$ and, since $\sin\alpha\sin\beta=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$, we obtain

$$\begin{aligned} S_n(f;x) &= \sum_{k=0}^{n} C_k(f)P_k(x) \\ &= \frac{2}{\pi\sin y}\int_0^{\pi} f(\cos t)\sin t\sum_{k=0}^{n}(\cos(k+1)(t-y) - \cos(k+1)(t+y)dt. \end{aligned} \tag{3.10}$$

Using the well-known quality

$$\frac{1}{2}+\sum_{k=0}^{n}\cos kx=\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}},$$

equation (3.10) can be written as

$$S_n(f;x)=\frac{2}{\pi\sin y}\int_0^{\pi} f(\cos t)\sin t\left(\frac{\sin(n+\frac{1}{2}+1)(t-y)}{\sin\frac{1}{2}(t-y)}-\frac{\sin(n+\frac{1}{2}+1)(t+y)}{\sin\frac{1}{2}(t+y)}\right)dt,$$

that is,

$$\begin{aligned} S_n(f;x) = \frac{1}{2\pi\sin y}\int_0^{\pi} &(f(\cos(t+y))\sin(t+y) \\ &-f(\cos(t-y))\sin(t-y))\frac{\sin(n+\frac{1}{2}+1)t}{\sin\frac{1}{2}t}dt. \end{aligned} \tag{3.11}$$

From equation (1.5), we have

$$\begin{aligned} V_{n,m}(f;x) &= \frac{1}{m+1}\sum_{k=n}^{n+m} S_k(f;x) \\ &= \frac{1}{m+1}\left(\sum_{k=0}^{n+m} S_k(f;x)-\sum_{k=0}^{n-1} S_k(f;x)\right). \end{aligned} \tag{3.12}$$

To obtain an estimation for $V_{n,m}(f;x)$, first, from the integral representation of $V_{n,m}(f;x)$, let

$$M_{n,m}=\frac{1}{\pi(m+1)}\int_{-\pi}^{\pi}\frac{|\sin\frac{2n+m+1}{2}t\sin\frac{m+1}{2}t|}{2\sin^2\frac{t}{2}}dt,$$

and, if $p=\frac{m+1}{2}$, $rp=\frac{2n+m+1}{2}$, for $p\geq\frac{1}{2}, r\geq 1$, we obtain

$$M_{n,m}=\frac{1}{\pi(m+1)}\int_0^{\pi}\frac{|\sin rpt\sin pt|}{2\sin^2\frac{t}{2}}dt.$$

Since the function $\frac{1}{\sin^2\frac{t}{2}} - \frac{1}{\frac{t^2}{4}}$ is bounded on $[0,\pi]$ and $p \geq \frac{1}{2}$, then

$$M_{n,m} = \frac{2}{\pi p}\int_0^{\pi} \frac{|\sin rpt \sin pt|}{t^2} dt + O(1).$$

Since

$$\int_0^{\pi} \frac{|\sin rpt \sin pt|}{t^2} dt = p\int_0^{p\pi} \frac{|\sin rt \sin t|}{t^2} dt,$$

and

$$\left|\int_0^{p\pi} \frac{|\sin rt \sin t|}{t^2} dt - \int_0^{\pi} \frac{|\sin rt \sin t|}{t^2} dt\right| \leq \int_{\frac{\pi}{2}}^{\infty} \frac{dt}{t^2},$$

then

$$M_{n,m} = \frac{2}{\pi}\int_0^{\pi} \frac{|\sin rt \sin t|}{t^2} dt + O(1),$$

and again, since the function $\frac{1}{\sin^2\frac{t}{2}} - \frac{1}{\frac{t^2}{4}}$ is bounded on $[0,\pi]$ and $p \geq \frac{1}{2}$, then

