THE STRUCTURE OF FINITE GROUPS
WITH $c^*$-NORMAL SUBGROUPS

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Abstract. Let $H$ be a subgroup of a finite group $G$. $H$ is said to be $c^*$-normal in $G$ if there exists a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K$ is $s$-quasinormally embedded in $G$. We fix in every non-cyclic Sylow subgroup $P$ of $G$ some subgroup $D$ satisfying $1 < |D| < |P|$ and study the structure of $G$ under the assumption that every subgroup $H$ of $P$ with $|H| = |D|$ is $c^*$-normal in $G$. Some recent results are generalized and unified.

Keywords: $c$-normal subgroup, $s$-quasinormally embedded subgroup, saturated formation.


1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation in [3]. $G$ always means a group, $|G|$ denotes the order of $G$ and $\pi(G)$ denotes the set of all primes dividing $|G|$. Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation, provided that (1) if $G \in \mathcal{F}$ and $H \leq G$, then $G/H \in \mathcal{F}$, and (2) if $G/M$ and $G/N$ are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for any normal subgroups $M$, $N$ of $G$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation.

A subgroup $H$ of $G$ is called $s$-quasinormal (or $s$-permutable, $\pi$-quasinormal) in $G$ provided $H$ permutes with all Sylow subgroups of $G$, i.e, $HP = PH$ for any Sylow subgroup $P$ of $G$. This concept was introduced by Kegel in [6] and has been studied extensively by Deskins [2] and Schmidt [14]. More recently, Ballester-Bolinches and Pedraza-Aguilera [1] generalized $s$-quasinormal subgroups to $s$-quasinormally embedded subgroups. A subgroup $H$ is said to

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be an $s$-quasinormally embedded in $G$ if for each prime $p$ dividing the order of $H$, a Sylow $p$-subgroup of $H$ is also a Sylow $p$-subgroup of some $s$-quasinormal subgroup of $G$. Clearly, every $s$-quasinormal subgroup of $G$ is an $s$-quasinormally embedded subgroup of $G$, but the converse does not hold. The authors of [18] obtained many interesting results. For example, they prove:

**Theorem 1.1** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ a Sylow $p$-subgroup of $G$. Then $G$ is $p$-nilpotent if and only if there is a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $s$-quasinormally embedded in $G$.

As another generalization of the normality, Wang [15] introduced the following concept: A subgroup $H$ of $G$ is called $c$-normal in $G$ if there is a normal subgroup $K$ such that $G = HK$ and $H \cap K \leq H_G$, where $H_G$ is the normal core of $H$ in $G$. Many authors consider some subgroups of a Sylow subgroup of a group when investigating the structure of $G$, such as in [5], [7]-[8], [17], etc. In [5], Jaraden and Skiba provide the following result.

**Theorem 1.2** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup $P$ of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $c$-normal in $G$, where $F^*(E)$ is the generalized Fitting subgroup of $E$. Then $G \in \mathcal{F}$.

There are examples to show $s$-quasinormally embedded and $c$-normal are two different properties of subgroups in [16]. There is no inclusion-relationship between the two concepts. Hence it is meaningful to unify and generalize the two concepts and related results.

In [16], the authors introduce a new subgroup embedding property called $c^*$-normal which is a generalization of both $c$-normality and $s$-quasinormal embedding.

**Definition 1.1** A subgroup $H$ of a group $G$ is called $c^*$-normal in $G$ if there exists a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K$ is $s$-quasinormally embedded in $G$.

It is clear from Definition 1.1 that each of $c$-normality and $s$-quasinormal embedding implies weakly $c^*$-normal. But the converses do not hold.

The aim of this article is to unify and improve above Theorems using $c^*$-normal subgroups. The following result can unify and generalize some related ones including the above two theorems.
Theorem 1.3 (i.e., Theorem 3.4) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup $P$ of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $c^*$-normal in $G$, where $F^*(E)$ is the generalized Fitting subgroup of $E$. Then $G \in \mathcal{F}$.

2. Basic definitions and preliminary results

In this section, we collect some known results that are useful later.

Lemma 2.1. ([16]) Let $H$ be a subgroup of a group $G$.

(i) If $H$ is $c^*$-normal in $G$ and $H \leq M \leq G$, then $H$ is $c^*$-normal in $M$;

(ii) Let $N < G$ and $N \leq H$. Then $H$ is $c^*$-normal in $G$ if and only if $H/N$ is $c^*$-normal in $G/N$;

(iii) Let $p$ be a set of primes, $H$ a $p$-subgroup of $G$ and $N$ a normal $\pi'$-subgroup of $G$. If $H$ is $c^*$-normal in $G$, then $HN/N$ is $c^*$-normal in $G/N$.

By Corollary 3.2 in [16], we have the following result.

