CHARACTERISTIC SEMIMODULES

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Abstract. In the paper, a particular class of semimodules typical for additively idempotent semirings possessing at least two right multiplicatively absorbing elements is investigated.

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The present note is a direct continuation of [1] and the reader is fully referred to [1] as concerns notation, terminology and further references. Here, we introduce and study a certain type of (left) semimodules that are typical for additively idempotent semirings possessing at least two right multiplicatively absorbing elements.

1. Preliminaries

Let $A = A(*)$ be a groupoid. An element $a \in A$ is called *left (right) neutral* if $a*x = x$ ($x*a = x$) for all $x \in A$, and *left (right) absorbing* if $a*x = a$ ($x*a = a$) for all $x \in A$. If $A = A(+)$ then $0_A \in A$ ($o_A \in A$) means that 0_A (o_A) is (the

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unique) left and right neutral (absorbing) element of $A(+)$ and $0_A \notin A$ ($o_A \notin A$) denotes the fact that $A(+)$ has no (left and right) neutral (absorbing) element.

Similarly, if $A = A(\cdot)$, then $1_A \in A$ means that 1_A is (the unique) left and right neutral element of $A(\cdot)$.

A semiring is a non-empty set equipped with two associative binary operations that are usually written as addition and multiplication. The addition is commutative and the multiplication distributes over the addition. Given a semiring S, a (left S-)semimodule (${}_SM =$) M is a commutative semigroup $M(+)$ together with a scalar multiplication $S \times M \to M$ such that $(a+b)x = ax + bx$, $a(x+y) = ax + ay$ and $a(bx) = (ab)x$ for all $a, b \in S$ and $x, y \in M$. If S is a semiring then $R = \underline{R}(S) = \{\, a \in S \mid Sa = \{a\} \,\}$ denotes the set of right multiplicatively absorbing elements. If $a \in \underline{R}(S)$ then $a + a = aa + aa = (a+a)a = a$ and $a(b+b) = ab + ab = ab$ for every $b \in S$. Consequently, the semiring S is additively idempotent, provided that the right semimodule $\underline{R}(S)_S$ is faithful, i.e., for all $a, b \in S$, $a \neq b$, there is at least one $x \in \underline{R}(S)$ with $xa \neq xb$.

The semiring S is called (*congruence-*)*simple* if it has just two congruence relations (then these are id_S and $S \times S$ and $|S| \geq 2$). If S is simple and the ideal $\underline{R}(S)$ contains at least two elements then $\underline{R}(S)_S$ is faithful and S is additively idempotent (see [1, 7.1]).

Throughout the paper, all semirings and semimodules are assumed to be additively idempotent. It means that the respective additive semigroups $M(+)$ are semilattices, where the basic order relation is given by $\alpha \leq \beta$ iff $\alpha + \beta = \beta$.

2. Characteristic semimodules (a)

Let S be a non-trivial semiring and M be a (left S-)semimodule. The semimodule M will be called

- *precharacteristic* if $|M| \geq 2$, $0_M \in M$, $o_M \in M$, $S0_M = \{0_M\}$ and $So_M = \{o_M\}$ (in this case, we put $N = M \setminus \{o_M\}$ and $L = M \setminus \{0_M\}$);
- *characteristic* if M is faithful (i.e., for all $a, b \in S$, $a \neq b$, there is $x \in M$ with $ax \neq bx$), precharacteristic and there is a mapping $\underline{\varepsilon} : N \to S$ such that $\underline{\varepsilon}(x)y = 0_M$ and $\underline{\varepsilon}(x)z = o_M$ for all $x, y, z \in M$, $y \leq x$, $z \nleq x$.

In the rest of this section, assume that M is a characteristic semimodule.

2.1 Lemma. *$|M| \geq 3$ and $|N| \geq 2$.*

Proof. Since $|S| \geq 2$ and M is faithful, there are $a, b \in S$ and $x \in M$ such that $ax \neq bx$. Then $x \neq 0_M$, $x \neq o_M$ and $|M| \geq 3$. ∎

2.2 Proposition. *The semimodule M is simple.*

Proof. Let α be a congruence of the semimodule M. If $(0_M, o_M) \in \alpha$ then $\alpha = M \times M$. If $(x, o_M) \in \alpha$ for at least one $x \in N$ then $(0_M, o_M) = (\underline{\varepsilon}(x)x, \underline{\varepsilon}(x)o_M) \in \alpha$. If $x, y \in N$ are such that $x \nleq y$ and $(x, y) \in \alpha$ then $(0_M, o_M) = (\underline{\varepsilon}(y)y, \underline{\varepsilon}(y)x) \in \alpha$. ∎

2.3 Proposition. *$\underline{\varepsilon}$ is an injective mapping of N into $R = \underline{R}(S)$ and $|R| \geq 2$.*

Proof. We have $a\underline{\varepsilon}(x)y = \underline{\varepsilon}(x)y$ for all $a \in S$, $x \in N$ and $y \in M$. Since M is faithful, we get $a\underline{\varepsilon}(x) = \underline{\varepsilon}(x)$ and $\underline{\varepsilon}(x) \in R$. If $\underline{\varepsilon}(x_1) = \underline{\varepsilon}(x_2)$ then $0_M = \underline{\varepsilon}(x_1)x_1 = \underline{\varepsilon}(x_2)x_1$ and so $x_1 \leq x_2$. Symmetrically, $x_2 \leq x_1$ and we get $x_1 = x_2$. ∎

2.4 Lemma.

(i) $R = \{\, a \in S \mid aM = \{0_m, o_M\}\,\}$.

(ii) $\underline{\varepsilon}(0_M) = o_S = o_R \in R$.

(iii) $o_S L = \{o_M\}$.

Proof. (i) If $a \in R$ and $x \in M$ are such that $ax \in N$ then $ax = \underline{\varepsilon}(ax)ax = 0_M$. Thus $aM = \{0_M, o_M\}$. Conversely, if $a \in S$ is such that $aM = \{0_M, o_M\}$ then $bax = ax$ for all $b \in S$ and $x \in M$. Since M is faithful, we get $ba = a$ and $a \in R$.
(ii) We have $(\underline{\varepsilon}(0_M) + a)x = \underline{\varepsilon}(0_M)x + ax = o_M + ax = o_M = \underline{\varepsilon}(0_M)x$ for every $x \in L$ and $a \in S$. Of course, $(\underline{\varepsilon}(0_M)+a)0_M = 0_M = \underline{\varepsilon}(0_M)0_M$. Since M is faithful, we get $\underline{\varepsilon}(0_M) + a = \underline{\varepsilon}(0_M)$ and $\underline{\varepsilon}(0_M) = o_S$. By 2.3, $o_S \in R$.
(iii) If $x \in L$ then $o_S x = \underline{\varepsilon}(0_M)x = o_M$, since $x \nleq 0_M$. ∎

2.5 Proposition. *The right S-semimodule R_S is faithful.*

Proof. Let $a, b \in S$, $a \neq b$. Since the left semimodule ${}_SM$ is faithful, there is $x \in M$ with $ax \neq bx$ and we can assume that $bx \nleq ax$. Then $ax \in N$, $c = \underline{\varepsilon}(ax) \in R$ by 2.3 and $cax = 0_M \neq o_M = cbx$. Consequently, $ca \neq cb$. ∎

2.6 Proposition. *The semiring S is simple if and only if $R + S = S$ and the right semimodule R_S is simple.*

Proof. First, assume that S is a simple semiring. The relation $\alpha = ((R+S) \times (R+S)) \cup \mathrm{id}_S$ is a congruence of S and $R \times R \subseteq \alpha$. Since $|R| \geq 2$ by 2.3, we have $\alpha \neq \mathrm{id}_S$, and hence $\alpha = S \times S$ and $R+S = S$. Let σ be a congruence of the right semimodule R_S. Define a relation β on S by $(a, b) \in \beta$ iff $(ca, cb) \in \sigma$ for every $c \in R$. Then β is a congruence of the semiring S and $\sigma = \beta \cap (R \times R)$. Since S is simple, we have either $\beta = \mathrm{id}_S$ and $\sigma = \mathrm{id}_R$ or $\beta = S \times S$ and $\sigma = R \times R$.