$$M_{n,m} = \frac{2}{\pi}\int_0^{\pi} \frac{|\sin rt|}{t} dt + O(1).$$

Next, we must show that

$$\int_0^{\pi} \frac{|\sin rt|}{t} dt = \frac{2}{\pi}\ln r + O(1),$$

where $k \leq r < k+1$, for $k \geq 1$. For this, let

$$\begin{aligned}
\int_0^{\pi} \frac{|\sin rt|}{t} dt &= \sum_{i=0}^{k-1} (-1)^i \int_{\frac{i\pi}{r}}^{(i+1)\frac{\pi}{r}} \frac{|\sin rt|}{t} dt + O(1) \\
&= \sum_{i=0}^{k-1} \int_0^{\frac{\pi}{r}} \frac{\sin rt}{t + \frac{i\pi}{r}} dt + O(1) \\
&= \int_0^{\frac{\pi}{r}} \sin rt \left(\sum_{i=0}^{k-1} \frac{1}{t + \frac{i\pi}{r}}\right) dt + O(1).
\end{aligned}$$

And, since for $0 \leq t \leq \frac{\pi}{r}$ and from $\sum_{k=0}^{n} \sin(k+\frac{1}{2})x = \frac{\sin^2(n+1)\frac{x}{2}}{\sin\frac{x}{2}}$, we have:

$$\sum_{i=1}^{k-1} \frac{1}{t + \frac{i\pi}{r}} = \frac{r}{\pi}\left(\sum_{i=1}^{k-1}\frac{1}{i} + O(1)\right) = \frac{r}{\pi}(\ln k + O(1)).$$

Then

$$\int_0^\pi \frac{|\sin rt|}{t}dt = \int_0^{\frac{\pi}{r}} \sin rt\left(\frac{r}{\pi}(\ln k + O(1))\right)dt = \frac{2}{\pi}\ln k + O(1) = \frac{2}{\pi}\ln r + O(1).$$

So

$$M_{n,m} = \frac{4}{\pi}\ln r + O(1) = \frac{4}{\pi}\ln\frac{2(n+1)+m+1}{m+1} + O(1) = \frac{4}{\pi}\ln\frac{n+m+1}{m+1} + O(1).$$

Therefore, using the integral form of $V_{n,m}$, we obtain our result for, i.e.,

$$\left|V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x)\right| \le \frac{CM}{\sqrt{1-x^2}}\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right).$$

In a similar way, we can prove the other three cases.

Theorem 2. *Suppose that* $f(x)$ *is continuous function on* $[-1,1]$. *Then*

$$\left|f(x) - V_{n,m}^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\right| \le CE_n(f)\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.13}$$

$$\left|f(x) - V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x)\right| \le \frac{C}{\sqrt{1-x^2}}E_n(f)\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.14}$$

$$\left|f(x) - V_{n,m}^{(\frac{1}{2},-\frac{1}{2})}(f;x)\right| \le \frac{C}{\sqrt{1-x}}E_n(f)\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.15}$$

$$\left|f(x) - V_{n,m}^{(-\frac{1}{2},\frac{1}{2})}(f;x)\right| \le \frac{C}{\sqrt{1+x}}E_n(f)\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.16}$$

Proof. First, we consider the case $\alpha = \beta = \frac{1}{2}$.

Let $Q_n(x)$ be the best uniform approximation of algebraic polynomial for function $f(x)$ of order not exceeded n on $[-1,1]$, then

$$\begin{aligned}
|f(x) - V_{n,m}(f;x)| &\le |f(x) - Q_n(x)| + |Q_n(x) - V_{n,m}(f;x)| \\
&= |f(x) - Q_n(x)| + |V_{n,m}(f - Q_n;x)| \\
&\le E_n(f) + \frac{C}{\sqrt{1-x^2}}E_n(f)\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right) \\
&\le \frac{C}{\sqrt{1-x^2}}E_n(f)\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right).
\end{aligned}$$

In a similar way, we can prove the other three cases.

