SOME GEOMETRIC AGGREGATION OPERATORS BASED ON PICTURE FUZZY SETS AND THEIR APPLICATION IN MULTIPLE ATTRIBUTE DECISION MAKING

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Abstract. In this paper, we investigate the multiple attribute decision making (MADM) problems with picture fuzzy information. Firstly, concepts and some operational laws of picture fuzzy sets are introduced. Then, we develop some picture fuzzy geometric operators and discuss their basic properties. Next, we apply the proposed operators to deal with multiple attribute decision making problems under picture fuzzy environment. Finally, an illustrative example is given to demonstrate the practicality and effectiveness of our proposed method.

Keywords: picture fuzzy set, picture fuzzy number, picture fuzzy geometric aggregation operator, multiple attribute decision making.

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1. Introduction

Fuzzy set (FS), proposed by Zadeh in 1965 [24], has achieved a great success in various fields since it appears. Since then, many new theories treating imprecision and uncertainty have been introduced. As an extension of FS, the intuitionistic fuzzy set (IFS) which was proposed by Atanassov [1], [2] has received much attention. As a powerful tool to deal with vagueness and uncertainty, the intuitionistic fuzzy set has been widely used from the application's view point in many fields, such as logic programming [3], decision making [4], [11]– [14], [20]– [22], pattern recognition [9], [10], medical diagnosis [8], [16], and cluster analysis [17], [18].

The prominent characteristic of intuitionistic fuzzy set is that it assigns to each element a membership degree and a nonmembership degree. Although the intuitionistic fuzzy set is a powerful tool to deal with uncertainty, it still has inherent drawbacks. Here, we can give voting as an example. The voters may be divided into four groups of those who: vote for, abstain, vote against, refusal of the voting. Nevertheless, the IFS only care of those who vote for or vote against, and consider those who abstain and refusal are equivalent.

In fact, it is more reasonable to consider those who abstain as they vote for and vote against at the same degree. When some one refuse to vote, we can interpret that he is not concerned about the current election. Following this way, Cuong [5], [6] generalized IFS to picture fuzzy sets (PFSs). As a result, models which base on picture fuzzy set may be adequate in situations when we face human opinions involving more answers of the type: yes, abstain, no, refusal.

In data analysis, information may be gathered in the form of PFSs rather than IFSs.

In order to deal with these information, we need to develop new methods. Singh [15] proposed the weighted correlation coefficient to calculate the degree of correlation between the picture fuzzy sets aiming at clustering different objects.

However, aggregation operators are more commonly used in multi-attribute decision making problems. As a new generation of IFS, PFS still has not any aggregation operators yet.

Thus, in this paper, we aim to introduce new operations on picture fuzzy numbers and develop some new aggregation operators on PFSs.

To facilitate our discussion, the remainder of this paper is organized as follows.

In the next section, we review some basic concepts related to picture fuzzy set. New operations on PFSs are studied in Section 3.

In Section 4, some picture fuzzy geometric operators are introduced. Some properties of these new operators are also investigated.

In Section 5, we develop a method for multiple attribute decision making based on new operators under picture fuzzy environment.

An illustrative example is also given to show the effectiveness of the developed approach in Section 6.

In Section 7, we give the conclusion and some remarks.

2. Preliminaries

Let us first review some basic concepts related to PFS.
FS was first proposed by Zadeh [24] in 1965 as the following.

Definition 2.1 [24]. Let X be an universe of discourse, then a fuzzy set is defined as: $A = \{\langle x, \mu_A(x)\rangle | x \in X\}$ which is characterized by a membership function $\mu_A : X \to [0,1]$, where μ_A denotes the degree of membership of the element x to the set A.

Atanassov [1] generalized FS to IFS as below.

Definition 2.2 [1]. An IFS in X is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$$

which is characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$, with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X,$$

where the numbers $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and the degree of non-membership of the element x to the set A, respectively.

If $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) = 0, \forall x \in X$, then the IFS is reduced to a common fuzzy set. To aggregate intuitionistic preference information, Xu [19] defined operations as below.

Definition 2.3 [19] Let $\alpha = (\mu_\alpha, \nu_\alpha)$ and $\beta = (\mu_\beta, \nu_\beta)$ be two intuitionistic fuzzy numbers, then

(1) $\alpha \cdot \beta = (\mu_\alpha \mu_\beta, \nu_\alpha + \nu_\beta - \nu_\alpha \nu_\beta)$;

(2) $\alpha^\lambda = (\mu_\alpha^\lambda, 1 - (1 - \nu_\alpha)^\lambda), \lambda > 0$.

