# A NOTE ON EXTRINSIC FRAME HOMOGENEITY OF HYPERQUADRICS

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**Abstract.** Let $p_i$ and $q_i$ belong to a hyperquadric $Q$ and $(e_{1_i}, \dots, e_{n_i})$ and $(f_{1_i}, \dots, f_{n_i})$ be orthonormal frames in $T_{p_i}Q$ and $T_{q_i}Q$, respectively, where $1 \leqslant i \leqslant m$. We study sufficient and necessary conditions for existence of an isometry $\varphi : \mathbb{R}^{n+1}_\nu \to \mathbb{R}^{n+1}_\nu$ such that $\varphi(Q) \subset Q$, $\varphi(p_i) = q_i$ and $d\varphi(e_{j_i}) = f_{j_i}$.

**Keywords:** hyperquadric, frame homogeneous.
**2010 MSC:** 53A35, 53B99.

## 1. Introduction

A semi-Riemannian manifold $M$ is said to be frame-homogeneous provided any frame on $M$ can be carried to any other by the differential map of an isometry of $M$. If $M$ is a connected frame-homogeneous Riemannian manifold with, $\dim M \geqslant 2$, then it is homothetic to $S^n$, $\mathbb{P}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$ (see [1, p.259]). For indefinite metrics, the list is longer (see [2]).

Let $M$ be a semi-Riemannian manifold and $N$ be a semi-Riemannian hypersurface in $M$. Then $N$ is called extrinsically frame-homogeneous if any frame on $N$ can be carried to any other by the differential map of a pair isometry $\varphi : (M, N) \to (M, N)$. Proposition 4.30 of [1] asserts that hyperquadrics in $\mathbb{R}^n_\nu$ are extrinsically frame-homogeneous. Assume that $Q$ is a hyperquadric in $\mathbb{R}^n_\nu$. Here we consider frames $(e_{1_i}, \dots, e_{n_i})$ and $(f_{1_i}, \dots, f_{n_i})$ in $T_{p_i}Q$ and $T_{q_i}Q$ respectively, where $1 \leqslant i \leqslant m$, and then we study conditions on the points and on the frames which insure the existence of a pair isometry $\varphi : (\mathbb{R}^n_\nu, Q) \to (\mathbb{R}^n_\nu, Q)$ with the further property that its differential map carries each frame $(e_{1_i}, \dots, e_{n_i})$ to $(f_{1_i}, \dots, f_{n_i})$, respectively (see Theorem 3.3). Then we conclude the extrinsic frame-homogeneity of hyperquadrics as a special case of the theorem.

## 2. Preliminaries

Throughout the following $\mathbb{R}^n_\nu$ denotes the $n$-dimensional real vector space $\mathbb{R}^n$ with a scalar product of signature $(\nu, n - \nu)$ given by

$$\langle x, y\rangle = -\sum_{i=1}^{\nu} x_i y_i + \sum_{j=\nu+1}^{n} x_j y_j, \tag{2.1}$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Let $q$ be the associated quadratic form to the above scalar product, i.e. $q(x) = \langle x, x\rangle$. For $\varepsilon \in \{-1, 1\}$ and fixed real number $r > 0$, let $Q = q^{-1}(\varepsilon r)$. Then $Q$ is a semi-Riemannian hypersurface of $\mathbb{R}^n_\nu$ with sign $\varepsilon$. These hypersurfaces are called the hyperquadrics of $\mathbb{R}^n_\nu$. An orthonormal basis for a tangent space $T_xQ$ is called a frame on $Q$ at $x$. For any frame $(e_1, \dots, e_n)$ on $Q$, we assume that $e_1, \dots, e_\nu$ are time-like and $e_{\nu+1}, \dots, e_n$ are space-like.

Let us identify an $n \times n$ real matrix $A$ with the linear operator $A : \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ such that

$$(Ax)_i = \sum_j A_{ij} x_j, \quad 1 \leqslant i \leqslant n.$$

By this identification, composition of functions becomes the matrix multiplication. Hence, the set (group) of all linear isometries $\mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ is the same as the set $O_\nu(n)$ of all matrices $A \in GL(n, \mathbb{R})$ that preserve the scalar product defined in (2.1). Clearly, $O_\nu(n)$ is a closed subgroup of $GL(n, \mathbb{R})$ and so is itself a Lie group. Also if $x \in \mathbb{R}^n_\nu$, the translation $T_x : \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ sending each $v$ to $v + x$ is an isometry. Since $T_x \circ T_y = T_{x+y} = T_y \circ T_x$, the set of all translations of $\mathbb{R}^n_\nu$ is an abelian subgroup of $Iso(\mathbb{R}^n_\nu)$ isomorphic to the additive group $\mathbb{R}^n$. In fact, each isometry of $\mathbb{R}^n_\nu$ has a unique expression as $T_aA$, with $a \in \mathbb{R}^n_\nu$ and $A \in O_\nu(n)$ (see [1, p.240]), where the group multiplication is given by

