ON Γ-BIACTS AND THEIR GREEN'S RELATIONS

Ali Reza Shabani
Hamid Rasouli[1]

Department of Mathematics
Science and Research Branch
Islamic Azad University
Tehran
Iran
e-mails: ashabani@srbiau.ac.ir
hrasouli@srbiau.ac.ir

Abstract. A well-known generalization of a semigroup S is called the Γ–semigroup. We generalize the notion of biacts over semigroups to Γ–biacts over Γ-semigroups. Green's relations on semigroups and biacts play an important role in these theories. In this paper, we study Green's relations on Γ–biacts.

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1. Introduction and preliminaries

The concept of Γ−semigroup, as a generalization of the notion of semigroup, was introduced by Sen [10]. Certain algebraic properties of Γ–semigroups have been studied by some authors, for example, one may see [2], [3]. Actions over a semigroup S, S–acts, play an important role in a variety of areas such as theoretical computer science (see [7]). We extended some classical notions of S–acts to $\Gamma - S$–acts in [12]. Green [5] introduced the Green's relations on semigroups in 1951. Green's relations for Γ–semigroups were studied by Chinram and Siammai [2]. Also, Green's relations on biacts have been studied in [8]. A generalization of acts over semigroups to Γ–acts over Γ–semigroups can be found in [11]. In this paper, we generalize the notion of biacts to Γ–biacts and consider Green's relations on Γ–biacts, which are in fact a generalization of Green's relations on biacts. Other classical algebraic structures such as modules can also be generalized to Γ–modules. For more information, see for example [1, 6]. As an application of (ordered) Γ–semigroups in connection with fuzzy sets, we refer to [4, 9].

In the following, we recall certain preliminaries on Γ–semigroups and $\Gamma - S$–acts needed in the sequel.

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[1]Corresponding author.

Let X be a non-empty set, $B(X)$ denote the set of relations and $\varepsilon(X)$ the set of equivalence relations on X. Also, the set $\{(x,x) \mid x \in X\}$, the diagonal relation on X, is denoted by Δ_X, and the universal relation $X \times X$ is denoted by ∇_X. If $\rho \in B(X)$, the *transitive closure* of ρ is the relation $\rho^\infty = \bigcup_{i=1}^{\infty} \rho^i \in B(X)$ which is the smallest transitive relation in the poset $(B(X), \subseteq)$ containing ρ. Moreover, $\rho^e = (\rho \bigcup \rho^{-1} \bigcup \Delta_X)^\infty$ is the *equivalence closure* of ρ, that is, an equivalence relation on X generated by ρ (see [8, Theorem I.1.6]). A *lattice* is a poset L for which the meet $a \wedge b$ (the greatest lower bound) and the join $a \vee b$ (the least upper bound) exist for every $a, b \in L$.

Corollary 1.1. [8] *For a non-empty set X, if $\rho \in B(X)$, then $(x,y) \in \rho^e$ if and only if $x{=}y$ or for some $n \in \mathbb{N}$ there exists a sequence of elements $x{=}z_1, z_2, ..., z_n{=}y$ in X such that for every $i \in \{1,2,...,n-1\}, (z_i, z_{i+1}) \in \rho \bigcup \rho^{-1}$. In particular, if ρ and σ are equivalence relations on a set X, then in $\varepsilon(X)$ their join $\rho \vee \sigma$ is the relation defined by $x(\rho \vee \sigma) y$ if and only if there exist $z_1, z_2, ..., z_n \in X$ such that $x = z_1, z_n = y$ and $(z_i, z_{i+1}) \in \tau_i, \tau_i \in \{\rho, \sigma\}, i \in \{1,2,...,n-1\}$.*

Definition 1.2. [10] Let S and Γ be non-empty sets. Then S is said to be a Γ*–semigroup* if there exists a mapping $S \times \Gamma \times S \to S$ written as $(s,\gamma,t) \mapsto s\gamma t$, satisfying $(s\gamma t)\beta u = s\gamma(t\beta u)$ for all $s,t,u \in S$ and $\gamma, \beta \in \Gamma$. An element e in a Γ–semigroup S is called a *left* (*right*) Γ*–identity* if $e\gamma s = s$ ($s\gamma e = s$) for all $s \in S$ and $\gamma \in \Gamma$. By a Γ*–identity* we mean an element of S which is both a left and a right Γ–identity. A Γ–semigroup with a Γ–identity 1 is called a Γ*–monoid.*

Definition 1.3. [12] Let S be a Γ–semigroup with a left Γ–identity e and A be a non-empty set. A mapping $\lambda : S \times \Gamma \times A \to A$ where $(s,\gamma,a) \mapsto s\gamma a := \lambda(s,\gamma,a)$ such that $(s\gamma t)\beta a = s\gamma(t\beta a)$ and $e\gamma a = a$ for all $a \in A, s,t \in S$ and $\gamma, \beta \in \Gamma$, is called a *left* $\Gamma - S$*–action* and A is said to be a *left* $\Gamma - S$*–act* which is denoted by $\Gamma - {}_SA$. Also, for a Γ–semigroup S with a right Γ–identity e, by a *right* $\Gamma - S$*–act* we mean a non-empty set A together with a mapping $\lambda : A \times \Gamma \times S \to A$ where $(a,\gamma,s) \mapsto a\gamma s := \lambda(a,\gamma,s)$ satisfying the properties $a\gamma(s\beta t) = (a\gamma s)\beta t$ and $a\gamma e = a$ for all $a \in A, s,t \in S$ and $\gamma,\beta \in \Gamma$. We denote a right $\Gamma - S-$ act by $\Gamma - A_S$.

Remark 1.4. If S is a Γ–monoid with Γ–identity 1 and $\Gamma - {}_SA$ is a left $\Gamma - S$–act, then for every $s,t \in S, a \in A, \gamma, \beta \in \Gamma$, we have $s\gamma t = s\beta t$ and $s\gamma a = s\beta a$. Indeed, $s\gamma t = (s\beta 1)\gamma t = s\beta(1\gamma t) = s\beta t$; and $s\gamma a = (s\beta 1)\gamma a = s\beta(1\gamma a) = s\beta a$. Therefore, it is more interesting to consider left $\Gamma - S$–acts for a Γ–semigroup S with a left Γ–identity (not a Γ–identity) and, likewise, right $\Gamma - S$–acts for a Γ–semigroup S with a right Γ–identity (not a Γ–identity).

2. Γ–biacts and some basic properties

The purpose of this section is to introduce the structure of Γ–biacts and investigate some of their properties.