Voting was given as a good example in [7]. Human voters can be divided into four groups: vote for, abstain, vote against, refusal of the voting. In order to describe this situation, Cuong et. al. [5], [6], [7] generalized FS and IFS to the picture fuzzy set as the new concept for computaional intelligence problems.

Definition 2.4 [7]. An picture fuzzy set A on a universe X is an object of the form

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle | x \in X\}$$

where $\mu_A(x)$ $(x \in [0,1])$ is called the *degree of positive membership of x in A*, $\eta_A(x)$ $(x \in [0,1])$ is called the *degree of neutral membership of x in A* and $\nu_A(x)$ $(x \in [0,1])$ is called the *degree of negative membership of x in A*. Besides, μ_A, η_A, ν_A satisfy the following condition

$$\forall x \in X, \quad \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1.$$

Then, for $x \in X$, $\rho_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ could be called the *degree of refusal membership of x in A*.

When $\eta_A(x) = 0, \forall x \in X$, the PFS is reduced to IFS. In the voting, those who are abstain can be interpreted as: on one hand, they vote for; on the other hand, they vote against. Meanwhile, those who are refusal of the voting can be explained as they are not care about this voting.

For convenience, we call $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \rho_\alpha)$ a picture fuzzy number (PFN), where $\mu_\alpha \in [0,1]$, $\eta_\alpha \in [0,1]$, $\nu_\alpha \in [0,1]$, $\rho_\alpha \in [0,1]$ and $\mu_\alpha + \eta_\alpha + \nu_\alpha + \rho_\alpha = 1$. Sometimes, we omit ρ_α and denote a PFN as $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha)$ for short.

For every two PFSs A and B, Cuong et. al. [7] also defined some operations as following.

(1) $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\eta_A(x) \le \eta_B(x)$ and $\nu_A(x) \ge \nu_B(x), \forall x \in X$;

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(3) $A \cup B = \{(x, max(\mu_A(x), \mu_B(x)), min(\eta_A(x), \eta_B(x)), min(\nu_A(x), \nu_B(x))) | x \in X\}$;

(4) $A \cap B = \{(x, min(\mu_A(x), \mu_B(x)), min(\eta_A(x), \eta_B(x)), max(\nu_A(x), \nu_B(x))) | x \in X\}$;

(5) $coA = \bar{A} = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) | x \in X\}$.

Several properties of these operations were also discussed in [5]:

(1) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$;

(2) $\bar{\bar{A}} = A$;

(3) Operations $\cap$ and $\cup$ are commutative, associative and distributive;

(4) Operations $\cap$, co and $\cup$ satisfy the De Morgan law.

3. New operations on PFSs

To aggregate picture fuzzy preference information, some new operators will be introduced in this section. In order to generate Definition 2.3 to PFNs, let us first explain it from a view point of probability. We also choose voting as a realistic example. For an IFN $\alpha = (\mu_\alpha, \nu_\alpha)$, it represents that the ratio who vote for α is μ_α and the ratio who vote against α is ν_α in a decision conference. When we consider both α and β, by the multiplication formula of probability $P(A \cap B) = P(A)P(B)$, we can get the ratio who vote for both α and β as $\mu_{\alpha\beta} = \mu_\alpha \cdot \mu_\beta$. Then, by the complementary formula of probability and $A \cap B = \overline{\bar{A} \cup \bar{B}}$, we can get the ratio who vote against α or vote against β as $\nu_\alpha + \nu_\beta - \nu_\alpha \nu_\beta = 1 - (1 - \nu_\alpha)(1 - \nu_\beta)$. Next, by explaining

$$\alpha^\lambda = \underbrace{\alpha \cdot \alpha \cdot \cdots \cdot \alpha}_{\lambda},$$

we can obtain (2) in Definition 2.3.

Table 1. Explain Definition 2.3 from the view point of probability (The joint probability)

	vote for α	vote against α
vote for β	$\mu_\alpha\mu_\beta$	$\nu_\alpha\mu_\beta$
vote against β	$\mu_\alpha\nu_\beta$	$\nu_\alpha\nu_\beta$

In PFS theory, voters are divided into four groups: vote for (its ratio is denoted as μ), abstain (its ratio is denoted as η), vote against (its ratio is denoted as ν), refusal (its ratio is denoted as ρ).

In order to combine two PFNs $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \rho_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \nu_\beta, \rho_\beta)$, we can construct the joint probability as Table 2.

