ON HOMOLOGICAL PROPERTIES OF SOME H_v-STRUCTURES

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Abstract. The main aim of this note is to investigate the fundamental homological properties of various module derivations for H_v-structures and get some functorial relations for these derivation sets as a continuous line of [6].

1. Introduction

Algebraic hyperstructures are a natural generalization of the ordinary algebraic structures with which was first initiated by Marty [7]. After the pioneered work, algebraic hyperstructures have been developed by many researchers. Let H be a nonempty set and $\mathcal{P}^*(H)$ be the family of nonempty subsets of H. Every function $* : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a *hyperoperation* on H and $(H, *)$ is called a *hyperstructure*. The concept of H_v-structures as a larger class than the well-known hyperstructures was introduced by T. Vougiouklis at the Fourth AHA congress [11]. The concept of an H_v-structure constitutes a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. Since the quotients of the H_v-structures with respect to the fundamental equivalence relations are always ordinary structures.

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The hyperstructure $(H, *)$ is called an H_v-*group* if

(1) the $*$ is *weak associative*, i.e., $x * (y * z) \cap (x * y) * z \neq \emptyset$,

(2) the *reproduction axiom* holds, i.e., $a * H = H * a = H$ for every $a \in H$.

We say H is *weak commutative* if for every $x, y \in H$, $x * y \cap y * x \neq \emptyset$.

Since then the study of H_v-structure theory has been pursued in many directions by T. Vougiouklis, B. Davvaz, S. Spartalis, A. Dramalidis, S. Hoskova, and others. For more definitions and applications on H_v-structures one can see the books and survey papers as [2], [3], [5], [10], and [1], [4], [12].

A multivalued system $(R, *, \cdot)$ is called an H_v-*ring* if the following axioms hold:

(1) $(R, *)$ is a weak commutative H_v-group,

(2) $(R, \cdot)$ is a weak associative, i.e., $x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset$ for every $x, y, z \in R$,

(3) the hyperoperation $\cdot$ is weak distributive with respect to $*$, i.e., for every $x, y, z \in R$, we have $x \cdot (y * z) \cap (x \cdot y * x \cdot z) \neq \emptyset$, $(x * y) \cdot z \cap (x \cdot z * y \cdot z) \neq \emptyset$.

For example, if $(H, +)$ is an H_v-group, then for every hyperoperation $\cdot$ such that $\{x, y\} \subseteq x \cdot y$ for every $x, y \in H$, the hyperstructure $(H, +, \cdot)$ is an H_v-ring. Therefore, we can construct some H_v-rings by a given H_v-group. Let $(H, \cdot)$ be an H_v-group with (left, right) identity elements. Then, H is called *(left, right) reversible* in itself when any relation $c \in a \cdot b$ implies the existence of a left inverse a' of a and a is a right inverse b' of b such that $b \in a' \cdot c$ and $a \in c \cdot b'$. Furthermore we call $(R, *, \cdot)$ is an H_v-*field* if $(R, *, \cdot)$ is a H_v-ring and $(R, \cdot)$ is a (left, right) reversible H_v-group.

Let $(R, *, \cdot)$ be an H_v-ring, $(M, \sharp)$ be a weak commutative H_v-group and there exists an external hyperoperation $\circ : R \times M \longrightarrow P^*(M)$ denoted by $(a, x) \longmapsto a \circ x$ such that for every $a, a_1, a_2 \in R$ and every $x, x_1, x_2 \in M$, we have

(1) $a \circ (x_1 \sharp x_2) \cap ((a \circ x_1) \sharp (a \circ x_2)) \neq \emptyset$,

(2) $(a_1 * a_2) \circ x \cap ((a_1 \circ x) \sharp (a_2 \circ x)) \neq \emptyset$,

(3) $(a_1 . a_2) \circ x \cap a_1 \circ (a_2 \circ x)$.

Then, M is called a *left* H_v-*module* over R.

A non-empty subset S of M is an H_v-submodule of M if $(S, +)$ is an H_v-subgroup of $(M, +)$ and $RS \subseteq S$. It is clear that an arbitrary ring (module) will be an H_v-ring (H_v-module) if we identify x with $\{x\}$. In the case of an H_v-field instead of an H_v-ring then the H_v-*vector space* is defined. In [9], Vougiouklis defined the concept of H_v-vector space which is a generalization of the concept of vector space. Note that every vector space is a H_v-vector space that is strongly left and right distributive and specially, every field is a H_v-vector space over itself.

Definition 1.1. Let $(R, \cdot, *)$ and $(S, \circ, \Box)$ be two H_v-rings and $(M, \sharp)$ be a weak commutative H_v-group. If M is a right $S - H_v$-module, a left $R - H_v$-module, and $r(sm) \cap (rs)m \neq \emptyset$ for all $r \in R$, $s \in S$ and $m \in M$, then M is called $R - S - H_v$-*bimodule.*

Definition 1.2. Let X and Y be H_v-vector spaces over F. A map $T : X \longrightarrow Y$ is called

(1) *H_v-linear* if and only if
$T(x+y) \cap (T(x)+T(y)) \neq \emptyset$ and $T(a \circ x) \cap (a \circ T(x)) \neq \emptyset$,
for all $x, y \in X$, $\alpha \in F$.