Definition 2.1. [8] Let T and S be monoids. A $T-S$*–biact* $_TA_S$ is a non-empty set A equipped with a left T-action $T\times A\to A,(t,a)\mapsto ta$, satisfying $(t_1t_2)a=t_1(t_2a)$ for all $t_1,t_2\in T,a\in A$, and a right S-action $A\times S\to A,(a,s)\mapsto as$, satisfying $a(s_1s_2)=(as_1)s_2$ for all $s_1,s_2\in S,a\in A$, for which $(ta)s=t(as)$ holds for all $t\in T,s\in S,a\in A$. For a $T-S$–biact $_TA_S$, a relation $\rho\in B(A)$, i.e. $\rho\subseteq A\times A$, is called $T-S$*–compatible* if $(a,b)\in\rho$ implies that $(tas,tbs)\in\rho$ for all $t\in T,a,b\in A$ and $s\in S$. Moreover, an equivalence relation $\rho\in\varepsilon(A)$ which is $T-S$–compatible is called a $T-S$*–congruence* on $_TA_S$. The set of all $T-S$–congruences on $_TA_S$ is denoted by $\mathbf{Con}(_TA_S)$.

Definition 2.2. Let $\Gamma-{_TA}$ be a left $\Gamma-T$–act and $\Gamma-A_S$ be a right $\Gamma-S$–act. We call A a $\Gamma-T-S$*–biact*, or simply a Γ*–biact*, and write $\Gamma-{_TA_S}$, if for all $t\in T$,$s\in S,a\in A$ and $\gamma,\beta\in\Gamma$,$(t\gamma a)\beta s=t\gamma(a\beta s)$.

From now on, $\Gamma-{_TA_S}$ stands for a $\Gamma-T-S$–biact where T and S are Γ–semigroups with a left and a right Γ–identity, respectively (see Remark 1.4), unless otherwise stated. If no confusion arises, we may use the same symbol 1 for a left Γ–identity and a right Γ–identity.

Remark 2.3. Every $T-S$–biact $_TA_S$ over semigroups T and S with a left identity and a right identity, respectively, can be made into a $\Gamma-T-S$–biact over the induced left Γ–semigroup T with a left Γ–identity by setting $t\gamma t':=tt',t,t'\in T$, and right Γ–semigroup S with a right Γ–identity by defining $s\gamma s':=ss',s,s'\in S$. Define mappings $T\times\Gamma\times A\to A$ by $t\gamma a=ta$ and $A\times\Gamma\times S\to A$ by $a\beta s=as$ for all $t\in T,a\in A,s\in S$ and $\gamma,\beta\in\Gamma$. It is easily seen that $_TA_S$ is a $\Gamma-T-S$–biact. Conversely, let A be a $\Gamma-T-S$–biact where T is a Γ–semigroup with a left Γ–identity and S is a Γ–semigroup with a right Γ–identity. Fix an element γ in Γ. First note that T and S are semigroups with the operations $tt':=t\gamma t'$ and $ss':=s\gamma s'$ for all $t,t'\in T$ and $s,s'\in S$ respectively. We define $T\times A\to A$ by $ta:=t\gamma a$ and $A\times S\to A$ by $as:=a\gamma s$ for all $t\in T,a\in A,s\in S$. Then one can show that A is a $T-S$–biact.

Example 2.4. Let $S=T=\{4n+3\mid n\in\mathbb{N}\}$, $\Gamma=\{4n+1\mid n\in\mathbb{N}\}$ and $A=\{4n\mid n\in\mathbb{N}\}$. Under the usual addition of natural numbers, S and T are Γ–semigroups and A is a $\Gamma-T-S$–biact, but not a $T-S$–biact.

Definition 2.5. Let $\Gamma-{_TA_S}$ be a $\Gamma-T-S$–biact. A relation $\rho\in B(A)$, i.e. $\rho\subseteq A\times A$, is called $\Gamma-T-S$*– compatible* if $(a,b)\in\rho$ implies that $(t\gamma a\beta s,t\gamma b\beta s)\in\rho$ for all $t\in T$, $a,b\in A,s\in S$ and $\gamma,\beta\in\Gamma$. For a $\Gamma-T-S$–biact $\Gamma-{_TA_S}$, an equivalence relation $\rho\in\varepsilon(A)$ which is $\Gamma-T-S$–compatible is called a $\Gamma-T-S$*–congruence*, or simply a Γ*–congruence*, on $\Gamma-{_TA_S}$. We denote the set of all Γ–congruences on $\Gamma-{_TA_S}$ by $\mathbf{Con}(\Gamma-{_TA_S})$. Clearly, under the usual inclusion of relations, $\mathbf{Con}(\Gamma-{_TA_S})$ is a poset.

Remark 2.6. If $|S|=1$, we have a definition of a $\Gamma-T$–compatible relation and a $\Gamma-T$–congruence on $\Gamma-{_TA}$; and if $|T|=1$, we have that of a $\Gamma-S$–compatible relation and a $\Gamma-S$–congruence on $\Gamma-A_S$.

Lemma 2.7. *For a $\Gamma - T - S$–biact $\Gamma -_T A_S$ and a relation $\rho \in B(A)$ (or $\rho \in \varepsilon(A)$), ρ is $\Gamma - T - S$–compatible (or a $\Gamma - T - S$–congruence) on $\Gamma -_T A_S$ if and only if ρ is both $\Gamma - T$–compatible (or a $\Gamma - T$–congruence) on $\Gamma -_T A$ and $\Gamma - S$–compatible (or a $\Gamma - S$–congruence) on $\Gamma - A_S$.*

Proof. We need only to show the assertion for the case $\rho \in B(A)$.

Necessity. Suppose that $\rho \in B(A)$ is $\Gamma - T - S$–compatible on $\Gamma -_T A_S$ and $(a, b) \in \rho$. For every $\gamma, \beta \in \Gamma$ and $t \in T, s \in S$ we have $(t\gamma a, t\gamma b) = (t\gamma a\beta 1, t\gamma b\beta 1) \in \rho$ and $(a\beta s, b\beta s) = (1\gamma a\beta s, 1\gamma b\beta s) \in \rho$ which means that ρ is both $\Gamma - T$–compatible and $\Gamma - S$–compatible.