Table 2. Generate Definition 2.3 from the view point of probability (The joint probability)

	vote for α	abstain for α	vote against α	refusal for α
vote for β	$\mu_\alpha\mu_\beta$	$\eta_\alpha\mu_\beta$	$\nu_\alpha\mu_\beta$	$\rho_\alpha\mu_\beta$
abstain for β	$\mu_\alpha\eta_\beta$	$\eta_\alpha\eta_\beta$	$\nu_\alpha\eta_\beta$	$\rho_\alpha\eta_\beta$
vote against β	$\mu_\alpha\nu_\beta$	$\eta_\alpha\nu_\beta$	$\nu_\alpha\nu_\beta$	$\rho_\alpha\nu_\beta$
refusal for β	$\mu_\alpha\rho_\beta$	$\eta_\alpha\rho_\beta$	$\nu_\alpha\rho_\beta$	$\rho_\alpha\rho_\beta$

To compute $\mu_{\alpha\cdot\beta}$, we come to choose those who vote for both α and β, then $\mu_{\alpha\cdot\beta} = \mu_\alpha\mu_\beta + \eta_\alpha\mu_\beta + \mu_\alpha\eta_\beta = (\mu_\alpha + \eta_\alpha)(\mu_\beta + \eta_\beta) - \eta_\alpha\eta_\beta$. Similarly, those who are abstain for α and abstain for β can be viewed as abstain for α and β. That is $\eta_{\alpha\cdot\beta} = \eta_\alpha\eta_\beta$. We can also choose those who vote against α or vote against β as $\nu_{\alpha\cdot\beta} = \nu_\alpha\eta_\beta + \nu_\alpha\mu_\beta + \eta_\alpha\nu_\beta + \mu_\alpha\nu_\beta + \nu_\alpha\nu_\beta + \rho_\alpha\nu_\beta + \nu_\alpha\rho_\beta = 1 - (1-\nu_\alpha)(1-\nu_\beta)$. The rest products in Table 2 are considered as $\rho_{\alpha\cdot\beta} : \rho_{\alpha\cdot\beta} = \mu_\alpha\rho_\beta + \eta_\alpha\rho_\beta + \rho_\alpha\mu_\beta + \rho_\alpha\eta_\beta + \rho_\alpha\rho_\beta$. Also by explaining $\alpha^\lambda = \underbrace{\alpha \cdot \alpha \cdot \cdots \cdot \alpha}_{\lambda}$, we can obtain that

$$\alpha^\lambda = ((\mu_\alpha + \eta_\alpha)^\lambda - \eta_\alpha^\lambda, \eta_\alpha^\lambda, 1 - (1 - \nu_\alpha)^\lambda).$$

After simplify these expressions, we can introduce the following definition.

Definition 3.1 Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \nu_\beta)$ be two picture fuzzy numbers, then

(1) $\alpha \cdot \beta = ((\mu_\alpha + \eta_\alpha)(\mu_\beta + \eta_\beta) - \eta_\alpha\eta_\beta, \eta_\alpha\eta_\beta, 1 - (1 - \nu_\alpha)(1 - \nu_\beta))$;

(2) $\alpha^\lambda = ((\mu_\alpha + \eta_\alpha)^\lambda - \eta_\alpha^\lambda, \eta_\alpha^\lambda, 1 - (1 - \nu_\alpha)^\lambda), \lambda > 0$.

When $\eta_\alpha = \eta_\beta = 0$, the above definition is reduced to Definition 2.3.

Based on Definition 3.1, we can verify the following properties easily.

Theorem 3.2 *Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \rho_\alpha)$, $\beta = (\mu_\beta, \eta_\beta, \nu_\beta, \rho_\beta)$ and $\gamma = (\mu_\gamma, \eta_\gamma, \nu_\gamma, \rho_\gamma)$ be three picture fuzzy numbers, then*

(1) $\alpha \cdot \beta = \beta \cdot \alpha$;

(2) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$;

(3) $\alpha^{\lambda_1+\lambda_2} = \alpha^{\lambda_1} \cdot \alpha^{\lambda_2}$;

(4) $(\alpha \cdot \beta)^\lambda = \alpha^\lambda \cdot \beta^\lambda$.

In order to compare two PFNs, we introduce the following comparison laws.

Definition 3.3 Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \rho_\alpha)$ be a picture fuzzy numbers, then a score function S can be defined as $S(\alpha) = \mu_\alpha - \nu_\alpha$ and the accuracy function H is given by $H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha$, where $S(\alpha) \in [-1, 1]$ and $H(\alpha) \in [0, 1]$. Then, for two picture fuzzy numbers α and β

(i) if $S(\alpha) > S(\beta)$, then α is superior to β, denoted by $\alpha \succ \beta$;

(ii) if $S(\alpha) = S(\beta)$, then

(1) $H(\alpha) = H(\beta)$, implies that α is equivalent to β, denoted by $\alpha \sim \beta$;

(2) $H(\alpha) > H(\beta)$, implies that α is superior to β, denoted by $\alpha \succ \beta$.