Lemma 2.2. ([16]) Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If all maximal subgroups of $P$ are $c^*$-normal in $G$. Then $G$ is $p$-nilpotent.

Lemma 2.3. Suppose that $H$ is a $p$-subgroup for some prime $p$ and $H$ is not $s$-quasinormally embedded in $G$. Assume that $H$ is $c^*$-normal in $G$. Then $G$ has a normal subgroup $M$ such that $|G : M| = p$ and $G = HM$.

Proof. By the hypothesis, $G$ has a normal subgroup $T$ such that $HT = G$ and $T \cap H < H$. Hence $G$ has a proper normal subgroup $K$ such that $T \leq K$. Since $G/K$ is a $p$-group, $G$ has a normal maximal subgroup $M$ such that $HM = G$ and $|G : M| = p$.

Lemma 2.4. ([9]) Let $H$ be a nilpotent subgroup of a group $G$. Then the following statements are equivalent:

(i) $H$ is $s$-quasinormal in $G$;

(ii) $H \leq F(G)$ and $H$ is $s$-quasinormally embedded in $G$.

Lemma 2.5. Suppose that $H \leq O_p(G)$ and that $H$ is $c^*$-normal in $G$. Then $H$ is weakly $s$-permutable in $G$.

Proof. By the hypothesis, $G$ has a normal subgroup $B$ such that $HB = G$ and $H \cap B$ is $s$-quasinormally embedded in $G$. Note that $H \cap B \leq H \leq O_p(G)$, then
by Lemma 2.4 $H \cap B$ is $s$-quasinormal in $G$, and thus $H \cap B \leq H_{*G}$. Hence $H$ is weakly $s$-permutable in $G$.

By Lemma 2.11 of [13] and Lemma 2.5, we have the following.

**Lemma 2.6.** Let $N$ be an elementary abelian normal $p$-subgroup of a group $G$. If there exists a subgroup $D$ in $N$ such that $1 < |D| < |N|$ and every subgroup $H$ of $N$ with $|H| = |D|$ is $c^*$-normal in $G$, then there exists a maximal subgroup $M$ of $N$ such that $M$ is normal in $G$.

By Lemma 2.12 of [13] and Lemma 2.5, we have the following.

**Lemma 2.7.** Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and let $G$ be a group with solvable $\mathcal{F}$-residual $P = G^\mathcal{F}$. Suppose that every maximal subgroup of $G$ not containing $P$ belongs to $\mathcal{F}$. Then $P$ is a $p$-group for some prime $p$. In addition, if every cyclic subgroup of $P$ with prime order or order $4$ (if $p = 2$ and $P$ is non-abelian) not having a supersolvable supplement in $G$ is $c^*$-normal in $G$, then $|P/\Phi(P)| = p$.

The generalized Fitting subgroup $F^*(G)$ of $G$ is the unique maximal normal quasinilpotent subgroup of $G$. Its definition and important properties can be found in [4, X, 13]. We would like to give the following basic facts we will use in our proof.

**Lemma 2.8.** ([4, X, 13]) Let $G$ be a group and $M$ a subgroup of $G$.

(i) If $M$ is normal in $G$, then $F^*(M) \leq F^*(G)$;

(ii) $F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \text{Soc}(F(G)C_G(F(G))/F(G))$;

(iii) $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.

**Lemma 2.9.** ([13]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup $P$ of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is weakly $s$-permutable in $G$, where $F^*(E)$ is the generalized Fitting subgroup of $E$. Then $G \in \mathcal{F}$.

3. Main results

In this section, we will prove our main results.

**Theorem 3.1** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $c^*$-normal in $G$. Then $G$ is $p$-nilpotent.
Proof. Suppose that the theorem is false and let $G$ be a counterexample of minimal order. We will derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then by Lemma 2.1, $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is $p$-nilpotent by the choice of $G$. Then $G$ is $p$-nilpotent, a contradiction.

Step 2. $|D| > p$.

Suppose that $|D| = p$. Then by Lemma 2.1, $G$ is a minimal non-$p$-nilpotent group, so $G = [P]Q$, where $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$, $p \neq q$. Set $\Phi = \Phi(P)$ and let $X/\Phi$ be subgroup of $P/\Phi$ of order $p$, $x \in X\Phi$ and $L = \langle x \rangle$. Then $L$ is order $p$ or $4$. $L$ is $c^*$-normal in $G$. Lemma 2.7 implies that $|P/\Phi| = p$, it follows that $G$ is $p$-nilpotent, a contradiction.


By Lemma 2.3.

Step 4. If $N \leq P$ and $N$ is minimal normal in $G$, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, $N$ is elementary abelian. By Lemma 2.6, $N$ has a maximal subgroup which is normal in $G$, contrary to the minimality of $N$.

Step 5. Suppose that $N \leq P$ and $N$ is minimal normal in $G$. Then $G/N$ is $p$-nilpotent.

If $|N| < |D|$, $G/N$ satisfies the hypotheses of the theorem by Lemma 2.1. Thus $G/N$ is $p$-nilpotent by the minimal choice of $G$. So we may suppose that $|N| = |D|$ by Step 4. Let $N \leq K \leq P$ such that $|K/N| = p$. By Step 2, $N$ is non-cyclic, so $K$ is also non-cyclic, it follows that $K$ has a maximal subgroup $L \neq N$ and $K = LN$. So $L$ is $c^*$-normal in $G$ (note that $|L| = |D|$), it follows that $K/N = LN/N$ is $c^*$-normal in $G/N$. If $P/N$ is abelian, then $G/N$ satisfies hypothesis. Next suppose that that $P/N$ is a non-abelian 2-group. So every subgroup of $P$ of order $2|D|$ is $c^*$-normal in $G$. In this case one can show as above that every subgroup $X$ of $P$ containing $N$ and such that $|X : N| = 4$ is $c^*$-normal in $G$. Therefore $G/N$ also satisfies the hypothesis. By the minimal choice of $G$, $G/N$ is $p$-nilpotent.

Step 6. $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup $N$ of $G$ contained in $O_p(G)$. By Step 5, $G/N$ is $p$-nilpotent. It is easy to see that $N$ is the unique minimal normal subgroup of $G$ contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. Hence $O_p(G)$ is an elementary abelian $p$-group. On the other hand, $G$ has a maximal subgroup $M$ such that $G = MN$ and $M \cap N = 1$. It is easy
to deduce that \(O_p(G) \cap M = 1\), \(N = O_p(G)\) and \(M \cong G/N\) is \(p\)-nilpotent. Then \(G\) can be written as \(G = N(M \cap P)M_p\), where \(M_p\) is the normal \(p\)-complement of \(M\). Pick a maximal subgroup \(S\) of \(M_p = P \cap M\). Then \(NSM_p\) is a subgroup of \(G\) with index \(p\). Since \(p\) is the minimal prime in \(\pi(G)\), we know that \(NSM_p\) is normal in \(G\). Now by Step 3 and the induction, we have \(NSM_p\) is \(p\)-nilpotent. Therefore, \(G\) is \(p\)-nilpotent, a contradiction. Thus \(O_p(G) = 1\).

**Step 7.** The minimal normal subgroup \(L\) of \(G\) is not \(p\)-nilpotent.

If \(L\) is \(p\)-nilpotent, then it follows from the fact that \(L_p\) char \(L < G\) that \(L_p \leq O_p(G) = 1\). Thus \(L\) is a \(p\)-group. Whence \(L \leq O_p(G) = 1\) by Step 6, a contradiction.

**Step 8.** \(G\) is a non-abelian simple group.

Suppose that \(G\) is not a simple group. Take a minimal normal subgroup \(L\) of \(G\). Then \(L < G\). If \(|L|_p > |D|\), then \(L\) is \(p\)-nilpotent by the minimal choice of \(G\), contrary to Step 7. If \(|L|_p \leq |D|\). Take \(P_s \geq L \cap P\) such that \(|P_s| = p|D|\). Hence \(P_s\) is a Sylow \(p\)-subgroup of \(P_sL\). Since every maximal subgroup of \(P_s\) is of order \(|D|\), every maximal subgroup of \(P_s\) is weakly \(c^*\)-normal in \(G\) by hypotheses, thus in \(P_sL\) by Lemma 2.1. Now applying Lemma 2.2, we get \(P_sL\) is \(p\)-nilpotent. Therefore, \(L\) is \(p\)-nilpotent, contrary to Step 7.

**Step 9.** All subgroups of \(P\) of order \(|D|\) and \(2|D|\) (if \(P\) is a non-abelian \(2\)-group and \(|P : D| > 2\) are \(s\)-quasinormally embedded in \(G\).

Let \(H \leq P\) and \(|H| = |D|\) or \(2|D|\). If \(H\) is not \(s\)-quasinormally embedded in \(G\), by Lemma 2.3, there is a normal subgroup \(M\) of \(G\) such that \(|G : M| = p\). By Step 3, \(M\) is \(p\)-nilpotent, it follows that \(G\) is \(p\)-nilpotent, a contradiction.

**Step 10.** The final contradiction.

By Theorem 1.1, \(G\) is \(p\)-nilpotent, a contradiction. The contradiction completes the proof.

The following corollary is immediate from Theorem 3.1.

**Corollary 3.2** Suppose that \(G\) is a group. If every non-cyclic Sylow subgroup of \(G\) has a subgroup \(D\) such that \(1 < |D| < |P|\) and every subgroup \(H\) of \(P\) with order \(|H| = |D|\) or with order \(2|D|\) (if \(P\) is a nonabelian \(2\)-group and \(|P : D| > 2\) is \(c^*\)-normal in \(G\), then \(G\) has a Sylow tower of supersolvable type.

**Theorem 3.3** Let \(\mathcal{F}\) be a saturated formation containing \(\mathcal{W}\), the class of all supersolvable groups and \(G\) a group with \(E\) as a normal subgroup of \(G\) such that \(G/E \in \mathcal{F}\). Suppose that every non-cyclic Sylow subgroup of \(E\) has a subgroup \(D\) such that \(1 < |D| < |P|\) and every subgroup \(H\) of \(P\) with order \(|H| = |D|\) or with order \(2|D|\) (if \(P\) is a nonabelian \(2\)-group and \(|P : D| > 2\) is \(c^*\)-normal in \(G\). Then \(G \in \mathcal{F}\).
Proof. Suppose that $P$ is a non-cyclic Sylow $p$-subgroup of $E$, $\forall p \in \pi(E)$. Since $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $c^*$-normal in $G$ by hypotheses, thus in $E$ by Lemma 2.1. Applying Corollary 3.2, we conclude that $E$ has a Sylow tower of supersolvable type. Let $q$ be the maximal prime divisor of $|E|$ and $Q \in \text{Syl}_q(E)$. Then $Q \leq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathcal{F}$. For any subgroup $H$ of $Q$ with $|H| = |D|$, since $Q \leq O_q(G)$, $H$ is weakly $s$-permutable in $G$ by Lemma 2.5. Since $F^*(Q) = Q$ by Lemma 2.8, we get $G \in \mathcal{F}$ by applying Lemma 2.9.

Theorem 3.4 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $c^*$-normal in $G$. Then $G \in \mathcal{F}$.

Proof. We distinguish two cases:

Case 1. $\mathcal{F} = \mathcal{U}$.

Let $G$ be a minimal counter-example.

Step 1. Every proper normal subgroup $N$ of $G$ containing $F^*(E)$ (if it exists) is supersolvable.

If $N$ is a proper normal subgroup of $G$ containing $F^*(E)$, then $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 2.8 (iii), $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup $P$ of $F^*(E \cap N) = F^*(E)$, $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $c^*$-normal in $G$ by hypotheses, thus in $N$ by Lemma 2.1. So $N$ and $N \cap H$ satisfy the hypotheses of the theorem, the minimal choice of $G$ implies that $N$ is supersolvable.

Step 2. $E = G$.

If $E < G$, then $E \in \mathcal{U}$ by Step 1. Hence $F^*(E) = F(E)$ by Lemma 2.8. It follows that every Sylow subgroup of $F^*(E)$ is normal in $G$. By Lemma 2.5, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is weakly $s$-permutable in $G$. Applying Lemma 2.9 for the special case $\mathcal{F} = \mathcal{U}$, $G \in \mathcal{U}$, a contradiction.

Step 3. $F^*(G) = F(G) < G$.

If $F^*(G) = G$, then $G \in \mathcal{F}$ by Theorem 3.3, contrary to the choice of $G$. So $F^*(G) < G$. By Step 1, $F^*(G) \in \mathcal{U}$ and $F^*(G) = F(G)$ by Lemma 2.8.
Step 4. The final contradiction.

Since $F^*(G) = F(G)$, each non-cyclic Sylow subgroup of $F^*(G)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is weakly $s$-permutable in $G$ by Lemma 2.5. Applying Lemma 2.9, $G \in \mathcal{U}$, a contradiction.

Case 2. $\mathcal{F} \neq \mathcal{U}$.

By hypotheses, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $c^*$-normal in $G$, thus in $E$ Lemma 2.1. Applying Case 1, $E \in \mathcal{U}$. Then $F^*(E) = F(E)$ by Lemma 2.8. It follows that each Sylow subgroup of $F^*(E)$ is normal in $G$. By Lemma 2.5, each non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is weakly $s$-permutable in $G$. Applying Lemma 2.9, $G \in \mathcal{F}$. These complete the proof of the theorem.

The following corollaries are immediate from Theorem 3.4.

**Corollary 3.5** (17, Theorem 3.1) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(H)$ is $c$-normal in $G$.

**Corollary 3.6** (17, Theorem 3.2) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is $c$-normal in $G$.

**Corollary 3.7** (12, Theorem 1.1) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is $\pi$-quasinormally embedded in $G$.

**Corollary 3.8** (12, Theorem 1.2) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is $\pi$-quasinormally embedded in $G$.

**Corollary 3.9** (10, Theorem 3.4) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(H)$ is $s$-quasinormal in $G$.

**Corollary 3.10** (11, Theorem 3.3) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$.
Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is $s$-quasinormal in $G$.

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References


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