$$T_aA.T_bB = T_{a+Ab}AB.$$

By this multiplication $\mathbb{R}^n$ is a normal subgroup of $Iso(\mathbb{R}^n_\nu)$. Define the one-to-one and onto function

$$\varphi : O_\nu(n) \times \mathbb{R}^n \to Iso(\mathbb{R}^n_\nu),$$

by $\varphi(A, a) = T_aA$. If we define the multiplication on $O_\nu(n) \times \mathbb{R}^n$ as follows

$$(A, a).(B, b) = (AB, a + Ab),$$

and denote the resulting group by $O_\nu(n) \ltimes \mathbb{R}^n$, then

$$\varphi : O_\nu(n) \ltimes \mathbb{R}^n \to Iso(\mathbb{R}^n_\nu)$$

is an isomorphism between Lie groups. Then the action of each isometry $(A, a)$ on $\mathbb{R}^n_\nu$ is defined as follows

$$(A, a)x = Ax + a, \quad x \in \mathbb{R}^n_\nu.$$

## 3. Main result

The main result of this section is Theorem 3.3. To prove the theorem first we recall the following lemma from [1, p.234].

**Lemma 3.1** *Let* $E = \begin{pmatrix} -I_\nu & 0 \\ 0 & I_{n-\nu} \end{pmatrix}$, *where* $I_\nu$ *denotes the* $\nu \times \nu$ *identity matrix.The following conditions on an* $n \times n$ *real matrix are equivalent.*

(1) $A \in O_\nu(n)$.
(2) ] $A^{tr} = EA^{-1}E$.
(3) *The columns (rows) of* $A$ *form an orthonormal basis for* $\mathbb{R}^n_\nu$ *(first* $\nu$ *vectors timelike).*
(4) $A$ *carries one (hence every) orthonormal basis for* $\mathbb{R}^n_\nu$ *to an orthonormal basis.*

**Lemma 3.2** *Let* $Q = q^{-1}(\varepsilon r^2)$, *where* $r$ *is a fixed positive real number and* $\varepsilon \in \{-1, 1\}$. *If* $\varphi : \mathbb{R}^{n+r}_\nu \to \mathbb{R}^{n+1}_\nu$ *is an isometry such that* $\varphi(Q) \subset Q$, *then* $\varphi$ *is a linear isometry.*

**Proof.** Let $\varphi : \mathbb{R}^{n+1}_\nu \to \mathbb{R}^{n+1}_\nu$ be an isometry such that $\varphi(Q) \subset Q$. By the fact that $Iso(\mathbb{R}^{n+1}_\nu) = O_\nu(n+1) \times \mathbb{R}^{n+1}$, there exists $A \in O_\nu(n+1)$ and $a \in \mathbb{R}^{n+1}$ such that $\varphi = (A, a)$. We need only to show that $a = 0$. For arbitrary $x \in Q$;

$$q(Ax + a) = q(x). \tag{3.1}$$

Since $A \in Iso(\mathbb{R}^{n+1}_\nu)$, so $A(Q) = Q$. Hence (3.1) implies that

$$\langle y, a \rangle = \frac{-1}{2} q(a); \ \forall y \in Q,$$

which is obviously impossible unless $a = 0$. ∎

**Theorem 3.3** *Let* $p_i$ *and* $q_i$ *belong to* $Q = q^{-1}(\varepsilon r^2)$, *where* $1 \le i \le m$ *and* $\varepsilon \in \{-1, 1\}$. *Let* $(e_{1_i}, \ldots, e_{n_i})$ *and* $(f_{1_i}, \ldots, f_{n_i})$ *be orthonormal frames in* $T_{p_i}Q$ *and* $T_{q_i}Q$, *respectively. Then there is an isometry* $\varphi : \mathbb{R}^{n+1}_\nu \to \mathbb{R}^{n+1}_\nu$ *such that*

$$(*) \qquad \varphi(Q) \subseteq Q, \quad \varphi(p_i) = q_i, \quad \text{and} \quad d\varphi(e_{j_i}) = f_{j_i},$$

*if and only if*

$$(**) \qquad \langle \widetilde{e}_{j_i}, \widetilde{e}_{k_i} \rangle = \langle \widetilde{f}_{j_i}, \widetilde{f}_{k_i} \rangle, \ \langle p_i, p_l \rangle = \langle q_i, q_l \rangle, \ \langle \widetilde{e}_{j_l}, p_i \rangle = \langle \widetilde{f}_{j_l}, q_i \rangle,$$

*where* $1 \leqslant i, l \leqslant m$, $1 \leqslant j, k \leqslant n$ *and* $\widetilde{e}_{j_i}$ *and* $\widetilde{f}_{j_l}$ *denote the elements of* $\mathbb{R}^{n+1}_\nu$ *canonically corresponding to* $e_{j_i}$ *and* $f_{j_i}$, *respectively.*