We also use voting as a good example to explain the above definition, where $S(\alpha) = \mu_\alpha - \nu_\alpha$ represents goal difference and $H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha$ can be interpreted as the effective degree of voting. When $S(\alpha)$ increases, we can know that there are more people who vote for α and people who vote against α become less. When $H(\alpha)$ increases, we can know that there are more people who vote for or against α and people who refuse to vote become less. So, $H(\alpha)$ depicts the effective degree of voting.

4. Picture fuzzy geometric operators

In this section, we will define some picture fuzzy geometric aggregation operators based on the geometric mean.

Definition 4.1 Let p_j $(j = 1, 2, ..., n)$ be a collection of PFNs, then we define the picture fuzzy weighted geometric (PFWG) operator as below:

$$PFWG_w(p_1, p_2, ..., p_n) = \prod_{j=1}^{n} p_j^{w_j}$$

where $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of p_j $(j = 1, 2, ..., n)$, and $w_j > 0$, $\sum_{j=1}^n w_j = 1$.

According to the operational laws of PFNs, we can get the following theorem.

Theorem 4.2 *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFWG operator is also a PFNs, and*

$$PFWG_w(p_1, p_2, \cdots, p_n) = (\prod_{i=1}^{n}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i}, \prod_{i=1}^{n}\eta_i^{w_i}, 1 - \prod_{i=1}^{n}(1-\nu_i)^{w_i}).$$

Proof. By using mathematics induction on n, we prove Theorem 4.2 as follows.

(1) For $n = 2$:

With the operational laws of PFNs, we can get

$$p_1^{w_1} = (\mu_1 + \eta_1)^{w_1} - \eta_1^{w_1}, \eta_1^{w_1}, 1 - (1-\nu_1)^{w_1}),$$
$$p_2^{w_2} = (\mu_2 + \eta_2)^{w_2} - \eta_2^{w_2}, \eta_2^{w_2}, 1 - (1-\nu_2)^{w_2}).$$

Then, it follows that

$$p_1^{w_1} \cdot p_2^{w_2} = (\mu_1 + \eta_1)^{w_1}(\mu_2 + \eta_2)^{w_2} - \eta_1^{w_1}\eta_2^{w_2}, \eta_1^{w_1}\eta_2^{w_2}, 1 - (1-\nu_1)^{w_1}(1-\nu_2)^{w_2}).$$

Thus, Theorem 4.2 holds.

(2) If Theorem 4.2 holds for $n = k$, that is,

$$PFWG_w(p_1, p_2, \cdots, p_k) = (\prod_{i=1}^{k}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{k}\eta_i^{w_i}, \prod_{i=1}^{k}\eta_i^{w_i}, 1 - \prod_{i=1}^{k}(1-\nu_i)^{w_i}),$$

then, when $n = k + 1$, by the operational laws in Theorem 3.2, we have

$$\begin{aligned}
\prod_{i=1}^{k+1} p_i^{w_i} &= \prod_{i=1}^{k} p_i^{w_i} \cdot p_{k+1}^{w_{k+1}} \\
&= \left(\prod_{i=1}^{k}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{k}\eta_i^{w_i}, \prod_{i=1}^{k}\eta_i^{w_i}, 1 - \prod_{i=1}^{k}(1-\nu_i)^{w_i}\right) \\
&\cdot((\mu_{k+1} + \eta_{k+1})^{w_{k+1}} - \eta_{k+1}^{w_{k+1}}, \eta_{k+1}^{w_{k+1}}, 1 - (1-\nu_{k+1})^{w_{k+1}}) \\
&= \prod_{i=1}^{k+1}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{k+1}\eta_i^{w_i}, \prod_{i=1}^{k+1}\eta_i^{w_i}, 1 - \prod_{i=1}^{k+1}(1-\nu_i)^{w_i}),
\end{aligned}$$

i.e., Theorem 4.2 holds for $n = k + 1$. Thus, by the principle of mathematical induction Theorem 4.2 holds for all n. Obviously,

$$\prod_{i=1}^{n}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i}, \prod_{i=1}^{n}\eta_i^{w_i}, 1 - \prod_{i=1}^{n}(1-\nu_i)^{w_i} \in [0, 1],$$
$$\left(\prod_{i=1}^{n}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i}\right) + \prod_{i=1}^{n}\eta_i^{w_i} + \left(1 - \prod_{i=1}^{n}(1-\nu_i)^{w_i}\right) \leq 1,$$

and the result of $PFWG_w(p_1, p_2, \cdots, p_n)$ is also a PFN.