Now, assume that $R + S = S$ and R_S is simple. Let $\varrho \neq \mathrm{id}_S$ be a congruence of the semiring S and $(a, b) \in \varrho$, $a \neq b$. Since R_S is faithful by 2.5, we have $ca \neq cb$ for at least one $c \in R$, and hence $\gamma = \varrho \cap (R \times R) \neq \mathrm{id}_R$. Clearly, γ is a congruence of R_S, so that $\gamma = R \times R$ and $R \times R \subseteq \varrho$. If $d \in S$ then $d = e + f$ for some $e \in R$ and $f \in S$, $(e, o_S) \in \varrho$ (see 2.4(ii)) and $(d, o_S) = (e + f, o_S + f) \in \varrho$. Thus $\varrho = S \times S$. ∎

2.7 Lemma. *Let $a \in S$ and $x \in M$. Then $ax = 0_M$ iff $x \in N$ and $a \leq \underline{\varepsilon}(x)$.*

Proof. If $ax = 0_M$ then $x \neq o_M$ and $(a + \underline{\varepsilon}(x))y = \underline{\varepsilon}(x)y$ for every $y \in M$. Since ${}_SM$ is faithful, $a + \underline{\varepsilon}(x) = \underline{\varepsilon}(x)$ and $a \leq \underline{\varepsilon}(x)$. Conversely, if $a \leq \underline{\varepsilon}(x)$, $x \in N$, then $ax \leq \underline{\varepsilon}(x)x = 0_M$. ∎

2.8 Lemma. *Let $x, y \in N$. Then $x \leq y$ iff $\underline{\varepsilon}(y) \leq \underline{\varepsilon}(x)$.*

Proof. If $x \le y$ then $\underline{\varepsilon}(y)x = 0_M$ and $\underline{\varepsilon}(y) \le \underline{\varepsilon}(x)$ by 2.7. Conversely, if $\underline{\varepsilon}(y) \le \underline{\varepsilon}(x)$ then $\underline{\varepsilon}(y)x \le \underline{\varepsilon}(x)x = 0_M$, and hence $x \le y$. ∎

2.9 Lemma. *If $a \in R\backslash\{o_S\}$ then $a \le \underline{\varepsilon}(x)$ for at least one $x \in K = M\backslash\{0_M, o_M\}$.*

Proof. Since $a \ne o_S = \underline{\varepsilon}(0_M)$ (see 2.4(ii)), there is $x \in K$ with $ax = 0_M$ (use 2.4). Now, 2.7 applies. ∎

2.10 Proposition. *The following conditions are equivalent for $a \in S$:*

(i) $a = 0_S$.

(ii) $a = 0_R$.

(iii) $a + \underline{\varepsilon}(x) = \underline{\varepsilon}(x)$ *for every* $x \in N$.

(iv) $aN = \{0_M\}$.

If $o_N \in N$ then these conditions are equivalent to $ao_N = 0_M$.

Proof. (i) implies (iii) and (ii) implies (iii) trivially.
(iii) implies (iv). $0_M = \underline{\varepsilon}(x)x = (\underline{\varepsilon}(x) + a)x = \underline{\varepsilon}(x)x + ax = ax$ for every $x \in N$.
(iv) implies (i) and (ii). First, $a \in R$ by 2.4(i). Further, $(a + b)x = ax + bx = 0_M + bx = bx$ for all $b \in S$ and $x \in N$. Then $(a + b)y = by$ for every $y \in M$ and, since M is faithful, we have $a + b = b$. Thus $a = 0_S = 0_R$. ∎

2.11 Lemma. *Let $0_S \in S$. Then:*

(i) $0_S \in R$.

(ii) $N + N = N$.

(iii) $\{\, a \in S \mid 0_S a = 0_S \,\} = \{\, a \in S \mid aN \subseteq N \,\}$.

(iv) $0_S = \underline{\varepsilon}(w)$ *for* $w \in N$ *iff* w *is the greatest element of* N.

(v) *For all* $a, b \in \underline{\varepsilon}(N)$ *there is* $c \in \underline{\varepsilon}(N)$ *with* $c \le a$ *and* $c \le b$.

(vi) *If* $c \in S$ *is such that* $0_S c = 0_S$ *then for every* $a \in \underline{\varepsilon}(N)$ *there is* $b \in \underline{\varepsilon}(N)$ *with* $bc \le a$.

(vii) *If* $c \in S$ *and* $x \in N$ *are such that* $0_S c = \underline{\varepsilon}(x)$ *then* $cx \in N$ *and* $0_S c = \underline{\varepsilon}(cx)c$.

(viii) *If* $0_S \notin \underline{\varepsilon}(N)$ *then the set* N *has no maximal element.*

(ix) *If* $c \in S$ *and* $u \in N$ *then* $\underline{\varepsilon}(u)c = 0_S$ *iff* $cN \le u$ *(then* $0_S c = 0_S$*).*

(x) *If* $c \in S$ *and* $v \in N$ *are such that* cv *is the greatest element of* cN *then* $\underline{\varepsilon}(cv)c = 0_S$.

(xi) *If* w *is the greatest element of* N *and* $c \in S$ *then* $\underline{\varepsilon}(cw)c = 0_S$ *and* cw *is the greatest element of* cN.

Proof. (i) Use 2.10(i),(ii).
(ii) Use 2.10(iv).
(iii) If $0_S a = 0_S$ then $0_S ax = 0_S x = 0_M$ for every $x \in N$ (see 2.10(iv)), and hence $ax \in N$. Conversely, if $aN \subseteq N$ then $0_S aN \subseteq 0_S N = \{0_M\}$, and hence $0_S a = 0_S$ by 2.10.

(iv) If $0_S = \underline{\varepsilon}(w)$ then $\underline{\varepsilon}(w)x = 0_S x = 0_M$ and $x \le w$ for every $x \in N$. Conversely, if w is the greatest element of N then $\underline{\varepsilon}(w)N = \{0_M\}$ and $\underline{\varepsilon}(w) = 0_S$ by 2.10.
(v) We have $a = \underline{\varepsilon}(u)$ and $b = \underline{\varepsilon}(v)$, $u, v \in N$. Now, $u + v \in N$ by (ii) and we put $c = \underline{\varepsilon}(u + v)$ (use 2.8).
(vi) We have $a = \underline{\varepsilon}(v)$, $v \in N$. If $cv = o_M$ then $0_S v = 0_S cv = 0_S o_M = o_M$, and hence $Sv = \{o_M\}$, a contradiction with $av = 0_M$. Thus $cv \in N$ and we put $b = \underline{\varepsilon}(cv)$ (use 2.7).
(vii) We have $0_S cx = \underline{\varepsilon}(x)x = 0_M$, and hence $cx \in N$. Furthermore, $\underline{\varepsilon}(cx)cx = 0_M$, and so $\underline{\varepsilon}(cx)c \le \underline{\varepsilon}(x) = 0_S c$ by 2.7. Since $0_S \le \underline{\varepsilon}(cx)$, we have $0_S c \le \underline{\varepsilon}(cx)c$. Thus $0_S c = \underline{\varepsilon}(cx)c$.
(viii) $N + N = N$ by 2.11(ii). If w is maximal in N and $x \in N$ then $w \le w + x \in N$, $w = w + x$, $x \le w$, and therefore w is the greatest element of N. By (iv), $0_S = \underline{\varepsilon}(w)$.
(ix), (x) and (xi). Use 2.10. ∎