(2) *H_v-antilinear* if and only if
$T(x+y) \cap (T(x)+T(y)) \neq \emptyset$ and $T(a \circ x) \cap (T(x) \circ a) \neq \emptyset$,
for all $x, y \in X$, $a \in F$.

(3) *H_v- strong linear* if and only if
$T(x+y) = T(x)+T(y)$ and $T(a \circ x) = a \circ T(x)$,
for all $x, y \in X$, $a \in F$.

Let $(A, +)$ be an H_v-vector space over an H_v-field $(F, *, \cdot)$. Then A is called an *H_v-algebra* over F if there exists a mapping $\# : A \times A \longrightarrow \mathcal{P}^*(A)$ (images to be denoted by $x\#y$ for $x, y \in A$) such that the following conditions hold:

(1) $((x+y)\#z) \cap (x\#z) + (y\#z) \neq \emptyset$;
(2) $(c \circ x)\#y \cap c \circ (x\#y) \cap x\#(c \circ y) \neq \emptyset$, for all $x, y, z \in A$ and $c \in F$.

Let $(X, +, \circ, K)$ be an H_v-vector space. Suppose that for every $a \in K$, $|a|$ denoted the valuation of a in K. An *H_v-norm* on X is a mapping $\| \cdot \| : X \longrightarrow K$ that for all $a \in K$ and $x, y \in X$ has the following properties:

(1) if $z \in x+y$, then $\|z\| \leq \|x\| + \|y\|$,
(2) $sup\|a \circ x\| =\mid a \mid \cdot \|x\|$.

Let A be an H_v-algebra and $(A, \|.\|)$ be a normed H_v-vector space. If $\|x.y\| \leq \|x\| \cdot \|y\|$ for all $x, y \in A$, then A is called a *normed H_v-algebra*.

Let $(M, +_1, \circ_1, \| \cdot \|_1, K)$ and $(N, +_2, \circ_2, \| \cdot \|_2, K)$ be two normed H_v-vector spaces over the same H_v-field K. A mapping $f : M \longrightarrow N$ is called an *H_v-homomorphism* or *weak homomorphism* if for all $x, y \in M$ and $r \in R$, the following relations hold:

$$f(x +_1 y) \cap (f(x) +_2 f(y)) \neq \emptyset \text{ and } f(r \circ_1 x) \cap r \circ_2 f(x) \neq \emptyset.$$

f is called an *inclusion homomorphism* if $f(x +_1 y) \subseteq f(x) +_2 f(y)$ and $f(r \circ_1 x) \subseteq r \circ_2 f(x)$ for all $x, y \in M$ and $r \in R$.

Finally, f is called a *strong homomorphism* if for all $x, y \in M$ and $x \in R$, we have $f(x +_1 y) = f(x) +_2 f(y)$ and $f(r \circ_1 x) = r \circ_2 f(x)$. If there exists a strong one to one homomorphism from M onto N, then M and N are called *isomorphic*.

Let R be a weak-commutative H_v-ring and $\mathbf{H}$ be the set of all H_v-modules and all strong R-homomorphisms. One can show that $\mathbf{H}$ is a category.

Suppose that M and N are two H_v-modules and $M[N]$ is the set of all functions on M with values in N. First, we equip $M[N]$ to appropriate hyperoperations to be an H_v-module. As in [8] the $M[N]$ with the following hyperoperations is an H_v-module

$$\begin{array}{rcl} f+g & = & \{h \in M[N] : h(x) \in f(x)+g(x),\ \forall x \in M\}, \\ r \cdot f & = & \{k \in M[N] : k(x) \in r \cdot f(x),\ \forall x \in M\}. \end{array}$$

For a normed H_v-vector space $(X, +, \circ, \| \cdot \|, K)$, if (x_n) is a sequence in X, then

$$\lim x_n = x \Leftrightarrow \lim \| x_n - x \| = 0.$$

Let $(X, +, \circ, \| \ . \ \|, K)$ be a normed H_v-vector space. A sequence (x_n) in X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$, there is $N \in N$ such that $\| x_n - x_m \| < \varepsilon$, for every $m, n \in N$.

Definition 1.3. Let A be a normed H_v-algebra. If every Cauchy sequence in A has a limit that is also in A, then A is called a *Banach H_v-algebra*.

Definition 1.4. Let S be an H_v-algebra over a commutative H_v-ring K. M is called an *S-H_v-bimodule*, if M is a left and right S-H_v-module such that $s(mt) = (sm)t$, $a(sm) = s(am)$, $am = ma$, for all $s, t \in S$, $a \in K$, $m \in M$.