Theorem 4.3 (Idempotency). *Let $p_j = (\mu_j, \eta_j, \nu_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs. If $p_1 = p_2 = \cdots = p_n = p$, then*

$$PFWG_w(p_1, p_2, ..., p_n) = p.$$

Proof. Let $p_1 = p_2 = \cdots = p_n = p = (\mu, \eta, \nu)$. By Theorem 4.2, we obtain

$$\begin{aligned} PFWG_w(p_1, p_2, \cdots, p_n) &= \left(\prod_{i=1}^{n}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i}, \prod_{i=1}^{n}\eta_i^{w_i}, 1 - \prod_{i=1}^{n}(1-\nu_i)^{w_i}\right) \\ &= \left(\prod_{i=1}^{n}(\mu + \eta)^{w_i} - \prod_{i=1}^{n}\eta^{w_i}, \prod_{i=1}^{n}\eta^{w_i}, 1 - \prod_{i=1}^{n}(1-\nu)^{w_i}\right) \\ &= ((\mu+\eta)^{\sum_{i=1}^{n} w_i} - \eta^{\sum_{i=1}^{n} w_i}, \eta^{\sum_{i=1}^{n} w_i}, 1 - (1-\nu)^{\sum_{i=1}^{n} w_i}). \end{aligned}$$

As $\sum_{i=1}^{n} w_i = 1$, we have

$$\widetilde{p} = PFWG_w(p_1, p_2, \cdots, p_n) = (\mu, \eta, \nu),$$

which completes the proof.

Theorem 4.4 (Boundedness). *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs. If $\eta_* = min_j\{\eta_j\}$, $\nu_* = min_j\{\nu_j\}$, $\rho_* = min_j\{\rho_j\}$, $\mu_* = 1 - \eta_* - \nu_* - \rho_*$, $p^* = (\mu_*, \eta_*, \nu_*, \rho_*)$ and $\eta^* = max_j\{\eta_j\}$, $\nu^* = max_j\{\nu_j\}$, $\rho^* = max_j\{\rho_j\}$, $\mu^* = 1 - \eta^* - \nu^* - \rho^*$, $p_* = (\mu^*, \eta^*, \nu^*, \rho^*)$ then*

$$p_* \le PFWG_w(p_1, p_2, \cdots, p_n) \le p^*.$$

Proof. By Theorem 4.2, we get

$$PFWG_w(p_1, p_2, \cdots, p_n) = \left(\prod_{i=1}^{n}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i}, \prod_{i=1}^{n}\eta_i^{w_i}, 1 - \prod_{i=1}^{n}(1-\nu_i)^{w_i}\right).$$

From the definition of p_*, p^*, we know

$$(\mu_i + \eta_i)^{w_i} = (1 - \nu_i - \rho_i)^{w_i} \le 1 - \nu_* - \rho_*$$

$$(\mu_i + \eta_i)^{w_i} = (1 - \nu_i - \rho_i)^{w_i} \ge 1 - \nu^* - \rho^*$$

and

$$\eta_* \le \eta \le \eta^*$$

so

$$\prod_{i=1}^{n}(1-\nu^*-\rho^*)^{w_i} - \prod_{i=1}^{n}\eta^{*w_i} \le \prod_{i=1}^{n}(\mu_i+\eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i} \le \prod_{i=1}^{n}(1-\nu_*-\rho_*)^{w_i} - \prod_{i=1}^{n}\eta_*^{w_i}.$$

Using the condition $\sum_{j=1}^{n} w_j = 1$, we acquire

$$\mu^* = 1 - \nu^* - \rho^* - \eta^* \le \prod_{i=1}^{n}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i} \le 1 - \nu_* - \rho_* - \eta_* = \mu_*.$$

Similarly, we obtain

$$\eta_* \leq \prod_{i=1}^{k} \eta_i^{w_i} \leq \eta^*$$

$$\nu_* \leq 1 - \prod_{i=1}^{n} (1-\nu_i)^{w_i} \leq \nu^*.$$

As

$$S(PFWG_w(p_1, p_2, \cdots, p_n)) = \left(\prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i} \right) - (1 - \prod_{i=1}^{n} (1-\nu_i)^{w_i})$$

we get

$$(1 - \eta^* - \nu^* - \rho^*) - \nu^* \leq S(PFWG_w(p_1, p_2, \cdots, p_n)) \leq (1 - \eta_* - \nu_* - \rho_*) - \nu_*.$$

In other words,

$$p_* \leq PFWG_w(p_1, p_2, \cdots, p_n) \leq p^*$$

which completes the proof.