2.12 Proposition. *The following conditions are equivalent for $e \in S$:*

(i) *$e = 1_S$ is multiplicatively neutral in S.*

(ii) *e is right multiplicatively neutral in S.*

(iii) *$\underline{\varepsilon}(x)e = \underline{\varepsilon}(x)$ for every $x \in N$.*

(iv) *$ey = y$ for every $y \in M$.*

Proof. (i) implies (ii) and (ii) implies (iii) trivially.
(iii) implies (iv). First, $\underline{\varepsilon}(x)ex = \underline{\varepsilon}(x)x = 0_M$ and $ex \le x$. Further, $0_M = \underline{\varepsilon}(ex)ex = \underline{\varepsilon}(ex)x$, so that $x \le ex$. Thus $x = ex$ for every $x \in N$. Of course, $eo_M = o_M$ anyway.
(iv) implies (i). Clearly, $aey = ay$ and $eay = ay$ for all $a \in S$ and $y \in M$. Since M is faithful, we get $ae = a = ea$. ∎

2.13 Lemma. *Assume that $Sv = M$ for at least one $v \in M$. If $e \in S$ is a left multiplicatively neutral element the $e = 1_S$ is multiplicatively neutral.*

Proof. We have $eav = av$ for every $a \in S$. By 2.12(i),(iv), we get $e = 1_S$. ∎

2.14 Lemma. *Assume that the semiring S is simple. If G is a subsemimodule of M then G is faithful if and only if $G \nsubseteq \{0_M, o_M\}$.*

Proof. If $x \in G \setminus \{0_M, o_M\}$ then $\underline{\varepsilon}(x)x = 0_M \ne o_M = \underline{\varepsilon}(0_M)x = o_S x$. ∎

2.15 Remark. Let ${}_SG$ be a faithful subsemimodule of ${}_SM$. Then $G \nsubseteq \{0_M, o_M\}$ and if $u \in G \setminus \{0_M, o_M\}$ then $\underline{\varepsilon}(u)M = 0_M \in G$. Of course, $o_S u = o_M$, and hence $\{0_M, o_M\} \subseteq G$. Now, it is clear that the semimodule ${}_SG$ is characteristic, too. We have $o_G = o_M$ and $\underline{\varepsilon}(G \setminus \{o_M\}) \subseteq \underline{\varepsilon}(M \setminus \{o_M\})$.

2.16 Remark. (i) Assume that $o_N \in N$ and that the set $G = M \setminus \{o_N\} = (N \setminus \{o_N\}) \cup \{o_M\}$ is a subsemimodule of ${}_SM$. If ${}_SG$ is not a faithful semimodule then there are elements $a_1, b_1 \in S$ such that $a_1 \ne b_1$ and $a_1 u = b_1 u$ for every $u \in G$. According to 2.5, there is $c \in R$ such that $a = ca_1 \ne cb_1 = b$. Of course, $a, b \in R$. We have $au = bu$ for every $u \in G$ and, since ${}_SM$ is faithful, we have

$ao_N \neq bo_N$. Using 2.4(i), we can assume that $o_M = ao_N$ and $0_M = bo_N$. Then $bN = \{0_M\}$ and $b = 0_S$ by 2.10. We have $(a + c)u = cu$ for all $c \in S$ and $u \in G$. If $d \in R \setminus \{0_S\}$ then $do_N = o_M$ (use 2.4(i) and 2.10). Consequently, $(a + d)v = dv$ for all $d \in R \setminus \{0_S\}$ and $v \in M$. It follows that $a + d = d$ and $a \leq d$. Thus a is the smallest element of the set $R \setminus \{0_S\}$. If $a = \underline{\varepsilon}(w)$ for some $w \in N$ then w is the greatest element of the set $N \setminus \{o_N\}$.

(ii) Assume that $0_S \in S$ and that the set $R \setminus \{0_S\}$ has the smallest element, say a. Then $o_N \in N$ and $ax = 0_S x$ for every $x \in M \setminus \{o_N\}$.

2.17 Lemma. *Assume that $R + S = S$. If the semiring S has a right multiplicatively neutral element then $0_S \in S$.*

Proof. If $e \in S$ is right multiplicatively neutral then $e = a + b$, $a \in R$, $b \in S$, and $c = ce = c(a + b) = ca + cb = a + cb$. Thus $a = 0_S$. ∎

2.18 Remark.

(i) If the right S-semimodule S_S is simple then the semiring S is simple and the right S-semimodule R_S is simple, too (see 2.6).

(ii) Assume that R_S is simple and define a relation α on R by $(a, b) \in \alpha$ iff $ac = bc$ for every $c \in S$. Clearly, *alpha* is a congruence of R_S, and hence either $\alpha = \mathrm{id}_R$ or $\alpha = R \times R$. If the latter is true then $ac = bc$ for all $a, b \in R$, $c \in S$, and it follows easily that every congruence of the additive semigroup $R(+)$ is a congruence of the semimodule R_S. Consequently, $R(+)$ is a simple semilattice, $|R| = 2$ and $|S| = 2$ or $|S| = 3$.

(iii) Assume that R_S is simple and $|S| \geq 3$. Let $e \in S$ be a left multiplicatively neutral element. Then $(ae, a) \in \alpha$ for every $a \in R$.

Let $\alpha = \mathrm{id}_R$. Then $ae = a$, and so $abe = ab$ for all $a \in R$ and $b \in S$. Define a relation ϱ on S by $(c, d) \in \varrho$ iff $ac = ad$ for every $a \in R$. It is easy to see that ϱ is a congruence of the semiring S and $\varrho \cap (R \times R) = \mathrm{id}_R$. Thus $\varrho \neq S \times S$. Of course, if $\varrho = \mathrm{id}_S$ then $e = 1_S$ is multiplicatively neutral in S.

Finally, if $\alpha = R \times R$ then $|R| = 2$, $|S| = 3$ and $1_S \in S$. Thus $e = 1_S$.

3. Characteristic semimodules (b)

3.1. Assume that $\underline{\varepsilon}(N) = R$.

3.1.1 Lemma. *Let $a, b \in R$ be such that $P = P_{a,b} = \{\, c \in R \mid c \leq a, c \leq b \,\} \neq \emptyset$. Then $o_P \in P$.*

Proof. We have $a = \underline{\varepsilon}(u)$, $b = \underline{\varepsilon}(v)$, $u, v \in N$. If $c \in P$, $c = \underline{\varepsilon}(w)$, $w \in N$, then (by 2.3) $u \leq w$, $v \leq w$, and hence $u + v \leq w$. Now, $u + v \in N$ and $o_P = \underline{\varepsilon}(u + v)$. ∎

3.1.2 Lemma. *Let $a \in R$ and $b \in S$ be such that $Q = Q_{a,b} = \{c \in R \mid cb \leq a\} \neq \emptyset$. Then $o_Q \in Q$.*

Proof. We have $a = \underline{\varepsilon}(u)$, $u \in N$, and if $c \in Q$ then $c = \underline{\varepsilon}(w)$, $w \in N$. Now, $\underline{\varepsilon}(w)b \leq \underline{\varepsilon}(u)$, and so $\underline{\varepsilon}(w)bu \leq \underline{\varepsilon}(u)u = 0_M$. Then $\underline{\varepsilon}(w) \leq \underline{\varepsilon}(bu)$ and we see that $o_Q = \underline{\varepsilon}(bu)$. ∎

3.1.3. Using 3.1.1, 3.1.2 and [1, 7.2], we get a left S-semimodule ${}^{1}_{S}R = R^+(*, \circ))$ defined on $R^+ = R \cup \{\omega\}$, where $\omega \notin S$, the "addition" $*$ for $a, b \in R$ is defined as $a * b = o_P$ if $P = P_{a,b} \neq \emptyset$ and $a * b = \omega$ otherwise, and the "scalar multiplication" $\circ$ for $a \in S$, $x \in R^+$ is defined as $a \circ x = o_Q$ if $x \in R$ and $Q_{a,x} \neq \emptyset$, and $a \circ x = \omega$ otherwise. Let $\varphi : M \to R^+$ be defined by $\varphi(x) = \underline{\varepsilon}(x)$ for $x \in N$ and $\varphi(o_M) = \omega$. We have $\varphi(N) = R$ and, due to 2.3, φ is a bijective mapping of M onto R^+. We check that, in fact, φ is an isomorphism of the left S-semimodules.