Definition 1.5. Let U be a Banach H_v-algebra. A Banach H_v-space X which is also an U-H_v-bimodule is called a *Banach U-H_v-bimodule* if there exists a constant $K > 0$ such that $\|\alpha \cdot x\| \leq K\|\alpha\|\|x\|$ and $\|x \cdot \alpha\| \leq K\|\alpha\|\|x\|$ for each $\alpha \in U$ and $x \in X$.

By using a certain type of equivalence relations, we can connect H_v-structures to usual structures. The smallest of these relations are called *fundamental relations* and denoted by β^*, γ^*, ε^*, so that if H is an H_v-group, (H_v-ring, H_v-module over an H_v-ring R) then H/β^* is a group (H/γ^* is a ring, H/ε^* is an R/γ^*-module). The fundamental relation ε^* on an H_v-module M can be defined as follows:

Consider the left H_v-module M over an H_v-ring R. If ϑ denotes the set of all expressions consisting of finite hyperoperations of either on R and M or of the external hyperoperations applying on finite sets of elements of R and M, a relation ε can be defined on M whose transitive closure is the fundamental relation ε^*. The relation ε is defined as follows: for every $x, y \in M$, $x\varepsilon y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \vartheta$.

Suppose that $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\varepsilon^*(x)$ is the equivalence class containing $x \in M$. On M/ε^* the $\oplus$ and the external product $\odot$ using the γ^* classes in R are defined as follows:

For every $x, y \in M$, and for every $r \in R$,

$$\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(c), \text{ for every } c \in \varepsilon^*(x) + \varepsilon^*(y),$$

$$\gamma^*(r) \odot \gamma^*(x) = \gamma^*(d), \text{ for every } d \in \gamma^*(r) + \gamma^*(x),$$

The kernel of canonical map $\phi : M \longrightarrow M/\varepsilon^*$ is called the *heart* of M and it is denoted by ω_M, i.e., $\omega_M = \{x \in M : \omega(x) = 0\}$, where 0 is the unit element of the group $(M/\varepsilon^*, \oplus)$. One can prove that the unit element of the group $(M/\varepsilon^*, \oplus)$ is equal to ω_M.

Let M and N be two H_v-modules over an H_v-ring R. The kernel of a strong H_v-homomorphism $f : M \longrightarrow N$ is defined as follows:

$$Ker(f) = \{a \in M : f(a) \in \omega_N\}.$$

A function $f : M_1 \longrightarrow M_2$ is called *weak-monic* if for every $m_1, m_2 \in M_1$, $f(m_1) = f(m_2)$ implies $\varepsilon^*(m_1) = \varepsilon^*(m_2)$ and f is called *weak-epic* if for every $m_2 \in M_2$ there exists $m_1 \in M_1$ such that $\varepsilon^*(m_2) = \varepsilon^*(f(m_1))$. Finally f is called *weak-isomorphism* if f is weak-monic and weak-epic.

Let M be an H_v-module and X, Y be non-empty subsets of M. We say X is *weak equal* to Y and write $X \overset{w}{\equiv} Y$ if and only if for every $x \in X$ there exists $y \in Y$ such that $\varepsilon^*(x) = \varepsilon^*(y)$ and for every $y \in Y$ there exists $x \in X$ such that $\varepsilon^*(x) = \varepsilon^*(y)$.

Let $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} ... \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n$ be a sequence of H_v-modules and strong H_v-homomorphisms. We say this sequence is *exact* if for every $2 \leq i \leq n$, $Im(f_{i-1}) \overset{w}{\equiv} Ker(f_i)$.

Definition 1.6. Let X and Y be normed H_v-vector spaces over F and $T : X \longrightarrow Y$ be an operator. T is said to be H_v-*bounded* if there exists a positive real number K such that we have

$$\|Tx\| \leq K\|x\|, \; x \in X.$$

Definition 1.7. Let X and Y be normed H_v-vector spaces over F and $T : X \longrightarrow Y$ be an operator. Then T is called H_v-*weak linear operator* if T is additive and satisfies

$$(T(Z_{a\circ x}) \cap (a \circ T(x))) \neq \emptyset; \; (a \in F; x \in X)$$

Here z_{aox} for $a \neq 0$ is that element of aox such that $x \in a^{-1} \circ z_{a\circ x}$. So the set of all these elements denoted by $Z_{a\circ x}$.

Let X and Y be H_v-vector spaces over F. Denote the set of all weak linear operators and the set of all bounded weak linear operators from X into Y by $L_w(X, Y)$ and $B_w(X, Y)$, respectively.

Let Y be normal. Then $L_w(X, Y)$ with the following sum and product is a weak H_v-vector space over F.

$(T + S)(x) = T(x) + T(y)$, $T, S \in L_w(X, Y)$, $x \in X$

$a \circ T = \{S \in L_w(X, Y) : Sx \in a \circ Tx, x \in X\}$, $(a \in F, T \in L_w(X, Y))$.