Theorem 4.5 (Monotonicity). *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ and $p_j' = (\mu_j', \eta_j', \nu_j', \rho_j')$ $(j = 1, 2, ..., n)$ be two collections of PFNs. If $\eta_j \leq \eta_j'$, $\nu_j \leq \nu_j'$, $\rho_j \leq \rho_j'$, $\forall 1 \leq j \leq n$, then*

$$PFWG_w(p_1, p_2, ..., p_n) \geq PFWG_w(p_1', p_2', ..., p_n').$$

Proof. Since $\eta_j \leq \eta_j'$, $\nu_j \leq \nu_j'$, $\rho_j \leq \rho_j'$, $\forall 1 \leq j \leq n$, we obtain that

$$\begin{aligned} \prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i} &= \prod_{i=1}^{n} (1 - \nu_i - \rho_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i} \\ &\geq \prod_{i=1}^{n} (1 - \nu_i' - \rho_i')^{w_i} - \prod_{i=1}^{n} \eta_i^{'w_i} = \prod_{i=1}^{n} (\mu_i' + \eta_i')^{w_i} - \prod_{i=1}^{n} \eta_i^{'w_i}. \end{aligned}$$

Apparently

$$1 - \prod_{i=1}^{n} (1-\nu_i)^{w_i} \leq 1 - \prod_{i=1}^{n} (1-\nu_i')^{w_i}.$$

Following this way, we have

$$S(PFWG_w(p_1, p_2, ..., p_n)) \geq S(PFWG_w(p_1', p_2', ..., p_n')).$$

Then, we complete the proof.

When we need to weight the ordered positions of the picture fuzzy arguments instead of weighting the arguments themselves, PFWG can be generalized to PFOWG.

Definition 4.6 . Let p_j $(j = 1, 2, ..., n)$ be a collection of PFNs, then we define the picture fuzzy ordered weighted geometric (PFOWG) operator as below:

$$PFOWG_w(p_1, p_2, ..., p_n) = \prod_{j=1}^{n} p_{\sigma(j)}^{w_j}$$

where $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of p_j $(j = 1, 2, ..., n)$, and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$.

According to the operational laws of PFNs, we can get the following theorems. As their proofs are similar to the ones listed above, we omit them here.

Theorem 4.7 *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFWG operator is also a PFNs, and*

$$\begin{aligned} &PFOWG_w(p_1, p_2, ..., p_n) \\ &= \left(\prod_{i=1}^{n} (\mu_{\sigma(i)} + \eta_{\sigma(i)})^{w_i} - \prod_{i=1}^{n} \eta_{\sigma(i)}^{w_i}, \prod_{i=1}^{n} \eta_{\sigma(i)}^{w_i}, 1 - \prod_{i=1}^{n} (1 - \nu_{\sigma(i)})^{w_i} \right). \end{aligned}$$

Theorem 4.8 (Idempotency). *Let p_j $(j = 1, 2, ..., n)$ be a collection of PFNs. If $p_1 = p_2 = \cdots = p_n = p$, then*

$$PFOWG_w(p_1, p_2, ..., p_n) = p.$$

Theorem 4.9 (Boundedness). *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs. If $\eta_* = min_j\{\eta_j\}$, $\nu_* = min_j\{\nu_j\}$, $\rho_* = min_j\{\rho_j\}$, $\mu_* = 1 - \eta_* - \nu_* - \rho_*$, $p^* = (\mu_*, \eta_*, \nu_*, \rho_*)$ and $\eta^* = max_j\{\eta_j\}$, $\nu^* = max_j\{\nu_j\}$, $\rho^* = max_j\{\rho_j\}$, $\mu^* = 1 - \eta^* - \nu^* - \rho^*$, $p_* = (\mu^*, \eta^*, \nu^*, \rho^*)$ then*

$$p_* \leq PFOWG_w(p_1, p_2, ..., p_n) \leq p^*.$$

Theorem 4.10 (Monotonicity). *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ and $p'_j = (\mu'_j, \eta'_j, \nu'_j, \rho'_j)$ $(j = 1, 2, ..., n)$ be two collections of PFNs. If $\eta_j \leq \eta'_j$, $\nu_j \leq \nu'_j$, $\rho_j \leq \rho'_j$, $\forall 1 \leq j \leq n$, then*

$$PFWOG_w(p_1, p_2, ..., p_n) \geq PFOWG_w(p'_1, p'_2, ..., p'_n).$$

Theorem 4.11 (Commutativity) *Let p_j $(j = 1, 2, ..., n)$ then*

$$PFOWG_w(p_{\sigma(1)}, p_{\sigma(2)}, ..., p_{\sigma(n)}) = PFOWG_w(p_1, p_2, ..., p_n)$$

where $(\sigma(1), \sigma(2), ..., \sigma(n))$ is any permutation of $(1, 2, ..., n)$.