Sufficiency. Let $\rho \in B(A)$ be both $\Gamma - T$–compatible and $\Gamma - S$–compatible on $\Gamma -_T A_S$, $(a, b) \in \rho$, $t \in T$, $s \in S$ and $\gamma, \beta \in \Gamma$. Then $(t\gamma a, t\gamma b) \in \rho$ by $\Gamma - T$–compatibility, and therefore $((t\gamma a)\beta s, (t\gamma b)\beta s) \in \rho$ by $\Gamma - S$–compatibility. Hence, ρ is $\Gamma - T - S$–compatible on $\Gamma -_T A_S$. ∎

Definition 2.8. Let $\Gamma -_T A_S$ be a Γ–biact and $\rho \in B(A)$. The relation

$$\rho^c := \{(t\gamma a_1\beta s, t\gamma a_2\beta s) \in A \times A \mid t \in T,\ (a_1, a_2) \in \rho,\ s \in S,\ \gamma, \beta \in \Gamma\}$$

is called the $\Gamma - T - S$*–compatible closure* of ρ. The unique smallest $\Gamma - T - S$–congruence on ${}_T A_S$ containing $\rho \in B(A)$ will be denoted by $\rho^\#$ and called the Γ*–congruence closure* of ρ.

Proposition 2.9. *Let $\rho, \sigma \in B(A)$ for a Γ–biact $\Gamma -_T A_S$. Then*

(1) $\rho \subseteq \rho^c$.
(2) $(\rho^c)^{-1} = (\rho^{-1})^c$.
(3) $\rho \subseteq \sigma$ *implies that* $\rho^c \subseteq \sigma^c$.
(4) $(\rho^c)^c = \rho^c$.
(5) $(\rho \bigcup \sigma)^c = \rho^c \bigcup \sigma^c$.
(6) $\rho = \rho^c$ *if and only if* ρ *is* $\Gamma - T - S$*–compatible.*

Proof. (1) Take $(a_1, a_2) \in \rho$. Then $(a_1, a_2) = (1\gamma a_1\beta 1, 1\gamma a_2\beta 1) \in \rho^c$ for all $\gamma, \beta \in \Gamma$. Hence, $\rho \subseteq \rho^c$.

(2) Take $(a_1'', a_2'') \in (\rho^c)^{-1}$. So $(a_2'', a_1'') \in \rho^c$ and then $a_2'' = t'\gamma' a_2'\beta' s'$, $a_1'' = t'\gamma' a_1'\beta' s'$ for some $t' \in T$, $s \in S, \gamma', \beta' \in \Gamma$ and $(a_2', a_1') \in \rho$ whence $(a_1', a_2') \in \rho^{-1} \subseteq (\rho^{-1})^c$. Therefore, $a_1' = t\gamma a_1\beta s$, $a_2' = t\gamma a_2\beta s$ for some $t \in T$, $s \in S$, $\gamma, \beta \in \Gamma$ and $(a_1, a_2) \in \rho^{-1}$. Hence, $a_1'' = (t'\gamma' t)\gamma a_1\beta(s\beta' s')$ and $a_2'' = (t'\gamma' t)\gamma a_2\beta(s\beta' s')$ that $t'\gamma' t \in T, s\beta' s' \in S$, i.e. $(a_1'', a_2'') \in (\rho^{-1})^c$. Hence, $(\rho^c)^{-1} \subseteq (\rho^{-1})^c$. Similarly, $(\rho^{-1})^c \subseteq (\rho^c)^{-1}$. Therefore, $(\rho^c)^{-1} = (\rho^{-1})^c$.

(3) Let $\rho \subseteq \sigma$. Take $(a_1'', a_2'') \in \rho^c$. Then $(a_1'', a_2'') = (t'\gamma' a'_1\beta' s', t'\gamma' a'_2\beta' s')$ for some $t' \in T, s' \in S, \gamma', \beta' \in \Gamma$ and $(a_1', a_2') \in \rho$. Therefore, $(a_1', a_2') \in \sigma$ which implies that $(a_1'', a_2'') \in \sigma^c$. Hence, $\rho^c \subseteq \sigma^c$.

(4) By (1), $\rho^c \subseteq (\rho^c)^c$. Conversely, let $(a_1'', a_2'') \in (\rho^c)^c$. Then $(a_1'', a_2'') = (t'\gamma' a'_1\beta' s', t'\gamma' a'_2\beta' s')$ for some $t' \in T, s' \in S, \gamma', \beta' \in \Gamma$ and $(a_1', a_2') \in \rho^c$. Then $(a_1', a_2') = (t\gamma a_1\beta s, t\gamma a_2\beta s)$ for some $t \in T, s \in S, \gamma, \beta \in \Gamma$,$(a_1, a_2) \in \rho$. Hence, $a''_1 = (t'\gamma' t)\gamma a_1\beta(s\beta' s')$ and $a''_2 = (t'\gamma' t)\gamma a_2\beta(s\beta' s')$, i.e. $(a_1'', a_2'') \in \rho^c$. Hence, $(\rho^c)^c \subseteq \rho^c$. Therefore, $(\rho^c)^c = \rho^c$.

(5) Using (3), we have $\rho^c \subseteq (\rho\bigcup\sigma)^c$ and $\sigma^c \subseteq (\rho\bigcup\sigma)^c$, and therefore $\rho^c\bigcup\sigma^c \subseteq (\rho\bigcup\sigma)^c$. Conversely, suppose that $(a_1', a_2') \in (\rho\bigcup\sigma)^c$. Then $a_1' = t\gamma a_1\beta s$,$a_2' = t\gamma a_2\beta s$ for some $t \in T, s \in S, \gamma, \beta \in \Gamma$ and $(a_1, a_2) \in \rho\bigcup\sigma$. Thus, $(a_1, a_2) \in \rho$ or $(a_1, a_2) \in \sigma$, and hence $(a_1', a_2') \in \rho^c$ or $(a_1', a_2') \in \sigma^c$. Thus, $(a_1', a_2') \in \rho^c\bigcup\sigma^c$. Hence, $(\rho\bigcup\sigma)^c \subseteq \rho^c\bigcup\sigma^c$. Therefore, $(\rho\bigcup\sigma)^c = \rho^c\bigcup\sigma^c$.