Let Y be normal. Then $L_w(X, Y)$ is a normal weak H_v-vector space and $B_w(X, Y)$ is a sub H_v-vector space of $L_w(X, Y)$.

Let $(X, +_1, \circ_1, \|.\|_1, K)$ and $(Y, +_2, \circ_2, \|.\|_2, K)$ be two normed H_v-vector spaces. For a H_v-bounded strong homomorphism $f : X \longrightarrow Y$, we define the norm of f by

$$\|f\| := sup\left\{sup\left\|f\left(\frac{1}{\|x\|}\circ_1 x\right)\right\|_2 : 0 \neq x \in X\right\}.$$

Let $(R, +, \cdot)$ be H_v-ring. The function $d : R \longrightarrow R$ is called H_v- derivation if for all $x, y \in R$,

(1) $d(x + y) \cap (d(x) + d(y)) \neq \emptyset$,

(2) $d(x \cdot y) \cap (d(x)y + xd(y)) \neq \emptyset$.

2. H_v-module derivations

2.1. Some structures

Throughout this paper, A and U are Banach H_v-algebras such that A is a Banach U-H_v-bimodule with the compatible actions, as follows:

$$\alpha \cdot (ab) \cap (\alpha \cdot a)b \neq \emptyset,\ (ab) \cdot \alpha \cap a(b \cdot \alpha) \neq \emptyset\ (\alpha \in U,\ a, b \in A)$$

Let X be a Banach A-H_v-bimodule and a Banach U-H_v-bimodule with the compatible actions, that is;

$$\alpha \cdot (a \cdot x) \cap (\alpha \cdot a) \cdot x \neq \emptyset,\ (\alpha \cdot x) \cdot a \cap \alpha(x \cdot a) \neq \emptyset,\ (a \cdot \alpha) \cdot x \cap a \cdot (\alpha \cdot x) \neq \emptyset$$

for every $\alpha \in U, a \in A, x \in X$ and similarly for the right or two-sided actions. Then, we say that X is a *Banach A-U-H_v-module*. If moreover $\alpha \cdot x \cap x \cdot \alpha \neq \emptyset$ $(\alpha \in U, \quad x \in X)$, then X is called a *commutative Banach A-U-H_v-module*. Furthermore if $a \cdot x \cap x \cdot a \neq \emptyset$ for all $x \in X$ and $a \in A$, then X is called a *bi-commutative Banach A-U-H_v-module*. Throughout this paper, by a A-U-H_v-module, we shall always mean a commutative Banach A-U-H_v-module.

Note that in general, A is not a Banach A-U-H_v-module because A does not satisfy the compatibility condition $a(\alpha \cdot b) \cap (a \cdot \alpha)b \neq \emptyset$ for $\alpha \in U$ and $a, b \in A$. But when A is a commutative U-H_v-bimodule and acts on itself by H_v-algebra multiplication from both sides, then it is also a Banach A-U-H_v-module.

If X is a (commutative) Banach A-U-H_v-module, then so is X^*, where the actions of A and U on X^* are defined as follows:

$$\begin{aligned} \langle f \cdot \alpha, x \rangle &= \langle f, \alpha \cdot x \rangle, & \langle f \cdot a, x \rangle &= \langle f, a \cdot x \rangle, \\ \langle \alpha \cdot f, x \rangle &= \langle f, x \cdot \alpha \rangle, & \langle a \cdot f, x \rangle &= \langle f, x \cdot a \rangle, \end{aligned}$$

for each $a \in A, \alpha \in U, x \in X$ and $f \in X^*$.

2.2. Some H_v-maps

Let A and B be Banach H_v-algebras and Banach U-H_v-bimodules with compatible actions, a *U-H_v-module map* is a H_v-bounded H_v-map $h : A \to B$ with

$$\begin{aligned} &h(a \pm b) \cap (h(a) \pm h(b)) \neq \emptyset, \\ &h(\alpha \cdot a) \cap (\alpha \cdot h(a)) \neq \emptyset, \\ &h(a \cdot \alpha) \cap (h(a) \cdot \alpha) \neq \emptyset \quad (\alpha \in U, a, b \in A). \end{aligned}$$

Here h is not necessarily H_v-linear, so it is not necessarily a U-H_v-module homomorphism.

h is called *multiplicative U-H_v-module map* (or called *U-H_v-module morphism*) if $h(ab) \cap h(a)h(b) \neq \emptyset$ $(a, b \in A)$. We denote by $Hom_U(A, B)$ the H_v-metric space of all multiplicative U-H_v-module maps from A into B, with the H_v-metric derived from the usual H_v-linear operator norm $\|.\|$ on $L_U(A, B)$;

the set of all H_v-bounded linear operators from A into B, and denote $Hom_U(A,A)$ by $Hom(A)$.

