ON SOME PROPERTIES OF ϕ-MULTIPLIERS

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Abstract. In this paper, we investigate some new properties of ϕ-multipliers studied recently by M.Adib and A.Riazi on faithful Banach algebras. Specially we justifies the existence of Helgason-Wang function for a ϕ-multipliers and give some characterizations. As corollary we obtain some results for classical multipliers.

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1. Introduction and notations

The general theory of multipliers on a faithful Banach algebras was originally introduced by Helgason [4] and has been developed by Wang [8] and Birtal [2]. A good reference for this theory is the monograph of Larsen [5] and Laursen and Neumann [6].

In the following, let $\mathcal{A}$ denote a complex Banach algebra. We recall that a mapping $T : \mathcal{A} \to \mathcal{A}$ is said to be a multiplier if for all $x,\ y \in \mathcal{A}$

$$x(Ty) = (Tx)y.$$

The set of multipliers of $\mathcal{A}$ is denoted by $M(\mathcal{A})$. It is well known that if $\mathcal{A}$ is faithful (see definition below) then $M(\mathcal{A})$ is a closed commutative subalgebra of $\mathcal{B}(\mathcal{A})$ the Banach algebra of all bounded linear operators of $\mathcal{A}$ (see [1], [5], [6]).

We shall say that two elements $x, y \in \mathcal{A}$ are orthogonal whenever $xy = yx = 0$. Given a nonempty subset B of $\mathcal{A}$ the orthogonal of B is defined to be the set

$$B^{\top} := \{x \in \mathcal{A} : xy = yx = 0 \, for\ each\ \ y \in B\}.$$

Trivially, $B^{\top}$ is a closed two sided ideal of $\mathcal{A}$.

Recall that the left annihilator and the right annihilator of B are respectively the sets:

$$lanB = \{x \in A \ : \ xB = \{0\}\} \ \ and \ ranB = \{x \in \mathcal{A} \ : \ Bx = \{0\}\}.$$

We say that $\mathcal{A}$ is faithful (or without order) if $lan\mathcal{A} = \{0\}$ or $ran\mathcal{A} = \{0\}$, $\mathcal{A}$ is semi prime if $\{0\}$ is the unique two-sided ideal J such that $J^2 = \{0\}$ and $\mathcal{A}$ is said semi simple if rad $(\mathcal{A}) = \{0\}$ where rad $\mathcal{A}$ is the (Jacobson) radical of $\mathcal{A}$, see [1], [5], [6].

Note that if $\mathcal{A}$ is semi prime then $\mathcal{A}$ is faithful. Moreover if $x\mathcal{A}x = \{0\}$ then $x = 0$. It is well known that each semi simple algebra is semi prime and therefore faithful.

If $\mathcal{A}$ is a faithful Banach algebra then the linearity and continuity of every $T \in M(\mathcal{A})$ are automatic and for all $x,\ y \in \mathcal{A}$ we have

$$T(xy) = x(Ty) = (Tx)y.$$

Let $\Delta(\mathcal{A})$ denote the set of all maximal regular ideals of a commutative Banach algebra $\mathcal{A}$ and let $\mathcal{A}^*$ denote the dual of $\mathcal{A}$ (see [9]).
Recall that a multiplicative linear functional on a complex Banach algebra $\mathcal{A}$ is a non-zero linear functional $m \in \mathcal{A}^*$ such that $m(xy) = m(x)m(y)$ for all $x, y \in \mathcal{A}$. It is well known that if $\mathcal{A}$ is a commutative Banach algebra $\mathcal{A}$ then

$$rad\,\mathcal{A} = \bigcap_{m\in\Delta(\mathcal{A})} \ker m.$$

Now, let $\widehat{x}$ denotes the Gelfand transform of $x \in \mathcal{A}$ defined by

$$\widehat{x}(m) := m(x) \, for\ each \, m \in \Delta(\mathcal{A}).$$

Recall that $\mathcal{A}$ is semi simple if and only if for every non-zero element x of $\mathcal{A}$ there exists some $m \in \Delta(\mathcal{A})$ such that $\widehat{x}(m) \neq 0$, obviously this condition implies that if $\widehat{x}(m)$ is zero for all $m \in \Delta(\mathcal{A})$ then $x = 0$ [9].