When we need to weight both the ordered positions of the picture fuzzy arguments and the arguments themselves, PFWG can be generalized to the following picture fuzzy hybrid geometric operator.

Definition 4.12 . Let p_j $(j = 1, 2, ..., n)$ be a collection of PFNs, then we define the picture fuzzy hybrid geometric (PFHG) operator as below:

$$PFHG_w(p_1, p_2, ..., p_n) = \prod_{j=1}^{n} \widetilde{p}_{\sigma(j)}^{w_j}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the associated weighting vector of p_j $(j = 1, 2, ..., n)$ with $w_j > 0, \sum_{j=1}^{n} w_j = 1$ and $p_{\sigma(j)}$ is the j-th largest element of the picture fuzzy aruguments ($\widetilde{p}_j = p_j^{n\omega_j}, \omega_j$ is the weighting vector of picture fuzzy arguments p_j with $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$ and n is the balancing coefficient).

Based on the operations of the PFNs, we can drive the following theorem which is similar to 4.2.

Theorem 4.13 *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFWG operator is also a PFNs, and*

$$\begin{aligned} &PFHG_w(p_1, p_2, ..., p_n) \\ &\quad= \left(\prod_{i=1}^{n} (\widetilde{\mu}_{\sigma(i)} + \widetilde{\eta}_{\sigma(i)})^{w_i} - \prod_{i=1}^{n} \widetilde{\eta}_{\sigma(i)}^{w_i}, \prod_{i=1}^{n} \widetilde{\eta}_{\sigma(i)}^{w_i}, 1 - \prod_{i=1}^{n} (1 - \widetilde{\nu}_{\sigma(i)})^{w_i} \right). \end{aligned}$$

5. An approach to multiple attribute decision making with picture fuzzy information

In this section, we shall utilize the proposed operators to multiple attribute decision making under picture fuzzy environment. As their procedures are similar, we only consider the PFWG operator here.

The following assumptions or notations are used to represent the MADM problems for evaluation of alternatives with picture fuzzy information. Let $A = \{A_1, A_2, ..., A_m\}$ be a set of m alternatives and $G = \{G_1, G_2, ..., G_n\}$ be a set of n attributes. If the decision makers provide values for the alternative A_i under the attribute G_j with anonymity, these values can be considered as a picture fuzzy element p_{ij}. Suppose that the decision matrix $P = (p_{ij})_{m \times n}$ is the picture fuzzy decision matrix, where p_{ij} $(i = 1, 2, ..., m,\ j = 1, 2, ..., n)$ are in the form of PFNs. In the following, we apply the PFWG operator to the MADM problems for evaluation of alternatives with picture fuzzy information.

Step 1. We utilize the decision information given in matrix P, and the PFWG operator

$$\begin{aligned} \widetilde{p}_i &= PFWG_w(p_{i1}, p_{i2}, ..., p_{in}) = \prod_{j=1}^{n} p_{ij}^{w_j} \\ &= \left(\prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i}, \prod_{i=1}^{n} \eta_i^{w_i}, 1 - \prod_{i=1}^{n} (1 - \nu_i)^{w_i} \right) \end{aligned}$$

to derive the overall preference values $\widetilde{p}_i$ $(i = 1, 2, ..., m)$ of the alternative A_i.

Step 2. Calculate the scores $S(\widetilde{p}_i)$ $(i = 1, 2, ..., m)$ of the overall picture fuzzy values $\widetilde{p}_i$ by Definition 3.3.

Step 3. Rank all the alternatives A_i $(i = 1, 2, ..., m)$ in accordance with the values of $S(\widetilde{p}_i)$ $(i = 1, 2, ..., m)$ and select the best one(s).

Step 4. End.

6. Numerical example

In this section, we will present a numerical example (adapted from [15]) to show evaluation of theses with picture fuzzy information in order to illustrate the proposed method.

Suppose there are five theses A_i $(i = 1, 2, ..., 5)$, and we want to select the best one. Four attributes are selected by experts to evaluate the theses: (1) G_1 is the language of a thesis; (2) G_2 is the innovation; (3) G_3 is the rigor; (4) G_4 is the structure of the thesis. In order to avoid influence each other, the experts are required to evaluate the five theses A_i $(i = 1, 2, ..., 5)$ under the above four attributes in anonymity. The decision matrix $P = (p_{ij})_{5\times4}$ is presented in Table 3, where p_{ij} $(i = 1, 2, 3, 4, 5,\ j = 1, 2, 3, 4)$ are in the form of PFNs.