Let A and U be as above and X be a Banach A-U-H_v-module. A *H_v-module derivation* $D : A \to X$ is a U-H_v-module map such that $D(ab) \cap (D(a) \cdot b + a \cdot D(b)) \neq \emptyset$ for all $a, b \in A$. The set of all H_v-module derivations from A to X is denoted by $Z^U(A,X)$ (abb. $Z(A,X)$). Note that D is not necessarily H_v-linear, but still its H_v-boundedness implies its norm H_v-continuity since D preserves subtraction.

Let f and g be H_v-module derivations from A to X and $\alpha \in U$, so are $f+g$ and αf. Since X is a Banach U-H_v-bimodule, we have that $Z(A,X)$ is a Banach U-H_v-bimodule.

For $x \in X$, define a map by $D_x : A \to X$, $a \longmapsto a \cdot x - x \cdot a$, $a \in A$. When X is a A-U-H_v-module, it is clear that D_x is a H_v-module derivation. H_v-module derivations of this kind are called *inner* and denoted by $InnZ(A,X)$.

An *H_v-Jordan module derivation* $D : A \to X$ is a U-H_v-module map such that $D(a^2) \cap (D(a) \cdot a + a \cdot D(a)) \neq \emptyset$ for all $a \in A$. The set of all Jordan H_v-module derivations from A to X is denoted by $JZ(A,X)$.

The U-H_v-module map $f : A \to X$ is said to be *H_v-Lie module derivation* if the identity

$$f([a,b]) \cap ([f(a),b] + [a,f(b)]) \neq \emptyset$$

holds for all $a, b \in A$. The set of all Lie H_v-module derivations is denoted by $LieZ(A,X)$. Here $[a,b] = ab - ba$.

By a *Brešar generalized H_v-module derivation* (f,D), we mean $f : A \to X$ is a U-H_v-module map such that $f(ab) \cap (f(a) \cdot b + a \cdot D(b)) \neq \emptyset$ for all $a, b \in A$, where D is a H_v-module derivation on A. We denote by $Z^B(A,X)$ is the set of Brešar generalized H_v-module derivation from A to X.

If (f_1, D_1) and (f_2, D_2) are Brešar generalized H_v-module derivations and $\alpha \in U$, then $(f_1 + f_2, D_1 + D_2)$ and $(\alpha f_1, \alpha D_1)$ are also Brešar generalized H_v-module derivations and hence, $Z^B(A,X)$ is a Banach U-H_v-bimodule.

For $x, y \in X$, a U-H_v-module map satisfies the identity

$$f_{x,y} : A \ni a \mapsto (x \cdot a + a \cdot y) \in X$$

for all $a \in A$ is called a *Brešar generalized inner H_v-module derivation.*

For a U-H_v-module map $f : A \to X$ is called a *Brešar generalized Jordan H_v-module derivation* if

$$f(a^2) \cap (f(a) \cdot a + a \cdot D(a)) \neq \emptyset$$

for all $a \in A$. Here D is a Jordan H_v-module derivation. We denote the set of Brešar generalized Jordan H_v-module derivations from A to X by $JZ^B(A,X)$.

The U-H_v-module map $f : A \to X$ is said to be *Brešar generalized Lie H_v-module derivation* if the identity

$$f([a,b]) \cap ([f(a),b] + [a,D(b)]) \neq \emptyset$$

holds for all $a, b \in A$. Here D is a Lie H_v-module derivation. We denote the set by $LieZ^B(A, X)$.

For a U-H_v-module map $f : A \to X$ and an element $x \in X$, a pair (f, x) is called an H_v-*generalized module derivation* in the sense of *Nakajima*, if

$$f(ab) \cap (f(a) \cdot b + a \cdot f(b) + a \cdot x \cdot b) \neq \emptyset$$

for all $a, b \in A$. We denote the set of this type of generalized H_v-module derivations by $Z^G(A, X)$. This is also a Banach U-H_v-bimodule for a Banach A-U-H_v-module X.

For $x, y \in X$, a U-H_v-module map $f_{x,y} : A \to X$ is called H_v-*generalized inner module derivation* if

$$f_{x,y}(ab) \cap (f_{x,y}(a) \cdot b + a \cdot f_{x,y}(b) + a \cdot (-x - y) \cdot b) \neq \emptyset$$

for all $a, b \in A$. We denote this H_v-derivation by $(f_{x,y}, -x - y)$.

A pair (f, x) is called a H_v-*generalized Jordan module derivation* if

$$f(a^2) \cap (f(a) \cdot a + a \cdot f(a) + a \cdot x \cdot a) \neq \emptyset$$

for all $a \in A$. We denote the set of generalized Jordan H_v-module derivations from A to X by $JZ^G(A, X)$.

The pair (f, x) is called a H_v-*generalized Lie module derivation* if the relation

$$f([a, b]) \cap ([f(a), b] + [a, f(b)] + a \cdot x \cdot b - b \cdot x \cdot a) \neq \emptyset$$

holds for all $a, b \in A$ and the set of generalized Lie H_v-module derivations from A to X can be denoted by $LieZ^G(A, X)$.