Recently, Riazi and Adib [7] have defined and studied the concept of ϕ-multipliers where ϕ is a homomorphism from $\mathcal{A}$ to $\mathcal{A}$ which is a generalization of multipliers if $\mathcal{A}$ is a faithful Banach algebra.

Definition 1.1 [7] Let $\mathcal{A}$ be a Banach algebra, ϕ be a homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{A}$ be a continuous linear mapping.

T is said ϕ-multiplier on $\mathcal{A}$ if for all $x, y \in \mathcal{A}$, we have

$$T(xy) = T(x)\phi(y) = \phi(x)T(y).$$

We denote $M_\phi(\mathcal{A})$ the collection of all ϕ-multipliers of $\mathcal{A}$ where ϕ is a homomorphism.

An immediate example of a ϕ-multiplier of a Banach algebra $\mathcal{A}$ is given by $L_a \circ \phi$ where $a \in com(\mathcal{A})$ ($com(\mathcal{A})$ the commutator of $\mathcal{A}$) and $L_a : x \in \mathcal{A} \to ax$ the left multiplication operator by a.

If $\mathcal{A}$ is a commutative Banach algebra with unit u, given a multiplier $T \in M_\phi(\mathcal{A})$ where ϕ is a homomorphism from $\mathcal{A}$ to $\mathcal{A}$, for each $x \in \mathcal{A}$ we have

$$\begin{aligned}(L_{Tu} \circ \phi)x &= (Tu)(\phi(x)) \\ &= \phi(u)(Tx) \\ &= uTx = Tx \ \ (T \in M_\phi(\mathcal{A}),\ \phi \text{ is homomorphism } : \phi(u) = u).\end{aligned}$$

Thus $T = L_{Tu} \circ \phi$.

The most important example of a ϕ-multipliers is obtained when we take $\mathcal{A} = L_1(G)$ the group algebra of a locally compact Abelian group G [3].

In the following section, we investigate some news properties of ϕ-multipliers and We extend some classical results for multipliers.

2. ϕ-Multipliers and their properties

Proposition 2.1 *Let $\mathcal{A}$ be a faithful Banach algebra, ϕ be a bounded onto homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{A}$ be a mapping.*
T is ϕ-multiplier on $\mathcal{A}$ if and only if for all $x, y \in A$ we have

$$T(x)\phi(y) = \phi(x)T(y).$$

Moreover $T(\mathcal{A})$ is a two-sided ideal in $\mathcal{A}$.

Proof. Directly, is trivial.

Conversely, assume that for all $x, y \in \mathcal{A}$ we have $T(x)\phi(y) = \phi(x)T(y)$. Since $\mathcal{A}$ is faithful then we can consider $\operatorname{ran} \mathcal{A} = \{0\}$. First, for any x, y, $z \in A$ we have

$$\begin{aligned} z[\phi(x)T(y)] &= \phi(z')[\phi(x)T(y)] \quad (\phi \text{ is onto } : \exists\, x' \in \mathcal{A} \,/\, \phi(x') = x) \\ &= \phi(z')[T(x)\phi(y)] \\ &= [\phi(z')T(x)]\phi(y)] \\ &= [T(z')\phi(x)]\phi(y) \\ &= T(z')\phi(xy) \\ &= \phi(z')T(xy) \\ &= zT(xy). \end{aligned}$$

Thus $[\phi(x)T(y) - T(xy)] \in \text{ran } \mathcal{A} = \{0\}$, and therefore the equalities

$$T(xy) = \phi(x)T(y) = T(x)\phi(y).$$

Now, we prove that T is linear, for any $x,\ y,\ z \in \mathcal{A}$ and $\lambda,\ \mu \in \mathbb{C}$ we have

$$\begin{aligned} z[T(\lambda x + \mu y)] &= \phi(z')T[\lambda x + \mu y] \quad (\phi \text{ is onto } : \exists z' \in \mathcal{A} \,/\, \phi(z') = z) \\ &= T(z')\phi[\lambda x + \mu y] \\ &= T(z')[\lambda\phi(x) + \mu\phi(y)] \\ &= \lambda T(z')\phi(x) + \mu T(z')\phi(y) \\ &= \lambda\phi(z')(Tx) + \mu\phi(z')(Ty) \\ &= z(\lambda Tx + \mu Ty) \qquad (z = \phi(z')), \end{aligned}$$

which implies that $z[T(\lambda x + \mu y) - (\lambda Tx + \mu Ty)] = 0$ for all $z \in \mathcal{A}$ and since ran $\mathcal{A} = \{0\}$ this implies $T(\lambda x + \mu y) = \lambda Tx + \mu Ty$.