Table 3. Picture fuzzy decision matrix P

	G_1	G_2	G_3	G_4
A_1	(0.2,0.3,0.1,0.4)	(0.7,0.1,0.1,0.1)	(0.1,0.2,0.6,0.1)	(0.4,0.1,0.2,0.3)
A_2	(0.4,0.2,0.3,0.1)	(0.1,0.6,0.1,0.2)	(0.3,0.2,0.4,0.1)	(0.3,0.1,0.4,0.2)
A_3	(0.2,0.5,0.1,0.2)	(0.6,0.1,0.1,0.2)	(0.5,0.1,0.2,0.2)	(0.5,0.1,0.3,0.1)
A_4	(0.2,0.3,0.1,0.4)	(0.6,0.2,0.1,0.1)	(0.5,0.3,0.2,0)	(0.5,0,0.3,0.2)
A_5	(0.6,0.1,0.2,0.1)	(0.4,0.2,0.3,0.1)	(0.6,0.1,0.2,0.1)	(0.3,0.4,0.2,0.1)

The information about the attribute weights is known as: $w = (0.2, 0.4, 0.1, 0.3)$.

6.1. The decision making steps

Now, we apply the developed approach to evaluate these theses with picture fuzzy information.

Step 1. Utilize the decision information given in matrix P and

$$\widetilde{p}_i = PFWG_w(p_{i1}, p_{i2}, ..., p_{in}),$$

we have

$$\begin{aligned}
\widetilde{p}_1 &= (0.439853, 0.133514, 0.198915),\\
\widetilde{p}_2 &= (0.302763, 0.252098, 0.272259),\\
\widetilde{p}_3 &= (0.520169, 0.137973, 0.175133),\\
\widetilde{p}_4 &= (0.632456, 0, 0.175133),\\
\widetilde{p}_5 &= (0.458142, 0.2, 0.241609).
\end{aligned}$$

Step 2. Calculate the scores $S(\widetilde{p}_i)$ $(i = 1, 2, 3, 4)$ of the overall picture fuzzy preference values $\widetilde{p}_i$ by Definition 3.3:

$$\begin{aligned} S(\widetilde{p}_1) &= 0.240938, \\ S(\widetilde{p}_2) &= 0.0305047, \\ S(\widetilde{p}_3) &= 0.345036, \\ S(\widetilde{p}_4) &= 0.457323, \\ S(\widetilde{p}_5) &= 0.216533. \end{aligned}$$

Step 3. Rank all the alternatives $A_i (i = 1, 2, ..., 4)$ in accordance with the values of $S(\widetilde{p}_i)$: $A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$. Note that $\succ$ means "preferred to". Thus, the best thesis is A_4.

6.2. Comparative analysis

The proposed method has several advantages as below.

First, our method can accommodate situations in which the input arguments are picture fuzzy numbers. As mentioned before, picture fuzzy set is a generalized set containing FS and IFS as its special cases. Thus, our method can be widely used.

Second, we can compare our method with IFWG [23]. Here, we have to translate data in Table 3 into intuitionistic fuzzy numbers (IFNs). For example, the fist PFN $(0.2, 0.3, 0.1, 0.4)$ should be changed into IFN $(0.2, 0.1)$. Then, we omit the process of calculation and list the results in Table 4.

Table 4. Scores for alternatives obtained by $IFWG$ operator

	The overall intuitionistic fuzzy values	Scores	Order
A_1	(0.379196, 0.198915)	0.18028	3
A_2	(0.204767, 0.272259)	-0.0674913	5
A_3	(0.447769, 0.175133)	0.272636	1
A_4	(0.447769, 0.175133)	0.272636	1
A_5	(0.414387, 0.241609)	0.172778	4

Obviously, A_3 and A_4 can not be distinguished by $IFWG$ while they can be distinguished by $PFWG$. This indicates that the picture fuzzy set takes much more information and our method is meaningful.

7. Conclusion

In this paper, we have investigated the multiple attribute decision making (MADM) problems based on the PFWG, PFOWG and PFHG operators with picture fuzzy information. Firstly, some basic concepts related to picture fuzzy set have been reviewed. Then, from the view of probability, some new operations on picture fuzzy sets have been developed. At the same time we have discussed their basic properties. Furthermore, we have discussed the picture fuzzy geometric operators and applied these new picture fuzzy operators to multiple attribute decision making problems in which attribute values take the form of picture fuzzy information. Finally, an illustrative example for evaluation of theses has been given to demonstrate the validity and applicability of the new approach. There are some other generalizations of these basic aggregation operators such as generalized weighted aggregation operator and Bonferroni mean, which can also be used to construct new operators for PFSs. The researches about these new operators may be interesting and meaningful.

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