If $x = 0$, then these definitions lead to the conventional notions of generalized Jordan and Lie H_v-module derivations.

Throughout this paper we use the following notations for the above sets:

- $Z(A, X)$, the set of H_v-module derivations,
- $InnZ(A, X)$, the set of inner H_v-module derivations,
- $JZ(A, X)$, the set of Jordan H_v-module derivations,
- $LieZ(A, X)$, the set of Lie H_v-module derivations,
- $Z^B(A, X)$, the set of Brešar generalized H_v-module derivations,
- $InnZ^B(A, X)$, the set of Brešar generalized inner H_v-module derivations,
- $JZ^B(A, X)$, the set of Brešar generalized Jordan H_v-module derivations,
- $LieZ^B(A, X)$, the set of Brešar generalized Lie H_v-module derivations,
- $Z^G(A, X)$, the set of generalized H_v-module derivations,
- $InnZ^G(A, X)$, the set of generalized inner H_v-module derivations,
- $JZ^G(A, X)$, the set of generalized Jordan H_v-module derivations,
- $LieZ^G(A, X)$, the set of generalized Lie H_v-module derivations.

A U-H_v-module map $f : A \to X$ is said to be *left H_v-module multiplier* if $f(ab) \cap (f(a) \cdot b) \neq \emptyset$ for all $a, b \in A$. We denote by $Mull^U(A, X)$ (abb. $Mull(A, X)$) the set of all left H_v-module multipliers from A to X. Especially if $f(a^2) \cap (f(a) \cdot a) \neq \emptyset$ for all $a \in A$, then f is called *Jordan left H_v-module multiplier* and we denote the set of these maps by $JMull(A, X)$. Furthermore, we can define the set

$$\begin{aligned} &LieMull(A, X) \\ &= \{f \mid f : A \to X,\ H_v\text{-module map and } f[a, b] \cap ([-f(b), a]) \neq \emptyset \text{ for all } a, b \in A\} \end{aligned}$$

which is called the set of *Lie left H_v-module multipliers*.

If f and g are left H_v-module multipliers in all types and $\alpha \in U$, then $f + g$ and αf are also left H_v-module multipliers in all types, hence all the above special sets are Banach U-H_v-bimodules.

At the end of this section, we want to give a well-known lemma which will be used several times in the next sections in our paper.

Lemma 2.1. [8, Theorem 4.7 (Five Short Lemma in H_v-module)] *Let*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram of H_v-modules and H_v-module homomorphisms over an H_v-ring R with exact rows, the followings hold:

(1) *If α_1 is an epimorphism and α_2, α_4 are monomorphisms, then α_3 is a monomorphism;*

(2) *If α_5 is a monomorphism and α_2, α_4 are epimorphisms, then α_3 is a monomorphism*

3. Homological properties of generalized H_v-module derivations

In this section, we first discuss the relation between the U-H_v-bimodules $Z^B(A, X)$ and $Z^G(A, X)$. Now, we give some elementary lemmas which show the relation between H_v-module derivations and our generalized H_v-module derivations.

Lemma 3.1.

(1) *If $(f, x) : A \to X$ is a generalized H_v-module derivation, then there exists an H_v-module derivation $d = f + l_x : A \to X$, where $l_x : A \to X$ is a left multiplication, i.e., $l_x(a) = xa$, such that $f(ab) = f(a)b + ad(b)$ for all $a, b \in A$. Moreover, if $\{x \in X \mid Ax = 0\} = 0$, then d is uniquely determined by f.*

(2) *If $D : A \to X$ is an H_v-module derivation, then for any nonzero element $x \in X$, $(f = D + l_x, -x) : A \to X$ is a generalized H_v-module derivation such that $f \neq D$ and D associates to f.*

(3) *If* $(f, x) : A \to X$ *is a generalized* H_v*-module derivation, then* $(f, f + l_x) : A \to X$ *is a Brešar generalized* H_v*-module derivation.*

(4) *If* A *contains a unit element and* $(f, D) : A \to X$ *is a Brešar generalized* H_v*-module derivation, then* $(f, -f(1)) : A \to X$ *is a generalized* H_v*-module derivation. It means that the notions of generalized* H_v*-module derivations of Nakajima and Brešar coincide when* A *contains an identity element.*

Proof. We only need to check the boundedness of the map $d = f + l_x : A \to X$. Since f is a U-H_v-module map and X is a A-U-H_v-bimodule, then we get

$$\|(f + l_x)(a)\| \leq \|f(a)\| + \|x \cdot a\| \leq M\|a\| + K\|x\|\|a\| = (M + K\|x\|)\|a\|,$$

for each $a \in A$. This means that the map $f + l_x$ isH_v- bounded. The other parts of the proof can be done easily. ∎

Remark 1. Throughout this paper, the most important thing which we have to check is the boundedness of the maps (for U-H_v-module maps). In the next parts of the paper, we have omitted the boundedness of the maps (because all of them are done similarly).