Theorem 3. *Suppose that $f(x)$ is continuous function on $[-1,1]$, then*

$$\left|f(x) - V_{n,m}^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\right| \leq \frac{C}{m+1}\sum_{k=n}^{n+m} E_k(f)\left(1+\ln\frac{k+1}{k-n+1}\right). \tag{3.17}$$

$$\left|f(x) - V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x)\right| \leq \frac{C}{(m+1)\sqrt{1-x^2}}\sum_{k=n}^{n+m} E_k(f)\left(1+\ln\frac{k+1}{k-n+1}\right). \tag{3.18}$$

$$\left|f(x) - V_{n,m}^{(\frac{1}{2},-\frac{1}{2})}(f;x)\right| \leq \frac{C}{(m+1)\sqrt{1-x}}\sum_{k=n}^{n+m} E_k(f)\left(1+\ln\frac{k+1}{k-n+1}\right). \tag{3.19}$$

(3.20)
$$\left|f(x) - V_{n,m}^{(\frac{-1}{2},\frac{1}{2})}(f;x)\right| \leq \frac{C}{(m+1)\sqrt{1+x}}\sum_{k=n}^{n+m} E_k(f)\left(1+1+\ln\frac{k+1}{k-n+1}\right).$$

Proof. We start the case $\alpha = \beta = \frac{1}{2}$.

Choose the integer p such that $2^p \leq m+1 < 2^{p+1}$, then

$$\begin{aligned} f(x) - V_{n,m}(f;x) &= \frac{1}{m+1}\sum_{k=n}^{n+m}(f(x) - S_k(f;x)) \\ &= \frac{1}{m+1}\left\{(f(x) - S_n(f;x)) + \sum_{k=1}^{p}\sum_{i=n+2^{k-1}}^{n+2^k-1}(f(x) - S_i(f;x))\right. \\ &\quad \left. + \sum_{k=n+2^p}^{n+m}(f(x) - S_k(f;x))\right\}. \end{aligned}$$

Since
$$\sum_{k=n}^{n+m} S_k(f;x) = (m+1)V_{n,m}(f;x),$$
we get
$$S_n(f;x) = V_{n,0}(f;x),$$
which yields
$$\sum_{i=n+2^{k-1}}^{n+2^k-1} S_i(f;x) = 2^{k-1}V_{n+2^{k-1},2^{k-1}-1}(f;x),$$
and
$$\sum_{i=n+2^p}^{n+m} S_i(f;x) = (m+1-2^p)V_{n+2^p,m-2^p}(f;x),$$

So

$$\begin{aligned} f(x)-V_{n,m}(f;x) &= \frac{1}{m+1}\{(f(x)-V_{n,0}(f;x)) \\ &+\sum_{k=1}^{p}2^{k-1}(f(x)-V_{n+2^{k-1},2^{k-1}-1}(f;x)) \\ &+(m+1-2^p)(f(x)-V_{n+2^p,m-2^p}(f;x)\}. \end{aligned}$$

Applying Theorem 2, we obtain

$$|f(x)-V_{n,0}(f;x)|\leq \frac{C}{\sqrt{1-x^2}}(1+\ln(n+1))E_n(f)$$

$$|f(x)-V_{n+2^{k-1},2^{k-1}-1}(f;x)|\leq \frac{C}{\sqrt{1-x^2}}\left(1+\ln\left(\frac{n+2^k}{2^{k-1}}\right)\right)E_{n+2^{k-1}}(f) \tag{3.21}$$

$$|f(x)-V_{n+2^p,m-2^p}(f;x)|\leq \frac{C}{\sqrt{1-x^2}}\left(1+\ln\left(\frac{n+m+1}{m+1-2^p}\right)\right)E_{n+2^p}(f).$$