(6) Let first $\rho = \rho^c$. Then $(a_1, a_2) \in \rho$ implies that $(t\gamma a_1\beta s, t\gamma a_2\beta s) \in \rho^c = \rho$, for all $t \in T, s \in S$ and $\gamma, \beta \in \Gamma$. Thus, ρ is $\Gamma - T - S$–compatible. Conversely, if ρ is a $\Gamma - T - S$–compatible relation and $(a_1', a_2') \in \rho^c$, then $a_1' = t\gamma a_1\beta s$, $a_2' = t\gamma a_2\beta s$ for some $t \in T, s \in S$, $(a_1, a_2) \in \rho, \gamma, \beta \in \Gamma$. Therefore, $(a_1', a_2') = (t\gamma a_1\beta s, t\gamma a_2\beta s) \in \rho$ by $\Gamma - T - S$–compatibility. Thus, $\rho^c \subseteq \rho$. But, by (1), $\rho \subseteq \rho^c$. Therefore, $\rho = \rho^c$. ∎

Lemma 2.10. *Let $\Gamma - {}_TA_S$ be a Γ–biact. If the relation $\rho \in B(A)$ is $\Gamma - T - S$–compatible, then ρ^n is also $\Gamma - T - S$–compatible for any $n \in \mathbb{N}$.*

Proof. Let $(a_1, a_2) \in \rho^n$. Then there exist $b_1, b_2, \ldots, b_{n-1} \in A$ such that $(a_1, b_1), (b_1, b_2), ..., (b_{n-1}, a_2) \in \rho$. Since ρ is $\Gamma{-}T{-}S$–compatible, $(t\gamma a_1\beta s, t\gamma b_1\beta s)$, $(t\gamma b_1\beta s, t\gamma b_2\beta s), ..., (t\gamma b_{n-1}\beta s, t\gamma a_2\beta s) \in \rho$ for all $t \in T$, $s \in S$ and $\gamma, \beta \in \Gamma$, and so $(t\gamma a_1\beta s, t\gamma a_2\beta s) \in \rho^n$. ∎

Definition 2.11. Let Γ–${}_TA_S$ be a Γ–biact and $\rho \in B(A)$. If $(a_1', a_2') \in (\rho\bigcup\rho^{-1})^c$, or equivalently, $a_1' = t\gamma a_1\beta s$ and $a_2' = t\gamma a_2\beta s$ for some $t \in T, s \in S$, $\gamma, \beta \in \Gamma$ and $(a_1, a_2) \in \rho$ or $(a_2, a_1) \in \rho$, then we say that a_1' *is connected with* a_2' *by an elementary* $\Gamma - T - S - \rho$*–transition*, and use the notation $a_1' \to a_2'$.

Theorem 2.12. *Let Γ–${}_TA_S$ be a Γ–biact and $\rho \in B(A)$. Then $\rho^\# = (\rho^c)^e$.*

Proof. Obviously, $\rho \subseteq \rho^c \subseteq (\rho^c)^e$. We show that $(\rho^c)^e \in \mathbf{Con}(\Gamma - {}_TA_S)$. In view of [8, Theorem I.1.6], $(\rho^c)^e = \theta^\infty$ where $\theta = \rho^c\bigcup(\rho^c)^{-1}\bigcup\Delta_A$. Let $(a_1, a_2) \in (\rho^c)^e$. Then $(a_1, a_2) \in \theta^n$ for some $n \in \mathbb{N}$. Using Proposition 2.9(2) and (5), and the clear fact $\Delta_A^c = \Delta_A$, we get

$$\theta = \rho^c\bigcup(\rho^{-1})^c\bigcup\Delta_A^c = (\rho\bigcup\rho^{-1}\bigcup\Delta_A)^c = \theta^c.$$

Therefore, by Proposition 2.9(6), θ is $\Gamma - T - S$–compatible and then so is θ^n by Lemma 2.10. Thus, $(t\gamma a_1\beta s, t\gamma a_2\beta s) \in \theta^n \subseteq (\rho^c)^e$ for every $t \in T$, $s \in S$, $\gamma, \beta \in \Gamma$. Hence, $(\rho^c)^e$ is a Γ– congruence on Γ–${}_TA_S$ containing ρ. Let σ be a Γ–congruence on Γ–${}_TA_S$ containing ρ. Then, by using Proposition 2.9(3) and (6), we get $\rho^c \subseteq \sigma^c = \sigma$ and so $(\rho^c)^e \subseteq \sigma^e = \sigma$. Hence, $\rho^\# = (\rho^c)^e$. ∎

Corollary 2.13. *Let $\rho \in B(A)$ for a Γ–biact $\Gamma{-}{}_TA_S$, $a_1, a_2 \in A$. Then $(a_1, a_2) \in \rho^\#$ if and only if $a_1 = a_2$ or for some $n \in \mathbb{N}$ there is a sequence $a_1 = z_1 \to z_2 \to \cdots \to z_n = a_2$ of elementary $\Gamma - T - S - \rho$–transitions connecting a_1 to a_2.*

Proof. Using Theorem 2.12, $(a_1, a_2) \in \rho^\#$ if and only if $(a_1, a_2) \in (\rho^c)^e$, and by Corollary 1.1, if and only if $a_1 = a_2$ or for some $n \in \mathbb{N}$ there exists a sequence of elements $a_1 = z_1, z_2, ..., z_n = a_2$ in A such that for every $i \in \{1, 2, ..., n{-}1\}$,

$(z_i, z_{i+1}) \in \rho^c \bigcup (\rho^c)^{-1} = (\rho \bigcup \rho^{-1})^c$ by Proposition 2.9(2) and (5) so that $z_i \to z_{i+1}$, which gives the required sequence $a_1 = z_1 \to z_2 \to \cdots \to z_n = a_2$ of elementary $\Gamma - T - S - \rho$–transitions. ∎

In what follows we shall often use a more explicit version of Corollary 2.13 in the case of $|T| = 1$, i.e. in the case of right $\Gamma - S$–acts.

Lemma 2.14. *Let $\Gamma - A_S$ be a right $\Gamma - S$–act and $\rho \in B(A)$. Then for any $a, b \in A$, $(a, b) \in \rho^{\#}$ if and only if $a = b$ or there exist $p_1, ..., p_n, q_1, ..., q_n \in A$, $w_1, ..., w_n \in S$, $\gamma_1, \gamma_2, ..., \gamma_n \in \Gamma$, where for $i = 1, ..., n$, $(p_i, q_i) \in \rho$ or $(q_i, p_i) \in \rho$, such that*

$$a = p_1\gamma_1 w_1, q_2\gamma_2 w_2 = p_3\gamma_3 w_3, ..., q_n\gamma_n w_n = b.$$
$$q_1\gamma_1 w_1 = p_2\gamma_2 w_2, q_3\gamma_3 w_3 = p_4\gamma_4 w_4, ...$$

Proof. Using Corollary 2.13, we have $(a, b) \in \rho^{\#}$ if and only if $a = b$ or for some $n \in \mathbb{N}$ there is a sequence $a = z_1 \to z_2 \to \cdots \to z_n = b$ of elementary $\Gamma - S - \rho$–transitions connecting a to b. If $a = b$, it is clear. If $a = z_1 \to z_2 \to \cdots \to z_n = b$, then $a = z_1 = p_1\gamma_1 w_1$, $z_2 = q_1\gamma_1 w_1$, such that $(p_1, q_1) \in \rho$ or $(q_1, p_1) \in \rho$ and $z_2 = p_2\gamma_2 w_2 = q_1\gamma_1 w_1$, $z_3 = q_2\gamma_2 w_2$ such that $(p_2, q_2) \in \rho$ or $(q_2, p_2) \in \rho$. Continuing the same way, we get $q_{n-1}\gamma_{n-1}w_{n-1} = p_n\gamma_n w_n$ and $q_n\gamma_n w_n = z_n = b$, for some $p_1, ..., p_n, q_1, ..., q_n \in A$, $w_1, ..., w_n \in S$, $\gamma_1, \gamma_2, ..., \gamma_n \in \Gamma$. ∎