To prove that T is bounded let $y, z \in \mathcal{A}$ and $(y_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\|y_n - y\| \to 0$ and $\|T(y_n) - z\| \to 0$, for $x \in \mathcal{A}$ arbitrary we have

$$\begin{aligned} \|xz - xT(y)\| &= \|xz - xT(y_n) + xT(y_n) - xT(y)\| \\ &= \|xz - xT(y_n) + \phi(x')T(y_n) - \phi(x')T(y)\| \\ &\qquad (\phi \text{ is onto } : \exists\, z' \in \mathcal{A} \,/\, \phi(z') = z) \\ &\leq \|xz - xT(y_n)\| + \|\phi(x')T(y_n) - \phi(x')T(y)\| \\ &\leq \|x\|\|z - T(y_n)\| + \|T(x')\phi(y_n) - T(x')\phi(y)\| \\ &\leq \|x\|\|z - T(y_n)\| + \|T(x')\|\|\phi\|\|y_n - y\| \ \to\ 0 \end{aligned}$$

which implies that $x(z - T(y)) = 0$, since ran $\mathcal{A} = \{0\}$ therefore $z = Ty$ and by the closed graph theorem we conclude that T is a bounded operator.

Finally let $y \in T(\mathcal{A})$ and $z \in \mathcal{A}$, then there exists $x \in \mathcal{A}$ such that $y = T(x)$. Since ϕ is onto then there exists $z' \in \mathcal{A}$ such that $z = \phi(z')$ we obtain that

$$yz = T(x)z = T(x)\phi(z') = T(xz') \in T(\mathcal{A})$$

and

$$zy = zT(x) = \phi(z')T(x) = T(z'x) \in T(\mathcal{A}).$$

Then $T(\mathcal{A})$ is a two-sided ideal in $\mathcal{A}$. ∎

Remark 2.1. In hypothesis of the Proposition 2.1 if we take $\mathcal{A}$ a semi simple Banach algebra it suffice to suppose that ϕ is onto homomorphism because in this case it is automatically continuous by the Johnson's theorem [9].

Theorem 2.1 *Let $\mathcal{A}$ be a semi prime Banach algebra, ϕ be an onto homomorphism. If $T \circ \phi = \phi \circ T$ for all $T \in M_\phi(\mathcal{A})$ then $M_\phi(\mathcal{A})$ is semi prime Banach algebra.*

Proof. Let J be a two-sided ideal of $M_\phi(\mathcal{A})$ such that $J^2 = \{0\}$ and $T \in J$. We have for all $x, y \in \mathcal{A}$:

$$\begin{aligned}
(Tx)(Ty) &= (Tx)(T[\phi(y')]) \quad (\phi \textit{ is onto } : \exists\, y' \in \mathcal{A}/\ \ y = \phi(y')) \\
&= (Tx)(\phi[T(y')]) \quad (T \circ \phi = \phi \circ T) \\
&= \phi(x)T^2(y') = 0 \quad\ (T \in M_\phi(\mathcal{A}) \textit{ and } T^2 \in J^2 = \{0\}).
\end{aligned}$$

Then the two-sided ideal $T(\mathcal{A})$ in A satisfy $[T(\mathcal{A})]^2 = \{0\}$, since $\mathcal{A}$ is semi prime we conclude that $T(\mathcal{A}) = \{0\}$.Therefore $T = 0$, and hence $J = \{0\}$.
We deduce that $M_\phi(\mathcal{A})$ is semi prime Banach algebra. ∎

As corollary of this theorem, we obtain the classical following result [1, Theorem 1.4.4 ch IV].