Let $x, y \in M$. We have to show that $\varphi(x + y) = \varphi(x) * \varphi(y)$. If $o_M \in \{x, y\}$ then $\omega \in \{\varphi(x), \varphi(y)\}$ and $\varphi(x + y) = \varphi(o_M) = \omega = \varphi(x) * \varphi(y)$. If $x, y \in N$ and $o_M = x + y$ then $P_{a,b} = \emptyset$, where $a = \underline{\varepsilon}(x)$ and $b = \underline{\varepsilon}(y)$ (use 2.8), and then $\varphi(x + y) = \varphi(o_M) = \omega = a * b = \varphi(x) * \varphi(y)$. Finally, if $x + y \in N$ then $P_{a,b} \neq \emptyset$ and $\varphi(x + y) = \underline{\varepsilon}(x + y) = o_P = a * b = \underline{\varepsilon}(x) * \underline{\varepsilon}(y) = \varphi(x) * \varphi(y)$ (see 3.1.1 and its proof).

Let $a \in S$ and $x \in M$. We have to show that $\varphi(ax) = a \circ \varphi(x)$. If $x = o_M$ then $\varphi(ax) = \varphi(o_M) = \omega = a \circ \omega = a \circ \varphi(x)$. If $x \in N$ and $Q_{\varphi(x),a} = Q_{\underline{\varepsilon}(x),a} = \emptyset$ then $a \circ \varphi(x) = a \circ \underline{\varepsilon}(x) = \omega$ and either $ax = o_M$, $\varphi(ax) = \omega$, or $ax \in N$, $0_M = \underline{\varepsilon}(ax)ax$ and $\underline{\varepsilon}(ax) \in Q_{\underline{\varepsilon}(x),a}$, a contradiction. Assume, finally, that $x \in N$ and $Q_{\underline{\varepsilon}(x),a} \neq \emptyset$. Then $ax \in N$ and $\varphi(ax) = \underline{\varepsilon}(ax) = a \circ \underline{\varepsilon}(x) = a \circ \varphi(x)$ (see 3.1.2 and its proof).

3.2. Assume that $0_S \in S$ (then $0_S \in R$) and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$.

3.2.1 Lemma. *Take $a, b \in R$ and put $P = P_{a,b} = \{\, c \in R \,|\, c \leq a, c \leq b \,\}$. Then $0_S \in P$ and $o_P \in P$.*

Proof. We can assume that $a \neq 0_S \neq b$. Now we can proceed similarly as in the proof of 3.1.1. ∎

3.2.2 Lemma. *Let $a \in R$, $a \neq 0_S$, $b \in S$ and $Q = Q_{a,b} = \{\, c \in R \,|\, cb \leq a \,\}$. Then:*

(i) *$Q \neq \emptyset$ iff $0_S b \leq a$.*

(ii) *If $Q \neq \emptyset$ then $o_Q \in Q$.*

Proof. (i) This is clear.
(ii) We have $a = \underline{\varepsilon}(u)$ for some $u \in N$. If $Q = \{0_S\}$ then $o_Q = 0_S$. If $c \in Q \setminus \{0_S\}$ then $c = \underline{\varepsilon}(v)$, $v \in N$, $\underline{\varepsilon}(v)b \leq \underline{\varepsilon}(u)$, $\underline{\varepsilon}(v)bu = 0_M$ and $c = \underline{\varepsilon}(v) \leq \underline{\varepsilon}(bu)$ by 2.7. Of course, $bu \in N$, $\underline{\varepsilon}(bu)bu = 0_M$ and $\underline{\varepsilon}(bu)b \leq \underline{\varepsilon}(u) = a$. Thus $o_Q = \underline{\varepsilon}(bu)$. ∎

3.2.3 Lemma. *For all $a, b \in R \setminus \{0_S\}$ there is $c \in R \setminus \{0_S\}$ with $c \leq a$ and $c \leq b$.*

Proof. We have $a = \underline{\varepsilon}(x)$ and $b = \underline{\varepsilon}(y)$ for $x, y \in N$. Then $x + y \in N$ by 2.11(ii) and we put $c = \underline{\varepsilon}(x + y)$. ∎

3.2.4 Lemma. *If $a \in S$ is such that $0_S a = 0_S$ then for every $b \in R \setminus \{0_S\}$ there is $c \in R \setminus \{0_S\}$ with $ca \leq b$.*

Proof. See 2.11(vi). ∎

3.2.5 Lemma. *If $a \in S$ is such that $0_S a \neq 0_S$ then $0_S a = ba$ for at least one $b \in R \setminus \{0_S\}$.*

Proof. See 2.11(vii). ∎

3.2.6. Using 3.2.1, 3.2.5 and [1, 5.5 and 5.6], we get a left S-semimodule ${}^{2}_{S}R = R(*, \triangle)$, where the "scalar multiplication" $\triangle$ for $a \in S$ and $x \in R$ is defined as $a \triangle x = o_Q$ if $x \in R$, $x \neq 0_S$ and $Q_{a,x} \neq \emptyset$, and $a \triangle x = 0_S$ otherwise. Let $\varphi : M \to R$ be defined by $\varphi(x) = \underline{\varepsilon}(x)$ for $x \in N$ and $\varphi(o_M) = 0_S$. Proceeding similarly as in 3.1.3, one checks easily that φ is an isomorphism of the semimodule ${}_SM$ onto the semimodule ${}^{2}_{S}R$.

3.3 Theorem. *The following conditions are equivalent:*

(i) *There is a characteristic semimodule ${}_SM$ such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R$.*

(ii) *The ordered set $R^+ = R \cup \{\omega\}$, where ω is the smallest element of the set, is a lattice and $o_Q \in Q = Q_{a,b} = \{\, c \in R \,|\, cb \leq a \,\}$ for all $a \in R$ and $b \in S$ such that $Q \neq \emptyset$.*

Moreover, if these conditions are satisfied then ${}_SM \cong {}^{1}_{S}R$.

Proof. i) implies (ii). See 3.1.
(ii) implies (i). See [1, 7.2]. ∎

3.4 Theorem. *The following conditions are equivalent:*

(i) *$0_S \in S$ and there is a characteristic semimodule ${}_SM$ such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R \setminus \{0_S\}$.*

(ii) *$0_R \in R$, the ordered set R is a lattice, $o_Q \in Q = Q_{a,b} = \{\, c \in R \,|\, cb \leq a \,\}$ for all $a \in R \setminus \{0_S\}$ and $b \in S$ such that $Q \neq \emptyset$ and the assertions* 3.2.3, 3.2.4 and 3.2.5 *are true.*

Moreover, if the two equivalent conditions are satisfied then ${}_SM \cong {}^{2}_{S}R$.

Proof. (i) implies (ii). See 3.2.
(ii) implies (i). See [1, 7.3]. ∎

3.5 Remark. Assume that the equivalent conditions of 3.4 are satisfied. Let $a_0 \in R$ and $b \in S$ be such that $Q = Q_{a_0,b} \neq \emptyset$ and $o_Q \notin Q$.