Thus

$$\begin{aligned} |f(x)-V_{n,m}(f;x)| \leq \frac{C}{(m+1)\sqrt{1-x^2}} &\{(1+\ln(n+1))E_n(f) \\ &+\sum_{k=1}^{p}2^{k-1}\left(1+\ln(\frac{n+2^k}{2^{k-1}})\right)E_{n+2^{k-1}}(f) \\ &+(m+1-2^p)\left(1+\ln\left(\frac{n+m+1}{m+1-2^p}\right)\right)E_{n+2^p}(f)\Bigg\}. \end{aligned} \tag{3.22}$$

Note that $\forall u,v>0$, we have the inequality $u+v\leq(1+u)(1+v)$ and then $\ln(u+v)\leq\ln(1+u)+\ln(1+v)$. Setting $u=\frac{x}{z}, v=\frac{y}{z}$, we obtain

$$ln\frac{x+y}{z}\leq\ln\left(1+\frac{x}{z}\right)+\ln(1+\frac{y}{z}).$$

So

$$\begin{aligned} I &= \sum_{k=1}^{p}2^{k-1}E_{n+2^{k-1}}(f)\ln\left(\frac{n+2^k}{2^{k-1}}\right) \\ &\leq \sum_{k=1}^{p}2^{k-1}E_{n+2^{k-1}}(f)\ln\left(1+\frac{n}{2^{k-1}}\right)+\sum_{k=1}^{p}2^{k-1}E_{n+2^{k-1}}(f)\ln\left(1+\frac{2^k}{2^{k-1}}\right) \\ &= I_1+I_2. \end{aligned} \tag{3.23}$$

For I_2, as mentioned above

$$I_2\leq C\sum_{k=n}^{n+2^p-1}E_k(f). \tag{3.24}$$

For I_1, note that

$$
\begin{aligned}
I_1 &= \sum_{k=1}^{p} 2^{k-1} E_{n+2^{k-1}}(f) \ln(1+\frac{n}{2^{k-1}}) \\
&= E_{n+1}(f)\ln(1+n) + 2\sum_{k=2}^{p} 2^{k-2} E_{n+2^{k-1}}(f) \ln(1+\frac{n}{2^{k-1}}) \\
&= E_{n+1}(f)\ln(1+n) + 2\sum_{k=2}^{p} \sum_{i=n+2^{k-2}}^{n+2^{k-1}-1} 2^{k-2} E_{n+2^{k-1}}(f) \ln(1+\frac{n}{2^{k-1}}) \\
&= E_{n+1}(f)\ln(1+n) + 2\left\{ \sum_{i=n+1}^{n+1} E_{n+2^1}(f) \ln\left(1+\frac{n}{2^1}\right) + \sum_{i=n+2}^{n+3} E_{n+2^2}(f) \ln\left(1+\frac{n}{2^2}\right) \right. \\
&\left. + \sum_{i=n+4}^{n+7} E_{n+2^3}(f) \ln(1+\frac{n}{2^3}) + \cdots + \sum_{i=n+2^{p-1}}^{n+2^{p-1}-1} E_{n+2^{p-1}}(f) \ln\left(1+\frac{n}{2^{p-1}}\right) \right\} \\
&\leq C \sum_{k=n}^{n+2^p-1} E_k(f) \ln\left(1+\frac{n}{k-n+1}\right).
\end{aligned}
\tag{3.25}
$$

Combining the estimates given in (3.25) and (3.24) for I_1 and I_2, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{p} 2^{k-1} E_{n+2^{k-1}}(f) \ln\left(\frac{n+2^k}{2^{k-1}}\right) \\
&\leq C \left\{ \sum_{k=n}^{n+2^p-1} E_k(f) + \sum_{k=1}^{n+2^p-1} E_k(f) \ln\left(1+\frac{k+1}{k-n+1}\right) \right\}
\end{aligned}
\tag{3.26}
$$