Corollary 2.1 *Let $\mathcal{A}$ be a semi prime Banach algebra then $M(\mathcal{A})$ is semi prime Banach algebra.*

Proof. Since $\mathcal{A}$ is semi prime then $M(\mathcal{A}) = M_{Id_\mathcal{A}}(\mathcal{A})$ ($Id_\mathcal{A}$ the identity operator) and by theorem 2.1 we deduce that $M(\mathcal{A})$ is semi prime. ∎

Lemma 2.1 *Let $\mathcal{A}$ be a semi prime Banach algebra, ϕ be an onto homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T \in M_\phi(\mathcal{A})$ such that $\phi \circ T = T \circ \phi$. Then we have*

$$\ker(T^2) \subseteq \ker(\phi \circ T).$$

Proof. Let $x \in \ker(T^2)$ and $a \in \mathcal{A}$ arbitrary, then

$$\begin{aligned}
(\phi \circ T)(x)a(\phi \circ T)(x) &= (\phi \circ T)(x)\phi(a')(\phi \circ T)(x) \\
&\qquad (\phi \text{ is onto, then } \exists\, a' \text{ such that } \phi(a') = a) \\
&= (T \circ \phi)(x)\phi(a')(\phi \circ T)(x) \quad (\phi \circ T = T \circ \phi) \\
&= [T(\phi(x))\phi(a')]\phi(T(x)) \\
&= [\phi^2(x)T(a')]\phi(T(x)) \\
&\qquad (T \in M_\phi(\mathcal{A}) : T(\phi(x))\phi(a') = \phi^2(x)T(a')) \\
&= \phi^2(x)[T(a')\phi(T(x))] \\
&= \phi^2(x)\phi(a')T^2(x) \\
&\qquad (T \in M_\phi(\mathcal{A}) : T(a')\phi(T(x)) = \phi(a')T^2(x)) \\
&= 0 \qquad (x \in \ker(T^2) : T^2(x) = 0),
\end{aligned}$$

which proves that $(\phi \circ T)(x)\mathcal{A}(\phi \circ T)(x) = \{0\}$, and since $\mathcal{A}$ is semi prime then $(\phi \circ T)(x) = 0$ and $x \in \ker(\phi \circ T)$. Finally we obtain $\ker(T^2) \subseteq \ker(\phi \circ T)$. ∎

Theorem 2.2 *Let $\mathcal{A}$ be a semi prime Banach algebra, ϕ be a bijective homomorphism from A to $\mathcal{A}$ and $T \in M_\phi(\mathcal{A})$ such that $\phi \circ T = T \circ \phi$. Then*

$$\ker(T^2) = \ker T.$$

Proof. It's clear that $\ker(T) \subseteq \ker(T^2)$. On the other hand, since ϕ is bijective by Lemma 2.1 we have

$$\ker(T^2) \subseteq \ker(\phi \circ T) = \ker(T).$$

Therefore

$$\ker(T^2) = \ker T.$$

∎

Corollary 2.2 *Let $\mathcal{A}$ be a semi prime Banach algebra and $T \in M(\mathcal{A})$ then*

$$\ker(T^2) = \ker T.$$

Proof. Since $\mathcal{A}$ is a semi prime Banach algebra and $T \in M(\mathcal{A})$ then $T \in M_{Id_{\mathcal{A}}}(\mathcal{A})$ and by Theorem 2.2, we obtain that

$$\ker(T^2) = \ker T.$$

∎

Theorem 2.3 *Let $\mathcal{A}$ be a semi prime Banach algebra, ϕ be an onto homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T \in M_{\phi}(\mathcal{A})$ such that $\phi \circ T = T \circ \phi$. Then we have*

$$T(\mathcal{A}) \cap \ker(T) \subseteq \ker(\phi) \subseteq \ker(T)$$

Proof. Let $x \in T(\mathcal{A}) \cap \ker(T)$ then $Tx = 0$ and $\exists\, z \in \mathcal{A}$ such that $x = Tz$. We have, for all $a \in \mathcal{A}$,

$$\begin{aligned}
\phi(x)a\phi(x) &= \phi(Tz)\phi(a')\phi(Tz) && (x = Tz \text{ and } \phi \text{ is onto } : \exists\, a' \in \mathcal{A} \,/\, \phi(a') = a)\\
&= T(\phi(z))\phi(a')\phi(Tz) && (\phi \circ T = T \circ \phi)\\
&= \phi^2(z)T(a')\phi(Tz) && (T \in M_{\phi}(\mathcal{A}) : T(\phi(z))\phi(a') = \phi^2(z)T(a'))\\
&= \phi^2(z)\phi(a')T^2(z) && (T \in M_{\phi}(\mathcal{A}) : T(a')\phi(Tz) = \phi(a')T^2(z))\\
&= 0 && (T^2(z) = T(Tz) = T(x) = 0).
\end{aligned}$$

