INJECTIVITY OF G-NOMINAL SETS

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Abstract. In this paper, we consider injectivity for nominal sets, G-sets for a subgroup G of the group $\mathrm{Perm}(\mathbb{D})$ of permutations on an infinite countable set $\mathbb{D}$, with finitely supported elements. Also, we study injectivity of nominal sets with respect to monomorphisms whose domains or codomains are single-orbit. Furthermore, we define the category of G_C-sets, for finite subsets C of $\mathbb{D}$, where G_C is the set of elements of G fixing the elements of C, and study injectivity in such categories.

Keywords: S-set, nominal set, single-orbit nominal set, injectivity.
2010 Mathematical Subject Classification: 08B30, 20B30, 20M35, 20M30, 18A20.

1. Introduction and preliminaries

The theory of nominal sets originates from the work of Fraenkel in 1922, and developed by Mostowski in the 1930s, and it is also known as the FM set theory. The FM set theory is an axiomatic set theory which provides a mathematical model for names in syntax. At that time, they used nominal sets to prove the independence of the axiom of choice with the other axioms (in the classical Zermelo-Fraenkel (ZF) set theory). In computer science, Gabbay and Pitts rediscovered nominal sets to properly model the syntax of formal systems involving variable binding operations, see [13]. Since then, nominal sets have become a lively topic in semantics. Bojanczyk et al., used nominal sets in automata theory as a framework for describing automata on data words [3]. Bojanczyk defined the monoids in the category of nominal sets (also called nominal monoids) and used them in the study of languages over infinite alphabets [3]. The theory of syntactic monoids for languages of data words represents the same theory as the theory of finite monoids in the category of nominal sets, and under certain conditions, a language of data words is definable in first-order logic if and only if its syntactic monoid is aperiodic, [4]. Shinwell and Pitts used nominal techniques in order to implement a functional programming language incorporating facilities for manipulating syntax involving names and binding operations [20]. Bojanczyk et al., have also defined computation in nominal sets by presenting a basic functional programming language, [5]. Alexandru and Ciobanu have presented nominal algebraic structures in terms of finitely supported objects, [2]. Nominal techniques have also been used in game theory [1], in logic ([12],[16]), in topology [15], in domain theory ([19],[21]), and in proof theory [22]. Also, in [17], nominal sets have been used to model language constructs for hiding the identity of a name outside a given scope. Because of the importance of nominal sets we are going to study injectivity, which is one of the central notions in many branches of mathematics, in this category. For injectivity, see, for example, [10, 8, 14].

1.1. The category G-Set

In this subsection, we briefly recall the ingredients of the basic category G-**Set**, of sets with the actions of a group G on them, which are needed in this paper to define nominal sets. For more information see [9], [14].

Definition 1.1. Let G be a group (or a monoid) with 1 as its identity. Then, a set X together with a function

$$\lambda : G \times X \to X,$$

called the *action* of G on X (or the G-action), is called a (left) G-*set* if (denoting $\lambda(g, x)$ by gx) $(gh)x = g(hx)$ and $1x = x$, for every $x \in X$ and $g, h \in G$.

Remark 1.2. A G-set X can be regarded as a unary algebra with the family of unary operations

$$L_g : X \to X,\ \ L_g(x) = gx,$$

for $g \in G$, called the *left translations*, such that $L_s \circ L_t = L_{st}$ and $L_1 = id_X$.

Example 1.3.

1. For a group G, any set X with the identity action of G on it is a G-set, called a *discrete* G-set.

2. Each group G can clearly be considered as a G-set with the action given by its binary operation $G \times G \to G$. Note that the unary algebra related to this G-set is $(G; (L_g)_{g \in G})$, where $L_g : G \to G$ is defined by $L_g(h) = gh$.

3. **This is a prime example in this paper**. Let $\mathbb{D}$ be an infinite countable set and $G = \mathrm{Perm}\mathbb{D}$ be the group of all permutations (bijective maps) on $\mathbb{D}$. Then, $\mathbb{D}$ itself is a G-set, with the canonical action as the evaluation, defined by

$$(\forall \pi \in \mathrm{Perm}\mathbb{D})\ (\forall d \in \mathbb{D}),\quad \pi \cdot d = \pi(d).$$

Definition 1.4.

1. A *homomorphism* (also called an *equivariant map* or a *G-map*) from a G-set X to a G-set Y is a function f from X to Y such that for each $x \in X$ and $g \in G$, $f(gx) = gf(x)$.

2. Let Y be a subset of a G-set X. We say that Y is a *sub-G-set* of X if for all $\pi \in G$, $\pi Y \subseteq Y$; which is equivalent to $\pi Y = Y$, since G is a group.

Since id_X and the composition of two G-maps are clearly G-maps, we have the category G-**Set** of all G-sets and G-maps between them.