We have $a_0 = 0_S$ (see 3.4(ii)) and $0_S \in Q = \{\, c \in R \,|\, cb = 0_S \,\}$. Of course, $Q \subseteq Q_{a,b} = \{\, c \in R \,|\, cb \leq a \,\}$ for every $a \in R$. If $a \in R_1 = R \setminus \{0_S\}$ then $o_{Q_{a,b}} \in Q_{a,b}$ (see 3.4(ii)), and hence $Q \neq Q_{a,b}$ and there is $c_a \in R$ with $0_S \neq c_a b \leq a$.

Now, assume that $R_2 b = 0_S$, where $R_2 = R \setminus \{o_S\}$. Since $o_Q \notin Q$, we have $Q = R_2$, $a_1 = o_S b \neq 0_S$, $Q_{a,b} = R$ for every $a \in R_1$ and $Rb \leq a$. But $Rb = \{0_S, a_1\}$ and it follows that a_1 is the smallest element of the set R_1. By 3.4(i), $a_1 = \underline{\varepsilon}(w)$ for some $w \in N$. According to 2.8, we see that w is the greatest element of the set N. By 2.11(iv), $a_1 = 0_S$, a contradiction.

We have proved that $c_0 b \neq 0_S$ for at least one $c_0 \in R \setminus \{o_S\}$.

3.6 Remark. Let $0_S \in S$ and let M be a characteristic semimodule. Let $b \in S$ be such that $R_2 b = \{0_S\}$, $R_2 = R \setminus \{o_S\}$ (cf. 3.5). We have $0_S N = \{0_M\}$, and hence $R_2 bN = \{0_M\}$. Consequently, $bN \subseteq N$. For every $x \in N$, we get $R_2 \leq \underline{\varepsilon}(bx)$ by 2.7. If $\underline{\varepsilon}(bx) = o_S$ then $bx = 0_M$. On the other hand, if $\underline{\varepsilon}(bx) \in R_2$ then $\underline{\varepsilon}(bx)$ is the greatest element of the set R_2 and bx is the smallest element of the set $M \setminus \{0_M\}$ (use 2.8).

Of course, if $b = 0_S$ then $R_2 b = Sb = S0_S = \{0_S\}$. Assume, henceforth, that $b \neq 0_S$. We have $0_S N = \{0_M\}$ and, since the semimodule M is faithful and $b \neq 0_S$, we see that $bN \neq \{0_M\}$. As we have proved, the set $R_2 = R \setminus \{o_S\}$ has the greatest element a_0 and $a_0 = \underline{\varepsilon}(v)$, where v is the smallest element in $M \setminus \{0_M\}$. Besides, if $x \in N$ then either $bx = 0_M$ or $bx = v$. Thus $bM = \{0_M, v, o_M\}$.

4. Characteristic semimodules (c)

Let M $(= {}_S M)$ be a characteristic (left S-)semimodule, $N = M \setminus \{o_M\}$ and $R = \underline{R}(S)$.

4.1 Lemma. *Let $a \in R$ and $A = A_a = \{\, x \in M \mid ax = 0_M \,\}$. Then:*

(i) $0_M \in A \subseteq N$ *and* $A(+)$ *is a subsemilattice of* $M(+)$.

(ii) *If* $w \in A$ *is maximal in* A *then* w *is the greatest element of* A *and* $a = \underline{\varepsilon}(w)$.

(iii) $a \leq \underline{\varepsilon}(x)$ *for every* $x \in A$.

(iv) *If* $a \notin \underline{\varepsilon}(N)$ *then* $a < \underline{\varepsilon}(x)$ *for every* $x \in A$.

Proof. The assertion (i) is obvious. If w is maximal in A then $w = o_A$, $\underline{\varepsilon}(w)y = 0_M = ay$ for every $y \leq w$ and $\underline{\varepsilon}(w)z = o_M$ for every $z \not\leq w$. Since $z \notin A$, we have $az \neq 0_M$, and hence $az = o_M$ by 2.4(i). Thus $\underline{\varepsilon}(w)u = au$ for every $u \in M$ and $a = \underline{\varepsilon}(w)$, since ${}_S M$ is faithful. We have proved (i) and (iii), (iv) follow from 2.7.∎

Consider the following three conditions:

(α) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in N$ there is $i \geq 1$ with $x \leq x_i$.

(β) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in N \setminus \{o_N\}$ there is $i \geq 1$ with $x \leq x_i$.

(γ) If $a_1 > a_2 > a_3 > \dots$ is an infinite strictly decreasing sequence of elements from R then for every $a \in R \setminus \{0_S\}$ there is $i \geq 1$ with $a \geq a_i$.

4.2 Proposition. *Assume that* (α) *or* (γ) *is true. Then either* $\underline{\varepsilon}(N) = R$ *or* $0_S \in S$ *and* $\underline{\varepsilon}(N) = R \setminus \{0_S\}$.

Proof. Let $a \in R \setminus \underline{\varepsilon}(N)$ and $A = A_a$ (see 4.1). By 4.1(ii), the set A has no maximal element, and hence there is at least one infinite strictly increasing sequence $x_1 < x_2 < x_3 < \dots$ of elements from A. By 2.8, we get the infinite strictly decreasing sequence $\underline{\varepsilon}(x_1) > \underline{\varepsilon}(x_2) > \underline{\varepsilon}(x_3) > \dots$ of elements from R. By

4.1(iv), $\underline{\varepsilon}(x_i) > a$. Now, if ($\gamma$) is true, we get $a = 0_S$. On the other hand, if (α) is true then $A = N$, $aN = \{0_M\}$ and $a = 0_S$ by 2.10. ∎

4.3 Lemma. *(α) implies (γ) and (γ) implies (β). If $o_N \notin N$ then the conditions (α), (β) and (γ) are equivalent.*

Proof. Use 4.2 and 2.11(iv). ∎

4.4 Proposition. *Assume that (β) is true. Then just one of the following three cases takes place:*

(i) $\underline{\varepsilon}(N) = R$.

(ii) $0_S \in S$ *and* $\underline{\varepsilon}(N) = R \setminus \{0_S\}$.

(iii) $o_N \in N$, $0_S \in S$, *the set* $R \setminus \{0_S\}$ *has the smallest element* a_0, $\underline{\varepsilon}(N) = R \setminus \{a_0\}$, $a_0(N \setminus \{o_N\}) = \{0_M\}$ *and* $a_0 o_N = o_M$.

Proof. Assume that neither (i) or (ii) is true. Let $a \in R \setminus \underline{\varepsilon}(N)$. By 4.1, the set $A = A_a$ has no maximal element. By 4.2, the condition (α) is not satisfied. But (β) is, and so $o_N \in N$. We have $A \subseteq N$, $o_N \notin A$, and therefore $A \subseteq N' = N \setminus \{o_N\}$. Using ($\beta$), we conclude that $A = N'$, $aN = \{0_M\}$. If $ao_N = 0_N = 0_M$ then $a = 0_S$ by 2.10 and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$, a contradiction. Thus $ao_N \neq 0_N$ and it follows from 2.4(i) that $ao_n = o_M$. By 4.1(iv), $a < \underline{\varepsilon}(x)$ for every $x \in N'$. But $\underline{\varepsilon}(N) = R \setminus \{a\}$ and $\underline{\varepsilon}(N') = R \setminus \{0_S, a\}$. It follows that a is the smallest element of the set $R \setminus \{0_S\}$. ∎

4.5 Lemma. *Let G be a proper faithful subsemimodule of the semimodule M. Then:*

(i) $\{0_M, o_M\} \subset G$ *and* $|M| \geq 4$.