We conclude that $\phi(x)\mathcal{A}\phi(x) = \{0\}$ and since $\mathcal{A}$ is semi prime then $\phi(x) = 0$ and $x \in \ker\phi$. Therefore, $T(\mathcal{A}) \cap \ker(T) \subseteq \ker\phi$.

Let $x \in \ker\phi$. We have, for all $a \in \mathcal{A}$,

$$\begin{aligned}
T(x)aT(x) &= T(x)\phi(a')T(x) && (\phi \text{ is onto } : \exists\, a' \in \mathcal{A} \,/\, \phi(a') = a))\\
&= \phi(x)T(a')T(x) && (T \in M_{\phi}(\mathcal{A}) : T(x)\phi(a') = \phi(x)T(a'))\\
&= 0 && (x \in \ker\phi \ : \ \phi(x) = 0).
\end{aligned}$$

We conclude that $T(x)\mathcal{A}T(x) = \{0\}$, since $\mathcal{A}$ is semi prime then $T(x) = 0$ and $x \in \ker T$. Therefore,

$$\ker\phi \subseteq \ker T.$$

∎

We obtain assertion (i) of [1, Theorem 1.4.32 Ch IV] as corollary of this theorem.

Corollary 2.3 *Let $\mathcal{A}$ be a semi prime Banach algebra and $T \in M(\mathcal{A})$ then*

$$T(\mathcal{A}) \cap \ker(T) = \{0\}.$$

Proof. Since $\mathcal{A}$ is a semi prime Banach algebra and $T \in M(\mathcal{A})$ then $T \in M_{Id_{\mathcal{A}}}(\mathcal{A})$ and by Theorem 2.3 we obtain $T(\mathcal{A}) \cap \ker(T) \subseteq \ker Id_{\mathcal{A}} = \{0\}$, consequently

$$T(\mathcal{A}) \cap \ker(T) = \{0\}. \quad \blacksquare$$

Theorem 2.4 *Let $\mathcal{A}$ be a faithful Banach algebra, ϕ be a homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T \in M_\phi(\mathcal{A})$. Then we have the following inclusions*

(1) $\overline{T(\mathcal{A})} \subseteq [\phi(\ker(T))]^\top$

(2) $\phi(\ker(T)) \subseteq [T(\mathcal{A})]^\top$.

Proof.

(1) Let $x \in \ker T$ and $y \in \overline{T(\mathcal{A})}$, then $\exists\ (z_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $y = \lim_{n\to\infty} Tz_n$ and we have

$$\phi(x)Tz_n = T(x)\phi(z_n) = 0 \quad (T \in M_\phi(\mathcal{A})\ \textit{and}\ x \in \ker(T)),$$

and hence $\phi(x)y = \lim_{n\to\infty} \phi(x)Tz_n = 0$ for all $x \in \ker T$ and $y \in \overline{T(\mathcal{A})}$. In same way, we show that $y\phi(x) = 0$ for all $x \in \ker T$ and $y \in \overline{T(\mathcal{A})}$. Therefore, $\overline{T(\mathcal{A})} \subseteq [\phi(\ker(T))]^\top$.

(2) Let $x \in \ker T$ and $y \in T(\mathcal{A})$, then $Tx = 0$ and $\exists\ z \in \mathcal{A}$ such that $y = Tz$ it follows that $\phi(x)y = \phi(x)Tz = T(x)\phi(z) = 0$. In same wa, we show that $y\phi(x) = 0$, therefore, $\phi(\ker(T)) \subseteq [T(\mathcal{A})]^\top$. ∎

As corollary of this theorem, we obtain assertion (i) of [1, Theorem 1.4.9, Ch. IV].