Proposition 2.15. *Let $\varepsilon \in \varepsilon(A)$ for a Γ–biact Γ–${}_TA_S$. Then*

$$\varepsilon^b := \{(a_1, a_2) \in A \times A \mid (t\gamma a_1\beta s, t\gamma a_2\beta s) \in \varepsilon \text{ for all } t \in T, s \in S, \gamma, \beta \in \Gamma\}$$

is the largest Γ–congruence on Γ–${}_TA_S$ contained in ε.

Proof. Taking $t = 1$ and $s = 1$ we see that $\varepsilon^b \subseteq \varepsilon$. Clearly, ε^b is an equivalence relation. If $(a_1, a_2) \in \varepsilon^b$ and $t' \in T$, $s' \in S$, $\gamma', \beta' \in \Gamma$, then we have

$$(t\gamma(t'\gamma' a_1\beta' s')\beta s, t\gamma(t'\gamma' a_2\beta' s')\beta s) = ((t\gamma t')\gamma' a_1\beta'(s'\beta s), \ (t\gamma t')\gamma' a_2\beta'(s'\beta s)) \in \varepsilon$$

for all $t \in T, s \in S$, $\gamma, \beta \in \Gamma$ and so $(t'\gamma' a_1\beta' s', t'\gamma' a_2\beta' s') \in \varepsilon^b$. This means that $\varepsilon^b \in \mathbf{Con}(\Gamma - {}_TA_S)$. If $\sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$ and $\sigma \subseteq \varepsilon$, then for all $a_1, a_2 \in A$, let $(a_1, a_2) \in \sigma$ so that for all $t \in T$, $s \in S$, $\gamma, \beta \in \Gamma$ we have $(t\gamma a_1\beta s, t\gamma a_2\beta s) \in \sigma \subseteq \varepsilon$. Thus, $(a_1, a_2) \in \varepsilon^b$ and then $\sigma \subseteq \varepsilon^b$, i.e. ε^b is the largest Γ–congruence on Γ–${}_TA_S$ contained in ε. ∎

Remark 2.16. [8] For a $T - S$–biact ${}_TA_S$, the poset $\mathbf{Con}({}_TA_S)$ is a lattice and for any $\rho, \sigma \in \mathbf{Con}({}_TA_S)$, $\rho \wedge \sigma$ is $\rho \bigcap \sigma$ and $\rho \vee \sigma$ is $(\rho \bigcup \sigma)^{\#} = (\rho \bigcup \sigma)^e$ where $(\rho \bigcup \sigma)^{\#}$ denotes the $T - S$–congruence closure of $\rho \bigcup \sigma$. Similarly, the poset $\varepsilon(A)$ of all equivalence relations on the set A as a subposet of $B(A)$ is also a lattice and for any $\rho, \sigma \in \varepsilon(A)$, $\rho \wedge \sigma$ is $\rho \bigcap \sigma$ and $\rho \vee \sigma$ is $(\rho \bigcup \sigma)^e$.

Proposition 2.17. *Let Γ–${}_TA_S$ be a Γ–T–S–biact. Then the poset $\mathbf{Con}(\Gamma - {}_TA_S)$ is a lattice and for any $\rho, \sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$, $\rho \wedge \sigma = \rho \bigcap \sigma$ and $\rho \vee \sigma = (\rho \bigcup \sigma)^{\#} = (\rho \bigcup \sigma)^e$.*

Proof. Let $\rho,\sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$. It is easily seen that $\rho\bigcap\sigma$ and $(\rho\bigcup\sigma)^{\#}$ are the meet and the join of ρ,σ in $\mathbf{Con}(\Gamma - {}_TA_S)$, respectively. Then $\mathbf{Con}(\Gamma - {}_TA_S)$ is a lattice. It remains to show that $(\rho\bigcup\sigma)^{\#} = (\rho\bigcup\sigma)^e$. Applying Theorem 2.12, we have $(\rho\bigcup\sigma)^{\#} = ((\rho\bigcup\sigma)^c)^e = (\rho^c\bigcup\sigma^c)^e = (\rho\bigcup\sigma)^e$ in which the last two identities follow from Proposition 2.9(5) and (6). ∎

Theorem 2.18. *Let $\rho,\sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$ for a $\Gamma - T - S$–biact $\Gamma - {}_TA_S$. Then $\rho \vee \sigma = (\rho\bigcup\sigma)^{\infty} = (\rho\circ\sigma)^{\infty}$. This means that if $a_1, a_2 \in A$, then $(a_1, a_2) \in \rho\vee\sigma$ if and only if for some $n \in \mathbb{N}$ there exist elements $b_1, b_2, ..., b_{n-1} \in A$ such that $(a_1, b_1) \in \tau_1, (b_1, b_2) \in \tau_2, ..., (b_{n-1}, a_2) \in \tau_n$, where $\tau_i \in \{\rho,\sigma\}$, $i = 1, ..., n$.*

Proof. Consider any $\rho,\sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$. By using Proposition 2.17, we have $\rho\vee\sigma = (\rho\bigcup\sigma)^e = [(\rho\bigcup\sigma)\bigcup(\rho\bigcup\sigma)^{-1}\bigcup\Delta_A]^{\infty} = (\rho\bigcup\sigma)^{\infty}$ of which last equality follows from the symmetry and reflexivity properties of $\rho\bigcup\sigma$. We claim that $(\rho\bigcup\sigma)^{\infty} = (\rho\circ\sigma)^{\infty}$. To this end, first note that since ρ,σ are reflexive, $\rho,\sigma \subseteq \rho\circ\sigma \subseteq (\rho\circ\sigma)^{\infty}$ and so $\rho\bigcup\sigma \subseteq (\rho\circ\sigma)^{\infty}$. This implies that $(\rho\bigcup\sigma)^{\infty} \subseteq (\rho\circ\sigma)^{\infty}$. For the reverse inclusion, we have $\rho,\sigma \subseteq \rho\bigcup\sigma \subseteq (\rho\bigcup\sigma)^{\infty}$ so that $\rho\circ\sigma \subseteq (\rho\bigcup\sigma)^{\infty}\circ(\rho\bigcup\sigma)^{\infty} \subseteq (\rho\bigcup\sigma)^{\infty}$ which the last inclusion follows from the transitivity property of $(\rho\bigcup\sigma)^{\infty}$. Then $(\rho\circ\sigma)^{\infty} \subseteq (\rho\bigcup\sigma)^{\infty}$, as claimed. The second assertion is an easy consequence of the identity $\rho\vee\sigma = (\rho\bigcup\sigma)^{\infty}$ in the first one. ∎