Corollary 3.2. *The following sequence of* U-H_v*-modules* $Z^G(A, X)$ *and* $Z(A, X)$ *is exact and splitting:*

$$0 \to X \xrightarrow{\varphi_1} Z^G(A, X) \xrightarrow{\varphi_2} Z(A, X) \to 0,$$

where $\varphi_1(x) = (l_x, -x)$ *and* $\varphi_2((f, x)) = f + l_x$ *are* U-H_v*-module maps. Hence, we get* $Z^G(A, X) \cong X \oplus Z(A, X)$.

Our aim is to give necessary and sufficient condition for $Z^B(A, X)$ to be isomorphic to $Z^G(A, X)$ as a Banach U-H_v-bimodule when A *does not have a unit element.*

Theorem 3.3. *Suppose that* $\Phi : Z^G(A, X) \to Z^B(A, X)$ *and* $\psi : X \to \mathrm{Mull(A, X)}$ *are* U-H_v*-module morphisms such that* $\Phi((f, x)) = (f, f + l_x)$ *and* $\psi(x) = l_x$*. Then* Φ *is a* U-H_v*-module isomorphism if and only if* ψ *is a* U-H_v*-module isomorphism.*

Proof. We have the following split exact sequence of Banach U-H_v-bimodules:

$$0 \to Mull(A, X) \xrightarrow{\psi_1} Z^B(A, X) \xrightarrow{\psi_2} Z(A, X) \to 0,$$

where $\psi_1(g) = (g, 0)$ and $\psi_2((f, D)) = D$.

Define a map $\psi_2' : Z(A, X) \to Z^B(A, X)$ by $\psi_2'(D) = (D, D)$. Then $\psi_2 \psi_2' = id_{Z(A,X)}$, and thus is split exact. This gives the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\varphi_1} & Z^G(A, X) & \xrightarrow{\varphi_2} & Z(A, X) & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow \Phi & & \downarrow id & & \\ 0 & \longrightarrow & Mull(A, X) & \xrightarrow{\psi_1} & Z^B(A, X) & \xrightarrow{\psi_2} & Z(A, X) & \longrightarrow & 0 \end{array}$$

Hence we complete the proof of the theorem by using Five Lemma. ■

Corollary 3.4. *The following sequence of Banach U-H_v-bimodules $JZ(A,X)$ and $JZ^G(A,X)$, is exact and splitting:*

$$0 \to X \xrightarrow{\psi_X} JZ^G(A,X) \xrightarrow{\phi_X} JZ(A,X) \to 0,$$

where $\psi_X(x) = (l_x, -x)$ and $\phi_X((f,x)) = f + l_x$.

Corollary 3.5. *Suppose that $\Phi : JZ^G(A,X) \longrightarrow JZ^B(A,X)$ and $\psi : X \longrightarrow$ JMull(A, X) are U-H_v-module morphisms such that $\Phi((f,x)) = (f, f + l_x)$ and $\psi(x) = l_x$. Then Φ is a U-H_v-module isomorphism if and only if ψ is a U-H_v-module isomorphism.*

Corollary 3.6. *The following sequence of Banach U-H_v-bimodules $LieZ(A,X)$ and $LieZ^G(A,X)$, is exact and splitting:*

$$0 \to X \xrightarrow{\psi_X} LieZ^G(A,X) \xrightarrow{\phi_X} LieZ(A,X) \to 0$$

where $\psi_X(x) = (l_x, -x)$ and $\phi_X((f,x)) = f + l_x$.

Corollary 3.7. *Suppose that $\Phi : LieZ^G(A,X) \longrightarrow LieZ^B(A,X)$ and $\psi : X \longrightarrow$ LieMull(A, X) are U-H_v-module morphisms such that $\Phi((f,x)) = (f, f + l_x)$ and $\psi(x) = l_x$. Let us define the set*

$$\mathcal{X}(A) = \{x \in X \mid [x,a] = 0 \text{ for all } a \in A\}.$$

If $\mathcal{X}(A) = X$ (If X is a bi-commutative Banach A-U-H_v-module), then Φ is a U-H_v-module isomorphism if and only if ψ is a U-H_v-module isomorphism.

Corollary 3.8. *Let X be a A-U-H_v-module, then the following diagram is commutative and the rows are split exact:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\psi_1} & InnZ^G(A,X) & \xrightarrow{\psi_2} & InnZ(A,X) & \longrightarrow & 0 \\ & & \downarrow i_0 & & \downarrow i & & \downarrow i_1 & & \\ 0 & \longrightarrow & X & \xrightarrow{\varphi_1} & Z^G(A,X) & \xrightarrow{\varphi_2} & Z(A,X) & \longrightarrow & 0 \end{array}$$

where i_0, i_1, i are the canonical H_v-module injections and $\psi_1(x) = (f_{x,0}, -x)$, $\psi_2(f_{x,y}, -x-y) = f_{x,y} + l_{(-x-y)}$.