Corollary 2.4 *Let $\mathcal{A}$ be a faithful Banach algebra and $T \in M(\mathcal{A})$ then the following inclusions holds*

(1) $\overline{T(\mathcal{A})} \subseteq [\ker T]^\top$

(2) $\ker T \subseteq [T(\mathcal{A})]^\top$.

Proof. Since $\mathcal{A}$ is a faithful Banach algebra and $T \in M(\mathcal{A})$ then $T \in M_{Id_{\mathcal{A}}}(\mathcal{A})$. Theorem 2.4 implies that

(1) $\overline{T(\mathcal{A})} \subseteq [\ker T]^\top$

(2) $\ker T \subseteq [T(\mathcal{A})]^\top$. ∎

In the next theorem, we justifies the existence of Helgason-Wang function (for details, see [1], [5] and [6]) of a ϕ-multiplier on $\mathcal{A}$.

Theorem 2.5 *Let $\mathcal{A}$ be a semi simple commutative Banach algebra and ϕ be a homomorphism from $\mathcal{A}$ to $\mathcal{A}$. Then, for each $T \in M_\phi(\mathcal{A})$, there exists an unique continuous function φ_T on $\Delta(\mathcal{A})$ such that the equation*

$$\widehat{Tx}(m) = \varphi_T(m)\widehat{\phi(x)}(m)$$

holds for all $x \in A$ and all $m \in \Delta(\mathcal{A})$.

Moreover, $|\varphi_T(m)| \leq \frac{\|m\|\|T\|}{\|m\circ\phi\|}$ for all $m \in \Delta(\mathcal{A})$.

Proof. For each $m \in \Delta(\mathcal{A})$ take $x \in A$ such that $\widehat{\phi(x)}(m) \neq 0$ and define

$$\varphi_T(m) := \frac{\widehat{Tx}(m)}{\widehat{\phi(x)}(m)}.$$

The definition of φ_T is independent of x because if $y \in \mathcal{A}$ such that $\widehat{\phi(y)}(m) \neq 0$ then since $(Tx)\phi(y) = \phi(x)(Ty)$ we obtain

$$\frac{\widehat{Tx}(m)}{\widehat{\phi(x)}(m)} = \frac{\widehat{Ty}(m)}{\widehat{\phi(y)}(m)}.$$

Hence the function φ_T is well defined.

Suppose $\widehat{\phi(x)}(m) = 0$ and let $y \in \mathcal{A}$ such that $\widehat{\phi(y)}(m) \neq 0$. Then we have

$$\widehat{Tx}(m)\widehat{\phi(y)}(m) = \widehat{\phi(x)}(m)\widehat{Ty}(m) = 0,$$

which implies $\widehat{Tx}(m) = 0$. Hence $\widehat{Tx}(m) = \varphi_T(m)\widehat{\phi(x)}(m)$ for all $x \in \mathcal{A}$ and $m \in \Delta(\mathcal{A})$.

On other hand, since ϕ is a homomorphism from semi-simple commutative Banach $\mathcal{A}$ then by Gelfant theorem ϕ is automatically continuous [9], consequently φ_T is a continuous function on $\Delta(\mathcal{A})$ with the Gelfand topology.

To prove the uniqueness of φ_T, suppose that there is an other complex-valued function defined on $\Delta(\mathcal{A})$ denote ψ for which $\widehat{Tx} = \widehat{\psi(x)}$. Then

$$(\varphi_T(m) - \psi(m))\widehat{\phi(x)}(m) = 0 \text{ for all } x \in \mathcal{A}$$

Therefore, $\varphi_T(m) = \psi(m)$.

Let us denote

$$\|m\| := sup\{|\widehat{x}(m)| : \|x\| = 1\}$$

and

$$\|m \circ \phi\| := sup\{|\widehat{\phi(x)}(m)| : \|x\| = 1\}$$

Because $0 < \|m\| \leq 1$ and $0 < \|m \circ \phi\| \leq \|\phi\| < \infty$, for each $x \in \mathcal{A}$ we have

$$|\varphi_T(m)||\widehat{\phi(x)}(m)| = |\varphi_T(m)\widehat{\phi(x)}(m)| = |\widehat{Tx}(m)| \leq \|m\|\|T\|\|x\|.$$