(ii) *If* $w \in M \setminus G$, $a = \underline{\varepsilon}(w)$, *then the set* $A_a \cap G = \{\, x \in G \mid ax = 0_M \,\}$ *has no maximal element.*

Proof. Since ${}_SG$ is faithful, we have $G \nsubseteq \{0_M, o_M\}$. If $u \in G$, $u \neq 0_M, o_M$, then $\underline{\varepsilon}(u)u = 0_M \in G$ and $o_S u = o_M \in G$, so that $\{0_M, o_M\} \subset G$ and $|M| \geq 4$. Furthermore, if v is a maximal element of the set $A_a \cap G$ then $\underline{\varepsilon}(w) = a = \underline{\varepsilon}(v)$, $w = v$ and $w \in G$, a contradiction. ∎

4.6 Proposition. *Assume that (β) is true. Let ${}_SG$ be a proper faithful subsemimodule of ${}_SM$. Then:*

(i) $o_N \in N$ *and* $\underline{\varepsilon}(o_N) = 0_S \in S$.

(ii) *M is infinite.*

(iii) $\underline{\varepsilon}(N) = R$.

(iv) $G = M \setminus \{o_N\}$.

Proof. By 4.5, $\{0_M, o_M\} \subset G$ and if $w \in M \setminus G$, $a = \underline{\varepsilon}(w)$, then the set $B = \{\, x \in G \mid ax = 0_M \,\}$ has no maximal element. Clearly, $B \subseteq N' = N \setminus \{o_N\}$ and there is an infinite strictly increasing sequence $x_1 < x_2 < x_3 < \dots$ of elements from B. For every $x \in N'$ there is $i \geq 1$ with $x \leq x_i$ and it follows that $aN' = \{0_M\}$. If $aN = \{0_M\}$ then $a = 0_S \in S$ by 2.10 and, by 2.11(iv), $w = o_N$. Besides, $\underline{\varepsilon}(N) = R$ by 4.4. Assume, therefore, that $aN \neq \{0_M\}$. Then $o_N \in N$ and $ao_N = o_M$ by 2.4(i). But $aw = \underline{\varepsilon}(w)w = 0_M$, and therefore $w \neq o_N$, $w \in N'$ and there is $j \geq 1$ with $w \leq x_j$. Since $w \notin G$, we have $w < x_j$ and $b = \underline{\varepsilon}(x_j) < \underline{\varepsilon}(w) = a$ (use 2.8). On the other hand, $(a+b)u = 0_M + bu = bu$ for every $u \in N'$ and $(a+b)o_M = o_M = bo_M$. Since $x_j < o_N$, we have $b \neq 0_S$, $bN \neq \{0_M\}$ and $bo_N = o_M$. Thus $(a+b)v = bv$ for every $v \in M$, $a + b = b$ and $a \leq b$, a contradiction. ∎

4.7 Remark. Another proof of 4.6 is available here. First, by 2.15, ${}_SG$ is a characteristic semimodule and $\underline{\varepsilon}(F) \subseteq \underline{\varepsilon}(N)$, $F = G \setminus \{o_M\}$ (we have $o_M = o_G$). If $\underline{\varepsilon}(F) = \underline{\varepsilon}(N)$ then $F = N$, since $\underline{\varepsilon}$ is injective, and we get $G = M$. Now, if $G \neq M$ then $\underline{\varepsilon}(F) \subset \underline{\varepsilon}(N)$ and it follows from 4.4 that $0_S \in S$ and $\underline{\varepsilon}(N) = R$. Then $0_S = \underline{\varepsilon}(o_N)$, $o_N \in N$, and if $\underline{\varepsilon}(F) = R \setminus \{0_S\}$ then $G = M \setminus \{o_N\}$. On the other hand, if $\underline{\varepsilon}(F) = R \setminus \{a_0\}$, a_0 being the smallest element of $R \setminus \{0_S\}$, then $a_0 \leq \underline{\varepsilon}(u)$ for every $u \in N \setminus \{o_N\}$, and so $a_0 u = 0_M$.Thus $a_0 v = 0_S v$ for every $v \in G$ and, since G is faithful, we get $a_0 = 0_S$, a contradiction.

4.8 Proposition. *If (α) is satisfied then no proper subsemimodule of ${}_SM$ is faithful.*

Proof. Let, on the contrary, ${}_SG$ be a proper faithful subsemimodule of ${}_SM$. By 4.6, $o_N \in N$ and $G = M \setminus \{o_N\}$. By 4.5(ii), there is at least one infinite strictly increasing sequence $x_1 < x_2 < x_3 < \dots$ of elements from G. Since (α) is true, we have $o_N \leq x_i$ for some $i \geq 1$. But then $x_i = o_N$, a contradiction. ∎

4.9 Proposition. *Assume that (β) is true and that a subsemimodule G of ${}_SM$ is faithful whenever $G \nsubseteq \{0_M, o_M\}$* (e.g., if S is simple - see 2.14). *Then M has at most five distinct subsemimodules and, namely, if H is a subsemimodule then either $H = \{0_M\}$ or $H = \{o_M\}$ or $H = \{0_M.o_M\}$ or $H = M$ or $o_N \in N$, $|M| \geq 4$ and $H = M \setminus \{o_N\}$ (M is infinite in this case).*

Proof. The result follows easily from 4.6. ∎

4.10 Remark. Consider the situation from 4.9.

(i) Put $F = \{\, x \in M \mid Sx \subseteq \{0_M, o_M\} \,\}$. Then F is a subsemimodule of M and $\{0_M, o_M \subseteq F$. If $F = M$ then $SM = \{0_M, o_M\}$ and $S = R$ by 2.4(i). If $o_N \in N$, $|M| \geq 4$ and $F = M \setminus \{o_N\}$ then F is faithful and infinite, $SF = \{0_M, o_M\}$ and $S = R$ again (notice that $F = M \setminus \{o_N\}$ is a characteristic semimodule, too).

(ii) Finally, assume that $F = \{0_M, o_M\}$ (e.g., if $S \neq R$ or if S is simple and $|S| \geq 3$). If $x \in M \setminus \{0_M, o_M\}$ then $x \notin F$, and hence $Sx = M$ or $Sx = M \setminus \{o_N\}$. Consequently, $Sx = M$ for every $x \in M \setminus \{0_M, o_M\}$, provided that either $o_N \notin N$ or $o_N \in N$ and $M \setminus \{o_N\}$ is not a subsemimodule.

(iii) Assume that $F = \{0_M, o_M\}$, $o_N \in N$, $|M| \geq 4$ and $G = M \setminus \{o_N\}$ is a subsemimodule. Then G is an (infinite) characteristic semimodule and $Sx = G$ for every $x \in G \setminus \{0_M, o_M\}$.

4.11 Proposition. *Assume that the semiring S is simple, $|S| \geq 3$ and (α) is true. Then:*

(i) *The semimodule ${}_SM$ has just four subsemimodules and these are $\{0_M\}$, $\{o_M\}$, $\{0_M, o_M\}$ and M.*

(ii) *$Sx = M$ for every $x \in M \setminus \{0_M, o_M\}$.*

Proof. See 2.14, 4.8, 4.9 and 4.10. ∎

In the remaining part of this section, assume that $K = M \setminus \{0_M, o_M\} \subseteq Sx$ for every $x \in K$. Let $\alpha \neq \mathrm{id}_R, R \times R$ be a congruence of the right semimodule R_S. Put $A = \{\, a \in R \mid (a, o_S) \notin \alpha \,\}$ and $B = R \setminus A$.