Corollary 2.19. *For a $\Gamma - T - S$–biact $\Gamma - {}_TA_S$, if $\rho,\sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$ are such that $\rho\circ\sigma = \sigma\circ\rho$, then $\rho\vee\sigma = \rho\circ\sigma$.*

Proof. By the assumption, $(\rho\circ\sigma)^i = \rho^i\circ\sigma^i$ for all $i \in \mathbb{N}$. On the other hand, since ρ,σ are reflexive and transitive, $\rho^i = \rho, \sigma^i = \sigma$ for all $i \in \mathbb{N}$. Then $(\rho\circ\sigma)^i = \rho\circ\sigma$. Hence, using Theorem 2.18, we get $\rho\vee\sigma = (\rho\circ\sigma)^{\infty} = \bigcup_{i=1}^{\infty}(\rho\circ\sigma)^i = \rho\circ\sigma$. ∎

3. Green's relations on Γ–biacts

This section is devoted to study Green's relations on Γ–biacts.

Definition 3.1. [8] Let ${}_TA_S$ be a biact. The *Green's equivalences* on ${}_TA_S$ are defined by the following rules:

$(a_1, a_2) \in {}_T\mathcal{L}$ if and only if $Ta_1 = Ta_2$,
$(a_1, a_2) \in \mathcal{R}_S$ if and only if $a_1S = a_2S$,
$(a_1, a_2) \in {}_T\mathcal{J}_S$ if and only if $Ta_1S = Ta_2S$,

for all $a_1, a_2 \in A$. Further,

${}_T\mathcal{H}_S := {}_T\mathcal{L}\wedge\mathcal{R}_S = {}_T\mathcal{L}\bigcap\mathcal{R}_S$,
${}_T\mathcal{D}_S := {}_T\mathcal{L}\vee\mathcal{R}_S = ({}_T\mathcal{L}\bigcup\mathcal{R}_S)^e$.

Definition 3.2. Let $\Gamma - {}_TA_S$ be a Γ–biact. We define *Green's relations* on $\Gamma - {}_TA_S$ as follows:

$(a_1, a_2) \in {}_T\mathcal{L}$ if and only if $T\Gamma a_1 = T\Gamma a_2$,
$(a_1, a_2) \in \mathcal{R}_S$ if and only if $a_1\Gamma S = a_2\Gamma S$,

$(a_1, a_2) \in {}_T\mathcal{J}_S$ if and only if $T\Gamma a_1 \Gamma S = T\Gamma a_2 \Gamma S$,
for all $a_1, a_2 \in A$. Note that it is clear that ${}_T\mathcal{L}, \mathcal{R}_S$ and ${}_T\mathcal{J}_S$ are equivalence relations on the set A. Thus, in view of Remark 2.16, we also define
${}_T\mathcal{H}_S := {}_T\mathcal{L} \wedge \mathcal{R}_S = {}_T\mathcal{L} \bigcap \mathcal{R}_S \in \varepsilon(A)$,
${}_T\mathcal{D}_S := {}_T\mathcal{L} \vee \mathcal{R}_S = ({}_T\mathcal{L} \bigcup \mathcal{R}_S)^e \in \varepsilon(A)$.

Lemma 3.3. *In terms of the previous definition we have ${}_T\mathcal{L} \in \mathbf{Con}(\Gamma - A_S)$ and $\mathcal{R}_S \in \mathbf{Con}(\Gamma - {}_TA)$.*

Proof. Let $a_1, a_2 \in A, (a_1, a_2) \in {}_T\mathcal{L}$. Take $s \in S$ and $\gamma \in \Gamma$. Then $T\Gamma a_1 = T\Gamma a_2$ and so $T\Gamma\,(a_1\gamma s) = (T\Gamma a_1)\,\gamma s = (T\Gamma a_2)\,\gamma s = T\Gamma\,(a_2\gamma s)$, i.e. $(a_1\gamma s, a_2\gamma s) \in {}_T\mathcal{L}$. This means that ${}_T\mathcal{L}$ is a $\Gamma - S-$ congruence on A_S. The proof for $\mathcal{R}_S$ is similar. ■

Theorem 3.4. *Let $\Gamma - {}_TA_S$ be a Γ–biact. If $\rho \in \mathbf{Con}(\Gamma - {}_TA)$ and $\rho \subseteq \mathcal{R}_S$, $\lambda \in \mathbf{Con}(\Gamma - A_S)$ and $\lambda \subseteq {}_T\mathcal{L}$, then $\lambda \circ \rho = \rho \circ \lambda$. In particular, ${}_T\mathcal{L} \circ \mathcal{R}_S = \mathcal{R}_S \circ {}_T\mathcal{L}$.*

Proof. Let $(a_1, a_2) \in \lambda \circ \rho$. So there exists $a_3 \in A$ with $a_1\lambda a_3\rho a_2$. Since $\lambda \subseteq {}_T\mathcal{L}$ and $\rho \subseteq \mathcal{R}_S$, we get $T\Gamma a_1 = T\Gamma a_3$, $a_3\Gamma S = a_2\Gamma S$. Then $a_3 = t_1\gamma_1 a_1$, $a_2 = a_3\beta_3 s_3$, $a_1 = t_3\gamma_3 a_3$ and $a_3 = a_2\beta_2 s_2$ for some $t_1, t_3 \in T$, $s_2, s_3 \in S$ and $\gamma_1, \gamma_3, \beta_2, \beta_3 \in \Gamma$. Let $d = a_1\beta_3 s_3$. Then $d = t_3\gamma_3 a_3\beta_3 s_3 = t_3\gamma_3 a_2$. Now $a_3\rho a_2$ implies that $(t_3\gamma_3 a_3)\,\rho\,(t_3\gamma_3 a_2)$. Thus, $a_1\rho d$. Also, $a_1\lambda a_3$ gives that $(a_1\beta_3 s_3)\,\lambda\,(a_3\beta_3 s_3)$ and then $d\lambda a_2$. Hence, $a_1\,(\rho \circ \lambda)\,a_2$ which follows that $\lambda \circ \rho \subseteq \rho \circ \lambda$. Analogously, the reverse inclusion also holds. Since ρ and λ are arbitrary in ${}_T\mathcal{L}$ and $\mathcal{R}_S$ respectively, using Lemma 3.3, ${}_T\mathcal{L} \circ \mathcal{R}_S = \mathcal{R}_S \circ {}_T\mathcal{L}$. ■