We obtain, for $x \in \mathcal{A}$, such that $\|x\| = 1$

$$\begin{aligned} |\varphi_T(m)| &\leq \inf_{\|x\|=1} \frac{\|m\|\|T\|}{|\widehat{\phi(x)}(m)|} \\ &\leq \frac{\|m\|\|T\|}{\sup_{\|x\|=1|} |\widehat{\phi(x)}(m)|} \\ &\leq \frac{\|m\|\|T\|}{\|m \circ \phi\|} \end{aligned}$$

So, $|\varphi_T(m)| \leq \frac{\|m\|\|T\|}{\|m\circ\phi\|}$ for all $m \in \Delta(\mathcal{A})$. ∎

Remark 2.2.

1) The function φ_T given by Theorem 2.5 to a ϕ-multiplier T will be called the Helgason-Wang function of T.

2) In Theorem 2.5, if we suppose that ϕ is homomorphism for which there is $\delta > 0$ such that $\delta\|m\| \leq \|m \circ \phi\|$ for all $m \in \Delta(\mathcal{A})$, then the function φ_T is bounded and satisfies $\|\varphi_T\|_\infty \leq \frac{1}{\delta}\|T\|$.

As corollary of this theorem, we obtain Wang's theorem [8], [1, Theorem 1.4.14 Ch IV].

Corollary 2.5 *Let $\mathcal{A}$ be a semi simple commutative Banach algebra. Then for each $T \in M(\mathcal{A})$ there exists an unique bounded continuous function φ_T on $\Delta(\mathcal{A})$ such that the equation*

$$\widehat{Tx}(m) = \varphi_T(m)\widehat{x}(m)$$

holds for all $x \in \mathcal{A}$ and all $m \in \Delta(\mathcal{A})$.

Moreover, $\|\varphi_T\|_\infty \leq \|T\|$ for all $T \in M(\mathcal{A})$.

Proof. Since $\mathcal{A}$ is commutative semi-simple Banach algebra and $T \in M(\mathcal{A})$. Then $T \in M_{Id_{\mathcal{A}}}(\mathcal{A}) = M(\mathcal{A})$ and, by Theorem 2.5, we obtain the existence of an unique continuous function φ_T on $\Delta(\mathcal{A})$ such that the equation

$$\widehat{Tx}(m) = \varphi_T(m)\widehat{Id_{\mathcal{A}}(x)}(m) = \varphi_T(m)\widehat{x}(m)$$

holds for all $x \in \mathcal{A}$ and all $m \in \Delta(\mathcal{A})$.

Moreover, by Theorem2.5, we have $|\varphi_T(m)| \leq \frac{\|m\|\|T\|}{\|m\circ Id_{\mathcal{A}}\|}$ for all $m \in \Delta(\mathcal{A})$. Then we conclude that $\|\varphi_T\|_\infty \leq \frac{\|m\|\|T\|}{\|m\|} = \|T\|$ for all $T \in M_{Id_{\mathcal{A}}}(\mathcal{A}) = M(\mathcal{A})$. ∎

Theorem 2.6 *Let $\mathcal{A}$ be a commutative semi simple Banach algebra and ϕ be a homomorphism from $\mathcal{A}$ to $\mathcal{A}$. If $T \in M_\phi(\mathcal{A})$ then we have*

$$T(\ker(m \circ \phi)) \subseteq \ker m \textit{ for each } m \in \Delta(\mathcal{A}).$$

Proof. Let $T \in M_\phi(\mathcal{A})$ and $m \in \Delta(\mathcal{A})$. Let $x \in \mathcal{A}$ such that $x \notin \ker(m \circ \phi)$, for every $y \in A$ such that $y \in \ker(m \circ \phi)$ we have

$$\widehat{Ty}(m)\widehat{\phi(x)}(m) = \widehat{\phi(y)}(m)\widehat{Tx}(m) = 0 \quad (\widehat{\phi(y)}(m) = m(\phi(y)) = 0)$$

Since $\widehat{\phi(x)}(m) \neq 0$ this implies $\widehat{Ty}(m) = m(Ty) = 0$ and hence $Ty \in \ker m$. Therefore, $T(\ker(m \circ \phi)) \subseteq \ker m$. ∎

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