Proof. All maps in the above diagram are U-H_v-module maps, and the commutativity of the diagram is easily seen. If $\psi_2(f_{x,y}, -x-y) = 0$, then we see that $f_{x+y,0} = f_{x,y}$. Thus $Ker\psi_2 = Im\psi_1$. The other part is clear by Corollary 3.2 using the definitions of φ_1 and φ_2. ■

4. Functorial relations

4.1. Functorial relations

Firstly, we give a functorial relation between $Z(A,-)$ and $Z^G(A,-)$ as follows:

Theorem 4.1. *Let X_1 and X_2 be Banach A-U-H_v-odules and $\gamma : X_1 \to X_2$ be a U-H_v-module morphism. Then γ induces a U-H_v-module map*

$$\gamma' : Z^G(A, X_1) \to Z^G(A, X_2)$$

such that

$$\gamma'((f, x_1)) = (\gamma f, \gamma(x_1))$$

and $Z^G(A, -)$ is a covariant functor from the category of Banach A-U-H_v-modules to the category of Banach U-H_v-bimodules.

Proof. The map

$$\gamma' : Z^G(A, X_1) \to Z^G(A, X_2),\ (f, x_1) \mapsto (\gamma f, \gamma(x_1))$$

is a U-hypermodule map.

Let $\mathcal{X}$ be a category of Banach A-U-H_v-modules and $\mathcal{M}$ be a category of Banach U-H_v-bimodules. Define the functor as follows:

$$Z^G(A, -) : \mathcal{X} \to \mathcal{M}, \quad X \mapsto \mathcal{X}(X) = Z^G(A, X).$$

If $\gamma : X_1 \to X_2, \quad \gamma_* : X_2 \to X_3$ are U-H_v-module morphisms, then, for the following map, the first condition is satisfied:

$$\begin{aligned} &Z^G(A, -)(\gamma_* \circ \gamma) : Z^G(A, X_1) \to Z^G(A, X_3), \\ &(f, x_1) \mapsto ((\gamma_* \circ \gamma) \circ f, (\gamma_* \circ \gamma)(x_1)) \end{aligned}$$

On the other hand,

$$\begin{aligned} Z^G(A, -)(\gamma_*)((Z^G(A, -)(\gamma))(f, x_1)) &= Z^G(A, -)(\gamma_*)(\gamma f, \gamma(x_1)) \\ &= (\gamma_* \circ (\gamma f), \gamma_*(\gamma(x_1)). \end{aligned}$$

Thus $Z^G(A, -)(\gamma_*) \circ Z^G(A, -)(\gamma) = Z^G(A, -)(\gamma_* \circ \gamma)$.

For the second condition, we use the map,

$$\begin{aligned} &Z^G(A, -)(1_X) : Z^G(A, X) \to Z^G(A, X), \\ &(f, x_1) \mapsto (1_X f, 1_X(x_1)) = (f, x_1) \end{aligned}$$

Therefore, $Z^G(A, -)(1_X) = 1_{Z^G(A,-)(X)}$. ∎

Theorem 4.2. *Let $\Phi : Z^G(A, -) \to Z(A, -) \oplus F$ be a map of functors where F is the forgetful functor from the category of Banach A-U-H_v-modules to the category of Banach U-H_v-bimodules. Then Φ assigns to each Banach A-U-H_v-module X of $\mathcal{X}$, a U-H_v-module isomorphism $\Phi_X : Z^G(A, X) \to Z(A, -) \oplus X$ of $\mathcal{M}$ such that $\Phi_X((f, x)) = (f + l_x, x)$ where $x \in X$ and $X = X_1, X_2$; in such a way that for every U-H_v-module morphism of Banach A-U-H_v-modules $\gamma : X_1 \to X_2$ of $\mathcal{X}$, the diagram*

$$\begin{array}{ccc} Z^G(A, X_1) & \xrightarrow{\alpha_*} & Z^G(A, X_2) \\ \downarrow \Phi_{X_1} & & \downarrow \Phi_{X_2} \\ Z(A, X_1) \oplus X_1 & \xrightarrow{\overline{\alpha_*}} & Z(A, X_2) \oplus X_2 \end{array}$$

in $\mathcal{M}$ is commutative, where $\Phi_X(d, x) = (\gamma d, d(x))$ and $\overline{\alpha_*}(f, x_1) = (\gamma f, f(x_1))$. Hence we can say that Φ is a natural transformation of functors.

Since Φ_X is an equivalence for every A-U-H_v-module X_1 in $\mathcal{X}$ by Theorem 4.2, we have the following corollary:

Corollary 4.3. *The functors $Z^G(A, -)$ and $Z^G(A, -) \oplus F$ from the category of Banach A-U-H_v-modules to the category of Banach U-H_v-bimodules are naturally equivalent.*

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