**Proof.** First assume that $\varepsilon = +1$. To simplify the notation, let $\widetilde{e}_{(n+1)_i} = \frac{p_i}{r}$ and $\widetilde{f}_{(n+1)_i} = \frac{q_i}{r}$, where $1 \le i \le m$. Let $A_i = [\widetilde{e}_{1_i}, \cdots, \widetilde{e}_{(n+1)_i}]$ and $B_i = [\widetilde{f}_{1_i}, \cdots, \widetilde{f}_{(n+1)_i}]$ be two matrices with columns $\widetilde{e}_{j_i}$ and $\widetilde{f}_{j_i}$. Since the columns of $A_i$ and $B_i$ are orthonormal, so they belong to $O_\nu(n+1)$. Hence, by Lemma 3.2, the statement $(*)$ is equivalent to the following;
There is $A \in O_\nu(n+1)$ such that

$$(*') \qquad AA_i = B_i \, , \quad 1 \le i \le m.$$

And this holds if and only if

$$A_iB_i^{-1} = A_lB_l^{-1}\ , \quad 1 \leq i, l \leq m,$$

equivalently

$$A_l^{-1}A_i = B_l^{-1}B_i\ , \quad 1 \leq i, l \leq m. \tag{3.2}$$

Let $E = \begin{pmatrix} -I_\nu & 0 \\ 0 & I_{n+1-\nu} \end{pmatrix}$, where $I_\nu$ denotes the $\nu \times \nu$ identity matrix. Then, by Lemma 3.1, one gets that $A_i^{-1} = EA_i^{tr}E$. Hence (3.2) holds if and only if

$$A_l^{tr}EA_i = B_l^{tr}EB_i, \quad 1 \leq i, l \leq m,$$

which is equivalent to $(**)$.

In the case that $\varepsilon = -1$, let $\widetilde{e}_{0_i} = \frac{p_i}{r}$, $\widetilde{f}_{0_i} = \frac{q_i}{r}$, $A_i = [\widetilde{e}_{0_i}, \widetilde{e}_{1_i}, \ldots, \widetilde{e}_{n_i}]$ and $B_i = [\widetilde{f}_{0_i}, \widetilde{f}_{1_i}, \ldots, \widetilde{f}_{n_i}]$, then follow the above proof. ∎

As an immediate consequence of Theorem 3.3, one gets the following corollary. Hence the theorem is a generalization of Proposition 4.30 of [1, p.113].

**Corollary 3.4** *Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be (tangent) frames respectively on points $p, q \in Q = q^{-1}(\varepsilon r^2)$, where $\varepsilon \in \{-1, 1\}$. Then there is a unique isometry $\varphi : \mathbb{R}^{n+1}_\nu \to \mathbb{R}^{n+1}_\nu$ carry $Q$ isometricaly to itself, with $\varphi(p) = q$ and $d\varphi(e_i) = f_i$, for $1 \leq i \leq n$. Furthermore, this isometry should be linear.*

Another interesting consequence is the following corollary which is coincide with one's intuition.

**Corollary 3.5** *Let $p_i$ and $q_i$ belong to $S^n$, where $S^n$ denotes the standard $n$-dimensional sphere and $1 \leqslant i \leqslant m$. Let $(e_{1_i}, \ldots, e_{n_i})$ and $(f_{1_i}, \ldots, f_{n_i})$ be orthonormal frames in $T_{p_i}S^n$ and $T_{q_i}S^n$, respectively. Then there is an isometry $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that*

$$\varphi(S^n) \subseteq S^n, \quad \varphi(p_i) = q_i, \quad \text{and} \quad d\varphi(e_{j_i}) = f_{j_i}, \tag{$*$}$$

*if and only if*

$$\theta(p_i, p_l) = \theta(q_i, q_l) \quad \text{and} \quad \theta(\widetilde{e}_{k_i}, p_l) = \theta(\widetilde{f}_{j_l}, q_i), \tag{$**$}$$

*where $1 \leqslant i, l \leqslant m$, $1 \leqslant j, k \leqslant n$, $\theta(x, y)$ is the angle between two vectors $x$ and $y$, and $\widetilde{e}_{j_i}$ and $\widetilde{f}_{j_l}$ denote the elements of $\mathbb{R}^{n+1}$ canonically corresponding to $e_{j_i}$ and $f_{j_i}$, respectively.*

**Proof.** Since $\langle \widetilde{e}_{j_i}, \widetilde{e}_{k_i} \rangle = \delta_{jk}$ and $\langle p_i, p_l \rangle = \cos\theta(p_i, p_l)$, the corollary is a direct consequence of Theorem 3.3. ∎

## References


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[2] Wolf, J.A., *Spaces of Constant Curvature*, Publish or Perish, Berkley, California, 2011.


Accepted: