NOTE ON YOUNG AND ARITHMETIC-GEOMETRIC MEAN INEQUALITIES FOR MATRICES

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Abstract. In this short note, we prove that the conjecture of singular value Young inequality holds when \( j = n \). Meanwhile, we also present a refinement of the arithmetic-geometric mean inequality for unitarily invariant norms.

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1. Introduction

Let \( M_n \) be the space of \( n \times n \) complex matrices. We shall always denote the singular values of \( A \) by \( s_1(A) \geq \cdots \geq s_n(A) \geq 0 \). If \( A \in M_n \) has real eigenvalues, we label them as \( \lambda_1(A) \geq \cdots \geq \lambda_n(A) \). Let \( \| \cdot \| \) denote any unitarily invariant norm on \( M_n \).

Let \( A, B \) be positive semidefinite. Ando proved in [1] that if \( v \in [0,1] \), then

\[ s_j \left( A^v B^{1-v} \right) \leq s_j \left( vA + (1-v)B \right), \quad j = 1, \ldots, n. \]

This is Young inequality for singular values. Recently, Lin [2] posed the following.

Conjecture. Let \( A, B \) be positive semidefinite and \( v \in [0,1] \). Then

\[ s_j \left( A^v B^{1-v} \right) \leq s_j \left( vA^{1/2} + (1-v)B^{1/2} \right)^2, \quad j = 1, \ldots, n. \]

If the conjecture holds, then it is a strength of Ando’s inequality. In this short note, we prove that the conjecture holds when \( j = n \).

Let \( A, B \) be positive semidefinite. Bhatia and Kittaneh [3] proved that

\[
\| AB \| \leq \frac{1}{4} \left\| (A + B)^2 \right\|,
\]

where \( \| \cdot \| \) is a unitarily invariant norm.
which is an arithmetic-geometric mean inequality for unitarily invariant norms. In this short note, we also obtain a refinement of inequality (1.1).

2. Main results

In this section, we first prove that Lin’s conjecture holds when $j = n$.

Theorem 2.1. Let $A, B$ be positive semidefinite and $v \in [0, 1]$. Then

$$s_n (A^v B^{1-v}) \leq s_n \left( v A^{1/2} + (1 - v) B^{1/2} \right)^2 .$$

Proof. This is obviously true if either $A$, or $B$ is not invertible. So assume $A$ and $B$ are invertible. Then

$$s_n (A^v B^{1-v}) = \lambda_n^{1/2} (B^{1-v} A^{2v} B^{1-v}) = \lambda_n^{1/2} (B^{v-1} A^{-2v} B^{v-1})$$

On the other hand, we have

$$\lambda_1^{1/2} (B^{v-1} A^{-2v} B^{v-1}) = s_1 (A^{-v} B^{v-1}) \geq \lambda_1 (A^{-v} B^{v-1})$$

It follows from (2.1) and (2.2) that

$$s_n (A^v B^{1-v}) \leq \lambda_n^{-1} (A^{-v} B^{v-1})$$

Next, we give a refinement of inequality (1.1). To do this, we need the following lemma [3].

Lemma 2.2. Let $A, B$ be positive semidefinite. Then

$$\| A^{1/2} (A + B) B^{1/2} \| \leq \frac{1}{2} \| (A + B)^2 \|$$

Theorem 2.2. Let $A, B$ be positive semidefinite. Then

$$\| AB \| + \left( \int_{1/2}^{3/2} \| A^v B^{2-v} + A^{2-v} B^v \| \, dv - 2 \| AB \| \right) \leq \frac{1}{4} \| (A + B)^2 \| .$$

Proof. It is known [4, p.265] that the function

$$g (r) = \| A^r B^{1-r} + A^{1-r} B^r \|$$
is convex on \([0,1]\). Replacing \(A\) by \(A^2\), \(B\) by \(B^2\), and \(2r\) by \(v\), we know that the function
\[
f(v) = \|A^v B^{2-v} + A^{2-v} B^v\|
\]
is convex on \([0,2]\); it follows that this function is also convex on \(\left[\frac{1}{2}, \frac{3}{2}\right]\). Therefore, if \(v \in \left[\frac{1}{2}, 1\right]\), then by the convexity of the function of \(f(v)\), we have
\[
(2.3) \quad f \left( \lambda \times \frac{1}{2} + (1 - \lambda) \times 1 \right) \leq \lambda f \left( \frac{1}{2} \right) + (1 - \lambda) f (1).
\]
Let
\[
v = \lambda \times \frac{1}{2} + (1 - \lambda) \times 1.
\]
Then, we know that inequality (2.3) is equivalent to
\[
f(v) \leq (2 - 2v) f \left( \frac{1}{2} \right) + (2v - 1) f (1),
\]
which yields
\[
(2.4) \quad \int_{1/2}^1 f(v) \, dv \leq \frac{1}{4} \left( f \left( \frac{1}{2} \right) + f (1) \right).
\]
On the other hand, if \(v \in \left[1, \frac{3}{2}\right]\), then by the convexity of the function of \(f(v)\), we have
\[
f \left( \lambda \times 1 + (1 - \lambda) \times \frac{3}{2} \right) \leq \lambda f (1) + (1 - \lambda) f \left( \frac{3}{2} \right),
\]
which implies
\[
(2.5) \quad \int_{1}^{3/2} f(v) \, dv \leq \frac{1}{4} \left( f (1) + f \left( \frac{3}{2} \right) \right).
\]
It follows from (2.4) and (2.5) that
\[
\int_{1/2}^{3/2} f(v) \, dv \leq \frac{1}{2} \left( f (1) + f \left( \frac{1}{2} \right) \right),
\]
which is equivalent to
\[
\|AB\| + \left( \int_{1/2}^{3/2} \|A^v B^{2-v} + A^{2-v} B^v\| \, dv - 2 \|AB\| \right) \leq \frac{1}{2} \|A^{1/2} (A + B) B^{1/2}\|.
\]
So, Lemma 2.2 and the last inequality complete the proof. \(\blacksquare\)

**Remark 2.1.** By the convexity of the function of \(f(v)\), we know that
\[
2 \|AB\| \leq \|A^v B^{2-v} + A^{2-v} B^v\|
\]
and hence
\[ \int_{1/2}^{3/2} \| A^v B^{2-v} + A^{2-v} B^v \| \, dv - 2 \| AB \| \geq 0. \]

So Theorem 2.2 is a refinement of the arithmetic-geometric mean inequality (1.1).

References


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