Thus, from the choice of p such that $m+1-2^p \leq 2^p$, we have:

$$(m+1-2^p) E_{n+2^p}(f) \leq \sum_{k=n}^{n+2^p-1} E_k(f).$$

In addition, note that for any natural numbers α, β such that $1 \leq \alpha \leq \beta - 1$, also we have

$$
\begin{aligned}
\alpha \ln \frac{\beta-\alpha}{\alpha} &= \ln\frac{\beta-\alpha}{\alpha} + \ln\frac{\beta-\alpha}{\alpha} + \cdots + \ln\frac{\beta-\alpha}{\alpha} \\
&\leq \ln\frac{\beta-\alpha}{1} + \ln\frac{\beta-\alpha}{2} + \cdots + \ln\frac{\beta-\alpha}{\alpha} \\
&= \sum_{k=\alpha+1}^{\beta} \ln\frac{\beta-\alpha}{k-\beta+\alpha}
\end{aligned}
$$

therefore,

$$\begin{aligned}
(m+1-2^p)\ln\frac{n+m+1}{m+1-2^p} &\le \sum_{k=n+m+2}^{n+2m+2-2^p} \ln\frac{n+m+1}{k-n-m-1} \\
&= \sum_{k=1}^{n+1-2^p} \ln\frac{n+m+1}{k} \\
&= \sum_{k=n}^{n+m-2^p} \ln\frac{n+m+1}{k-n+1} \\
&\le \sum_{k=n}^{n+m-2^p} \ln\left(1+\frac{n}{k-n+1}\right) + \sum_{k=n}^{n+m-2^p} \ln\left(1+\frac{m+1}{k-n+1}\right).
\end{aligned}$$

where $\alpha = m+1-2^p$, $\beta = 2m+n-2^p$, but

$$\begin{aligned}
\sum_{k=n}^{n+m-2^p} \ln\left(1+\frac{m+1}{k-n+1}\right) &= \sum_{k=1}^{n+m-2^p} \ln\left(1+\frac{n}{k-n+1}\right) \\
&\le \sum_{k=1}^{m+1} \ln\left(1+\frac{m+1}{k}\right) \\
&\le C(m+1) \\
&\le C2^p,
\end{aligned}$$

where we used

$$\ln(1+x) = \ln x + O\left(\frac{1}{x}\right), \ (x \ge 1),$$

and

$$\sum_{k=1}^{m} \ln\left(\frac{m}{k}\right) = \ln\frac{m^n}{m!} \le \ln e^m \le m \le C2^p,$$

so

$$(m+1-2^p)\ln\frac{n+m+1}{m+1-2^p} \le C\left[2^p + \sum_{k=n}^{n+m-2^p} \ln\left(1+\frac{n}{k-n+1}\right)\right].$$

Since $2^p \le m+1$, then

$$E_n(f) + E_{n+1}(f) + \cdots + E_{n+m}(f) \ge (m+1)E_{n+2^p}(f) \ge 2^p E_{n+2^p}(f),$$

therefore,

$$
\begin{aligned}
&(m+1-2^p)E_{n+2^p}(f)\ln\tfrac{n+m+1}{m+1-2^p} \\
&\quad\le C\left[2^pE_{n+2^p}(f)+\sum_{k=n}^{n+m-2^p}E_{n+2^p}(f)\ln\left(1+\frac{n}{k-n+1}\right)\right] \\
&\quad\le C\left[\sum_{k=n}^{n+m}E_k(f)+\sum_{k=n}^{n+m-2^p}E_{n+2^p}(f)\ln\left(1+\frac{n}{k-n+1}\right)\right] \\
&\quad\le C\left[\sum_{k=n}^{n+m}E_k(f)+\sum_{k=n}^{n+m}E_k(f)\ln\left(1+\frac{n}{k-n+1}\right)\right].
\end{aligned}
\tag{3.27}
$$

Combining all of the above estimates (3.22)–(3.27), we get the desired result. This ends of the proof of the theorem.

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