Remark 3.5. Since ${}_T\mathcal{L}$ and $\mathcal{R}_S$ commute by Theorem 3.4, it follows from Corollary 2.19 that ${}_T\mathcal{D}_S = {}_T\mathcal{L} \vee \mathcal{R}_S = {}_T\mathcal{L} \circ \mathcal{R}_S = \mathcal{R}_S \circ {}_T\mathcal{L}$. It is clear that $|T| = 1$ implies that ${}_T\mathcal{J}_S = \mathcal{R}_S$ and $|S| = 1$ implies that ${}_T\mathcal{J}_S = {}_T\mathcal{L}$. Moreover, we have ${}_T\mathcal{D}_S \subseteq {}_T\mathcal{J}_S$. Indeed, first note that ${}_T\mathcal{L} \subseteq {}_T\mathcal{J}_S$ and $\mathcal{R}_S \subseteq {}_T\mathcal{J}_S$ whence ${}_T\mathcal{L} \bigcup \mathcal{R}_S \subseteq {}_T\mathcal{J}_S$. Since ${}_T\mathcal{J}_S \in \varepsilon(A)$, ${}_T\mathcal{D}_S = {}_T\mathcal{L} \vee \mathcal{R}_S = ({}_T\mathcal{L} \bigcup \mathcal{R}_S)^e \subseteq ({}_T\mathcal{J}_S)^e = {}_T\mathcal{J}_S$.

Here we generalize the notion of periodic semigroup to the Γ–semigroups which is needed in the sequel.

Definition 3.6. [8] A *monogenic* (*cyclic*) semigroup is a semigroup generated by a singleton. A semigroup is called *periodic* if all of its monogenic subsemigroups are finite.

Definition 3.7. Let S be a Γ–semigroup and $\gamma \in \Gamma$. An element $e \in S$ is called a *γ–idempotent* if $e_\gamma^2 = e$ where e_γ^2 means $e\gamma e$. A subset T of a S is called a *γ–subsemigroup* of S if for every $x, y \in T, x\gamma y \in T$. A Γ–semigroup S is said to be *periodic* if all of its monogenic γ–subsemigroups are finite for every $\gamma \in \Gamma$. Here, a monogenic γ–subsemigroup of S generated by $s \in S$ is denoted by $\langle s\rangle_\gamma$, and $\langle s\rangle_\gamma = \left\{s_\gamma^n \mid n \in \mathbb{N}\right\}$ where $s_\gamma^1 = s,\ s_\gamma^2 = s\gamma s\ , ..., s_\gamma^n = s_\gamma^{n-1}\gamma s$.

Lemma 3.8. [8] *Every finite semigroup includes an idempotent element.*

Lemma 3.9. *Among the powers s_γ^n of elements of a periodic Γ–semigroup S for $\gamma \in \Gamma$, there is a γ–idempotent.*

Proof. Let $s \in S$ and $\gamma \in \Gamma$. Consider the monogenic γ–subsemigroup $\langle s\rangle_\gamma$. For every $x, y \in \langle s\rangle_\gamma$, define $xy := x\gamma y \in \langle s\rangle_\gamma$. Then $\langle s\rangle_\gamma$ is made into a semigroup by this operation. Since S is periodic, $\langle s\rangle_\gamma$ is a finite Γ–semigroup and then a finite semigroup. Then, using Lemma 3.8, there is an idempotent element e in the semigroup $\langle s\rangle_\gamma$. Thus, there exists $k \in \mathbb{N}$, $e = s^k_\gamma$. Using the operation, $e = e^2 = e\gamma e = e^2_\gamma$. Then $e = s^k_\gamma$ is a γ–idempotent element of S. ▪

Notation 3.10. Let S be a Γ–semigroup, $s_1, s_2 \in S$, $\gamma, \beta \in \Gamma$. Then we put $(s_1\gamma s_2)^2_\beta := (s_1\gamma s_2)\beta(s_1\gamma s_2)$.

Theorem 3.11. *Let $\Gamma - {}_TA_S$ be a Γ–biact over periodic Γ–semigroups T and S. Then on $\Gamma - {}_TA_S$ we have ${}_T\mathcal{D}_S = {}_T\mathcal{J}_S$.*

Proof. In view of Remark 3.5, it suffices to prove that ${}_T\mathcal{J}_S \subseteq {}_T\mathcal{D}_S$. Assume that $a_1, a_2 \in A$ and $(a_1, a_2) \in {}_T\mathcal{J}_S$, i.e. $T\Gamma a_1\Gamma S = T\Gamma a_2\Gamma S$. Thus, $a_1 = t_2\alpha a_2\lambda s_2$ and $a_2 = t_1\gamma a_1\beta s_1$ for some $s_1, s_2 \in S$, $t_1, t_2 \in T$ and $\alpha, \lambda, \gamma, \beta \in \Gamma$. Then

$$\begin{aligned} a_1 &= t_2\alpha\,(t_1\gamma a_1\beta s_1)\,\lambda s_2 = (t_2\alpha t_1)\,\gamma a_1\beta\,(s_1\lambda s_2) = (t_2\alpha t_1)\,\gamma t_2\alpha a_2\lambda s_2\beta\,(s_1\lambda s_2) \\ &= (t_2\alpha t_1)\,\gamma t_2\alpha t_1\gamma a_1\beta s_1\lambda s_2\beta\,(s_1\lambda s_2) = (t_2\alpha t_1)^2_\gamma\,\gamma a_1\beta\,(s_1\lambda s_2)^2_\beta = \cdots \end{aligned}$$

Analogously, we obtain

$$a_2 = (t_1\gamma t_2)\,\alpha a_2\lambda\,(s_2\beta s_1) = (t_1\gamma t_2)^2_\alpha\,\alpha a_2\lambda\,(s_1\beta s_2)^2_\lambda = \cdots$$

Since T and S are periodic Γ–semigroups, we can find $m \in \mathbb{N}$ such that $(t_2\alpha t_1)^m_\gamma$ is a γ– idempotent by Lemma 3.9. Let now $c = t_1\gamma a_1 \in A$. Then

$$\begin{aligned} a_1 &= (t_2\alpha t_1)^m_\gamma\,\gamma a_1\beta\,(s_1\lambda s_2)^m_\beta = (t_2\alpha t_1)^m_\gamma\,\gamma\,(t_2\alpha t_1)^m_\gamma\,\gamma a_1\beta\,(s_1\lambda s_2)^m_\beta \\ &= (t_2\alpha t_1)^m_\gamma\,\gamma a_1 = \left((t_2\alpha t_1)^{m-1}_\gamma\,\gamma t_2\right)\alpha\,(t_1\gamma a_1) = \left((t_2\alpha t_1)^{m-1}_\gamma\,\gamma t_2\right)\alpha c. \end{aligned}$$