4.12 Lemma. *For every $x \in K$ there is $a \in R$ with $\underline{\varepsilon}(x) < a$ and $(\underline{\varepsilon}(x), a) \in \alpha$.*

Proof. There is a pair $(b, c) \in \alpha$ such that $b < c$. Since ${}_SM$ is faithful, $bu \neq cu$ for at least one $u \in M$. Of course, $bu < cu$, and hence $bu = 0_M$ and $cu = o_M$ follows from 2.4(i). Consequently, $u \in K$ and $u = dx$, $d \in S$. Now, $bdx = bu = 0_M$ and $(bd + \underline{\varepsilon}(x))y \leq (bd + \underline{\varepsilon}(x))x = 0_M$ for $y \in M$, $y \leq x$. If $z \in M$, $z \nleq x$, then $(bd + \underline{\varepsilon}(x))z = o_M$. Thus $(bd + \underline{\varepsilon}(x))v = \underline{\varepsilon}(x)v$ for every $v \in M$ and we see that $bd \leq \underline{\varepsilon}(x)$. We have $(\underline{\varepsilon}(x), a) = (bd + \underline{\varepsilon}(x), cd + \underline{\varepsilon}(x)) \in \alpha$, where $a = cd + \underline{\varepsilon}(x) \in R$, $\underline{\varepsilon}(x) \leq a$. Since $ax = cdx + \underline{\varepsilon}(x)x = cu = o_M \neq 0_M = \underline{\varepsilon}(x)x$, we get $\underline{\varepsilon}(x) < a$. ∎

4.13 Lemma.

(i) *$A \neq \emptyset \neq B$, $A \cap B = \emptyset$ and $A \cup B = R$.*

(ii) *The set B is a block of the congruence α.*

(iii) *If a_0 is maximal in A then $a_0 \notin \underline{\varepsilon}(N)$.*

Proof. Only (iii) needs a proof. If $a_0 = \underline{\varepsilon}(v)$, $v \in N$, then $v \in K$, since $a_0 \neq o_S = \underline{\varepsilon}(0_M)$, and, by 4.12, $(a_0, a) \in \alpha$ for some $a \in R$, $a_0 < a$. Since a_0 is maximal in A, we have $a \notin A$ and $(a, o_S) \in \alpha$. Since $(a_0, a) \in \alpha$, we have $(a_0, o_S) \in \alpha$, a contradiction with $a_0 \in A$. ∎

4.14 Lemma. *Let a_0 be maximal in A. Put $C = \{\, x \in M \mid a_0 x = 0_M \,\}$. Then:*

(i) *$0_M \in C \subseteq N$ and $C(+)$ is a subsemilattice of $M(+)$.*

(ii) *The set C has no maximal element.*

(iii) *$a_0 < \underline{\varepsilon}(x)$ for every $x \in C$.*

(iv) *If $C = N$ then $a_0 = 0_S$ and $\alpha = \alpha_1 = (R_1 \times R_1) \cup \mathrm{id}_R$, where $R_1 = R \setminus \{0_S\}$.*

Proof. If $w \in C$ is maximal in C then $a_0 = \underline{\varepsilon}(w)$ by 4.1(ii), a contradiction with 4.13 (iii). Thus C has no maximal element and we have $a_0 < \underline{\varepsilon}(x)$ for every $x \in C$ by 2.7. If $C = N$ then $a_0 = 0_S$ by 2.10, $A = \{0_S\}$ and $\alpha = \alpha_1$. ∎

4.15 Lemma. *Assume that (β) is true and let a_0 be maximal in A. Then just one of the following three cases holds:*

1. $0_S \in S$ *and* $\alpha = \alpha_1$ (see 4.14(iv)).
2. $0_S \in S$, *the set* $R_1 = R \setminus \{0_S\}$ *has the smallest element* a_0 *and* $\alpha = (R_1 \setminus \{a_0\}) \times (R_1 \setminus \{a_0\}) \cup \mathrm{id}_R$.
3. $0_S \in S$, R_1 *has the smallest element* a_0 *and* $\alpha = (R_1 \setminus \{a_0\}) \times (R_1 \setminus \{a_0\}) \cup (\{a_0, 0_S\} \times \{a_0, 0_S\})$.

Proof. We have $a_0 \notin \underline{\varepsilon}(N)$ by 4.13(iii), and hence $0_S \in S$ by 4.4. Now, assume that $\alpha \neq \alpha_1$ (see 4.14(iv)). Then $a_0 \neq 0_S$ and, by 4.4, $o_N \in N$, a_0 is the smallest element of $R_1 = R \setminus \{0_S\}$ and $\underline{\varepsilon}(N) = R \setminus \{a_0\}$. Consequently, $B = R \setminus \{0_S, a_0\}$ and the rest is clear. ∎

4.16 Lemma. *Assume that* (γ) *is true and let* a_0 *be maximal in* A. *Then* $0_S \in S$ *and* $\alpha = \alpha_1$.

Proof. Combining 4.13(iii) and 4.2, we get $a_0 = 0_S$. The rest is clear. ∎

4.17 Lemma. *Assume that* (β) *is true and the set* A *has no maximal element. Then:*

(i) *The set* $A' = A \cap \underline{\varepsilon}(N)$ *is infinite, has no maximal element and the set* $A \setminus A'$ *contains at most one element.*

(ii) *For every* $a \in A'$ *there is an infinite strictly increasing sequence* $a_1 < a_2 < a_3 < \dots$ *of elements from* A' *such that* $a_1 = a$ *and all the elements* a_i *belong to the same block of* α.

(iii) $a_i = \underline{\varepsilon}(x_i)$, $x_i \in N$ *and* $x_1 > x_2 > x_3 > \dots$.

Proof. (i) See 4.4.
(ii) We have $a_1 = a = e(x_1)$, $x_1 \in K$. By 4.12, there is $a_2 \in R$ with $a_1 < a_2$ and $(a_1, a_2) \in \alpha$. Then $a_2 \in A$ and, since $a_1 \leq a_2$, we have $a_2 \in A'$ (use 4.4). The rest is clear. ∎

Consider the following two conditions:

(δ) If $x_1 > x_2 > x_3 > \dots$ is an infinite strictly decreasing sequence of elements from M then for every $x \in M \setminus \{0_M\}$ there is $i \geq 1$ with $x \geq x_i$.

(ε) If $a_1 < a_2 < a_3 < \dots$ is an infinite strictly increasing sequence of elements from R then for every $a \in R \setminus \{o_S\}$ there is $i \geq 1$ with $a \leq a_i$.

4.18 Lemma.

(i) (ε) *implies* (δ).

(ii) *If* (β) *is true then the conditions* (δ) *and* (ε) *are equivalent.*

Proof. Use 4.4. ∎

4.19 Lemma. *Assume that* (ε) *is true and the set* A *has no maximal element. Then* $B = \{o_S\}$.

Proof. Since A has no maximal element, there is an infinite strictly increasing sequence $a_1 < a_2 < a_3 < \dots$ of elements from A. If $b \in B \setminus \{o_S\}$ then $b \le a_i$ for some $i \ge 1$. Now $(a_i, o_S) = (a_i + b, a_i + o_S) \in \alpha$, a contradiction. ∎

4.20 Lemma. *Assume that* (β) *and* (δ) *(or* (γ) *and* (ε)*) are true and the set* A *has no maximal element. Then just one of the following two cases holds:*

(1) $\alpha = \alpha_2 = (R_2 \times R_2) \cup \mathrm{id}_R$, *where* $R_2 = R \setminus \{o_S\}$.

(2) $0_S \in S$ *and* $\alpha = \alpha_3 = (R_3 \times R_3) \cup \mathrm{id}_R$, *where* $R_3 = R \setminus \{0_S, o_S\}$.