Therefore, $(a_1, c) \in {}_T\mathcal{L}$. Moreover, we have $c\beta s_1 = t_1\gamma a_1\beta s_1 = a_2$, and, using Lemma 3.9, if we choose $n \in \mathbb{N}$ such that $(s_2\beta s_1)^n_\lambda$ is a λ–idempotent, then we get

$$\begin{aligned} c &= t_1\gamma a_1 = t_1\gamma(t_2\alpha t_1)^{n+1}_\gamma\gamma a_1\beta(s_1\lambda s_2)^{n+1}_\beta = (t_1\gamma t_2)^{n+1}_\alpha\alpha(t_1\gamma a_1\beta s_1)\lambda(s_2\beta s_1)^n_\lambda\lambda s_2 \\ &= (t_1\gamma t_2)^{n+1}_\alpha\alpha a_2\lambda(s_2\beta s_1)^{2n}_\lambda\lambda s_2 = ((t_1\gamma t_2)^{n+1}_\alpha\alpha a_2\lambda(s_2\beta s_1)^{n+1}_\lambda)\lambda(s_2\beta s_1)^{n-1}_\beta\lambda s_2 \\ &= a_2\lambda(s_2\beta s_1)^{n-1}_\beta\lambda s_2. \end{aligned}$$

Hence, $(c, a_2) \in \mathcal{R}_S$ and so, using Remark 3.5, $(a_1, a_2) \in {}_T\mathcal{L} \circ \mathcal{R}_S = {}_T\mathcal{D}_S$. ▪

Definition 3.12. Let $\rho \in \mathbf{Con}(\Gamma - {}_TA_S)$ for a Γ–biact $\Gamma - {}_TA_S$. The set $\frac{\Gamma - {}_TA_S}{\rho} = \{[a]_\rho \mid a \in A\}$ with the left $\Gamma - T$–action $t\gamma[a]_\rho := [t\gamma a]_\rho$ and the right $\Gamma - S$–action $[a]_\rho\gamma s := [a\gamma s]_\rho$ for every $t \in T, s \in S$ and $\gamma \in \Gamma$ is clearly a Γ–biact which is called the *factor* Γ*–biact* of $\Gamma - {}_TA_S$ by ρ.

Proposition 3.13. *Let $\Gamma - {}_TA_S$ be a Γ–biact and $\rho \in \mathbf{Con}(\Gamma - {}_TA_S)$. Then*

(i) *If $\rho \subseteq {}_T\mathcal{L}$, then for all $a, b \in A$, $a\ {}_T\mathcal{L}\ b$ if and only if $[a]_\rho\ {}_T\mathcal{L}\ [b]_\rho$ in $\frac{\Gamma - {}_TA_S}{\rho}$.*

(ii) *If* $\rho \subseteq \mathcal{R}_S$*, then for all* $a, b \in A$*,* $a\ \mathcal{R}_S\ b$ *if and only if* $[a]_\rho\ \mathcal{R}_S\ [b]_\rho$ *in* $\frac{\Gamma - _TA_S}{\rho}$.

(iii) *If* $\rho \subseteq {}_T\mathcal{H}_S$*, then for all* $a, b \in A$*,* $a\ {}_T\mathcal{H}_S\ b$ *if and only if* $[a]_\rho\ {}_T\mathcal{H}_S\ [b]_\rho$ *in* $\frac{\Gamma - _TA_S}{\rho}$.

Proof. (i) Let $a, b \in A$. If $a\ {}_T\mathcal{L}\ b$, then there exist $t, u \in T$ and $\gamma, \beta \in \Gamma$ such that $a = t\gamma b$ and $b = u\beta a$. Then $[a]_\rho = [t\gamma b]_\rho = t\gamma[b]_\rho$ and $[b]_\rho = [u\beta a]_\rho = u\beta[a]_\rho$. Therefore, $T\Gamma[a]_\rho = T\Gamma[b]_\rho$ which means that $[a]_\rho\ {}_T\mathcal{L}\ [b]_\rho$ in $\frac{\Gamma - _TA_S}{\rho}$. Conversely, assume that $[a]_\rho\ {}_T\mathcal{L}\ [b]_\rho$ in $\frac{\Gamma - _TA_S}{\rho}$. Then $T\Gamma[a]_\rho = T\Gamma[b]_\rho$ so that there exist $t, u \in T$ and $\gamma, \beta \in \Gamma$ such that $[a]_\rho = t\gamma[b]_\rho = [t\gamma b]_\rho$ and $[b]_\rho = u\beta[a]_\rho = [u\beta a]_\rho$, i.e. $a\rho\,(t\gamma b)$ and $b\rho\,(u\beta a)$. Since $\rho \subseteq {}_T\mathcal{L}$, $a\ {}_T\mathcal{L}\ t\gamma b$ and $b\ {}_T\mathcal{L}\ u\beta a$. Then $T\Gamma a = T\Gamma\,(t\gamma b)$ and $T\Gamma b = T\Gamma\,(u\beta a)$. This implies that $a \in T\Gamma\,(t\gamma b) = (T\Gamma t)\gamma b \subseteq T\Gamma b$ and $b \in T\Gamma\,(u\beta a) = (T\Gamma u)\beta b \subseteq T\Gamma a$. Therefore, $T\Gamma a = T\Gamma b$, i.e. $a\ {}_T\mathcal{L}\ b$.

(ii) It is similar to (i).

(iii) Let $a, b \in A$. Assume that $\rho \subseteq {}_T\mathcal{H}_S$. Since ${}_T\mathcal{H}_S = {}_T\mathcal{L} \bigcap \mathcal{R}_S$, $\rho \subseteq {}_T\mathcal{L}$ and $\rho \subseteq \mathcal{R}_S$. Using (i) and (ii), $a\ {}_T\mathcal{H}_S\ b$ if and only if $a\ {}_T\mathcal{L}\ b$ and $a\ \mathcal{R}_S\ b$ if and only if $[a]_\rho\ {}_T\mathcal{L}\ [b]_\rho$ and $[a]_\rho\ \mathcal{R}_S\ [b]_\rho$ if and only if $[a]_\rho\ {}_T\mathcal{H}_S\ [b]_\rho$ in $\frac{\Gamma - _TA_S}{\rho}$. ∎

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