Proof. The conditions (β) and (ε) are satisfied (see 4.3 and 4.18). By 4.19, $B = \{o_S\}$. Let $a, b \in A' = A \cap \underline{\varepsilon}(N)$. By 4.17(ii), there are infinite strictly increasing sequences $a = a_1 < a_2 < a_3 < \dots$ and $b = b_1 < b_2 < b_3 < \dots$ of elements from A' such that all a_i belong to one block of α and the same is true for the elements b_i. Using (ε), we find $i, j \ge 1$ such that $b \le a_i$ and $a \le b_j$. Then $(a + b, a_i) = (a + b, a_i + b) \in \alpha$, $(a + b, b_j) = (a + b, a + b_j) \in \alpha$, $(a_i, b_j) \in \alpha$ and, finally $(a, b) \in \alpha$. Consequently, $A' \times A' \subseteq \alpha$. To finish the proof, we have to use 4.4. If $\underline{\varepsilon}(N) = R$ then $A' = A = R \setminus \{o_S\}$ and (1) is true. If $0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$ then $A' = A \setminus \{0_S\}$ and either (1) or (2) is true. Finally, if $\underline{\varepsilon}(N) = R \setminus \{a_0\}$ then $A' = A \setminus \{a_0\}$, $(0_S, a) \in \alpha$ for every $a \in A'$, $a > 0_S$, and we get $(a_0, a) = (0_S + a_0, a + a_0) \in \alpha$. Thus $\alpha = \alpha_2$ in this case. ∎

4.21 Proposition. *Assume that the conditions* (γ) *and* (ε) *(or* (α) *and* (δ)*) are true. Then just one of the following three cases holds:*

(1) $0_S \in S$ *and* $\alpha = \alpha_1 = (R_1 \times R_1) \cup \mathrm{id}_R$, $R_1 = R \setminus \{0_S\}$.

(2) $\alpha = \alpha_2 = (R_2 \times R_2) \cup \mathrm{id}_R$, $R_2 = R \setminus \{0_S\}$.

(3) $0_S \in S$ *and* $\alpha = \alpha_3 = (R_3 \times R_3) \cup rmid_R$, $R_3 = R \setminus \{0_S, o_S\}$.

Proof. First, if (α) and (δ) are true then (β), (γ) and (ε) follow from 4.3 and 4.18. Now, if the set A has at least one maximal element, then $\alpha = \alpha_1$ and $0_S \in S$ is proved in 4.16. On the other hand, if there is no maximal element in A, then $\alpha = \alpha_2, \alpha_3$ is proved in 4.20. ∎

5. Main results (summary)

5.1 Let S be a non-trivial semiring and M be a characteristic (left S-)semimodule. By 2.3, the mapping $\underline{\varepsilon}$ is an injective mapping of $N = M \setminus \{o_M\}$ into $R = \underline{R}(S)$ and we have $|R| \ge 2$, $|N| \ge 2$ and $|M| \ge 3$.

5.1.1 Theorem.

(i) *The (left* S*-)semimodule* M *is simple.*

(ii) *The right* S*-semimodule* R_S *is faithful and* $o_S = o_R \in R$.

(iii) *The semiring* S *is simple if and only if* $R + S = S$ *and the semimodule* R_S *is simple.*

(iv) $0_S \in S$ *if and only if* $0_R \in R$ *(and then* $0_S = 0_R$*).*

(v) $a = 0_S$ *if and only if* $aN = \{0_M\}$*.*

(vi) $\underline{\varepsilon}(w) = 0_S$ *if and only if* w *is the greatest element of the set* N*.*

(vii) *If* $e \in S$ *is a right multiplicatively neutral element of* S *then* $e = 1_S$ *is multiplicatively neutral. Moreover, if* $R + S = S$ *then* $0_S \in S$*.*

(viii) *If* $Sx = M$ *for at least one* $x \in M$ *and if* $e \in S$ *is a left multiplicatively neutral element then* $e = 1_S$ *is multiplicatively neutral.*

(ix) *If the semiring* S *is simple,* $|S| \geq 3$ *and if* $e \in S$ *is left multiplicatively neutral then* $e = 1_S$ *is multiplicatively neutral.*

Proof. See 2.2, 2.4(ii), 2.5, 2.6, 2.10, 2.11, 2.12, 2.13, 2.17 and 2.18. ∎

5.1.2 Theorem. *Assume that the condition* (α) *is true. Then:*

(i) *Either* $\underline{\varepsilon}(N) = R$ *or* $0_S \in S$ *and* $\underline{\varepsilon}(N) = R \setminus \{0_S\}$*.*

(ii) *No proper subsemimodule of* M *is faithful.*

(iii) *If* S *is simple then* $Sx = M$ *for every* $x \in M \setminus \{0_M, o_M\}$*.*

Proof. See 4.2, 4.8 and 4.11. ∎

5.1.3 Remark. Assume that the condition (α) is satisfied. Then either $\underline{\varepsilon}(N) = R$ or $0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$. Now, let M' be any characteristic semimodule. According to 4.3, condition (γ) is fulfilled, and hence we can use 4.2 to show that $\underline{\varepsilon}(N') = R$ or $\underline{\varepsilon}(N') = R \setminus \{0_S\}$, where $N' = M' \setminus \{o_{M'}\}$.

(i) If $\underline{\varepsilon}(N) = R = \underline{\varepsilon}(N')$ then ${}_SM \cong {}^1_SR \cong {}_SM'$, and so ${}_SM \cong {}_SM'$ (see 3.3 and [1, 7.2]).

(ii) If $0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\} = \underline{\varepsilon}(N')$ then ${}_SM \cong {}^2_SR \cong {}_SM'$, and so ${}_SM \cong {}_SM'$ (see 3.4 and [1, 7.3]).

(iii) Assume that $0_S \in S$, $\underline{\varepsilon}(N) = R$ and $\underline{\varepsilon}(N') = R \setminus \{0_S\}$. By 3.3 and 3.4, we have $M \cong {}^1_SR$ and $M' \cong {}^2_SR$. By [1, 7.3.7], the semimodule ${}^2_SR$ (or M') is isomorphic to a (proper) subsemimodule of ${}^1_SR$ (or M). But this is a contradiction with 5.1.2(ii).

(iv) Assume, finally, that $0_S \in S$, $\underline{\varepsilon}(N) = R \setminus \{0_S\}$ and $\underline{\varepsilon}(N') = R$. Again, we have $M \cong {}^2_SR$, $M' \cong {}^1_SR$ and M is isomorphic to a (proper) subsemimodule of M'. But M' satisfies (β), and hence $o_{N'} \in N'$ and $G = M' \setminus \{o_{N'}\}$ (see 4.6).

5.2. Let S be a semiring such that $|R| \geq 2$.

5.2.1 Theorem. *Assume that the condition* (γ) *is satisfied. The following two conditions are equivalent:*

(i) *There is a characteristic semimodule* M *such that* $0_S \in \underline{\varepsilon}(M) \setminus \{o_M\}$ *in case when* $0_S \in S$*.*

(ii) *Condition* 3.3(ii) *is satisfied.*

Proof. Combine 3.3 and 4.2. ∎

5.2.2 Theorem. *Assume that the condition* (γ) *is satisfied and* $0_S \in S$. *The following two conditions are equivalent:*

(i) *There is a characteristic semimodule* M *such that* $0_S \notin \underline{\varepsilon}(M) \setminus \{o_M\}$.

(ii) *Condition* 3.4(ii) *is satisfied.*

Proof. Combine 3.4 and 4.2. ∎

5.2.3 Remark. Assume that (γ) is true. If $0_S \notin S$ then there is (up to isomorphism) at most one characteristic semimodule. If $0_S \in S$ then there are (up to isomorphism) at most two characteristic semimodules.

5.2.4 Remark. Assume that $0_S \in S$. Clearly, condition 3.3(ii) implies condition 3.4(ii).

Now, assume that (ε) and 3.4(ii) are true (in fact, if (ε) is true then both $R = \underline{R}(S)$ and $\underline{R}(S)^+$ are lattices). If $b \in S$ is such that $o_Q \notin Q = Q_{0_S,b} = \{c \in R \mid cb = o_S\}$ then (ε) yields $Q = R_2 = R \setminus \{o_S\}$. But this gives a contradiction (see 3.5). It follows that the condition 3.3(ii) is true.

References

[1] Batíková, B., Kepka, T., Němec, P., *On how to construct left semimodules from the right ones*, Ital. J. Pure Appl. Math., 32 (2014), 561–578.

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