Notice that, the class of all G-sets is equational, and so the category G-**Set** is complete and cocomplete (that is, has all products, equalizers, pullbacks, coproducts, coequalizers, and pushouts). In fact, limits and colimits in this category are computed as in the category **Set** of sets, with natural actions. Also, monomorphisms (epimorphisms) in G-**Set** are exactly one-one (onto) G-maps. Therefore, we do not distinguish between monomorphisms of G-sets and inclusions, and call a G-set X containing (an isomorphic copy of) a G-set Y an extension of X (for more information, see [9], [14]).

We now recall some other notions about G-sets X, needed in the sequel.

Definition 1.5.

1. Let X be a G-set. For each $x \in X$, the set

$$G_x \doteq \mathrm{fix}_G x = \{g \in G \mid gx = x\}$$

is a subgroup of G, called the *fixed* or the *stabilizer* subgroup of x in G. Also, for $C \subseteq X$,

$$G_C \doteq \mathrm{Fix}_G C = \{g \in G \mid (\forall x \in C)\ gx = x\} = \bigcap_{x \in C} G_x$$

is the *fixed* or the *stabilizer* of C in G.

2. An element x of a G-set X is called a *zero element* if for all $g \in G$, $gx = x$; equivalently, $G_x = \mathrm{fix}_G x = G$.

3. The class $[x] = \{gx \mid g \in G\} = Gx$ of the equivalence relation

$$x \sim x' \text{ if there exists } g \in G \text{ such that } gx = x'.$$

is called the *orbit* of x and is denoted by $\mathrm{Orb}_G x$ or Orb x, if G is known.

4. A G-set X is called *orbit finite* if $X/\sim$ is finite and it is called *single-orbit* if $X/\sim$ is singleton.

5. A G-set X is called *indecomposable* G-set if it is not the coproduct (disjoint union) of two proper sub-G-sets.

Remark 1.6. Recall that for a category $\mathfrak{C}$ and a subclass $\mathcal{M}$ of monomorphisms in $\mathfrak{C}$, the category $\mathfrak{C}$ is said to satisfy the $\mathcal{M}$-*transferability property* if for all $f \in \mathfrak{C}$ and $m \in \mathcal{M}$ with a common domain there is a commutative diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{m} & \bullet \\ f\big\downarrow & & \big\downarrow g \\ \bullet & \xrightarrow{u} & \bullet \end{array}$$

with $u \in \mathcal{M}$.

Notice that, since pushouts exist in the category M-**Set**, the above condition is equivalent to pushouts transfer monomorphisms; that is, the pushout map corresponding to a morphism in $\mathcal{M}$ again belongs to $\mathcal{M}$. For more information see [9], [11], [14].

2. The categories of G-nominal sets and $G_{\mathcal{P}_f(\mathbb{D})}$-nominal sets

As before, let $\mathbb{D}$ be an infinite countable set, whose elements are called *directions* (or *atomic names*, *data values*) and Perm$\mathbb{D}$ be the group of all permutations (bijection maps) on $\mathbb{D}$. A permutation $\pi \in \text{Perm}\mathbb{D}$ is *finitary* if $\{d \in \mathbb{D} \mid \pi d \neq d\}$ is finite. Clearly, the set $\text{Perm}_{\text{f}}\mathbb{D}$ of all finitary permutations is a subgroup of Perm$\mathbb{D}$. *In this paper we* **fix** *the infinite countable set* $\mathbb{D}$ *and a subgroup* G *of* Perm$(\mathbb{D})$*, which is denoted by* $G \leq \text{Perm}(\mathbb{D})$.

Recall (Example 1.3(3)) that the set $\mathbb{D}$ is itself a G-set with the *canonical action* given by the evaluation

$$\forall \pi \in G,\ d \in \mathbb{D}, \pi \cdot d = \pi(d).$$

We denote this action by $\pi \cdot d = \pi(d)$ and if X is any other G-set, we use the usual juxtaposition notation πx for the action of $\pi \in G$ on $x \in X$. The following definition gives the interplay between these two actions by introducing the notion of a, so called, *support*, which is the centeral notion to define *G-nominal sets.*

Definition 2.1. Let $G \leq \text{Perm}\mathbb{D}$. Consider $\mathbb{D}$ to be the canonical G-set, and let X be any G-set. A subset $C \subseteq \mathbb{D}$ is called "a" G-*support* for an element $x \in X$ if whenever $\pi \in G$ fixes all the elements of C then it fixes x. That is,

$$\text{Fix}_G C = G_C \subseteq G_x = \text{fix}_G x,$$

or, in other words, for all $\pi \in G$,

$$(\forall c \in C, \pi \cdot c = \pi(c) = c) \quad \Rightarrow \quad \pi x = x.$$

If C is finite (possibly empty) then we say that x is *finitely G-supported.*

Remark 2.2.

1. Let X be a G-set and x a finitely G-supported element of X with a support C. Then $\{x\}$ is a G_C-set.
2. If H is a subgroup of G and X a G-set then X is also an H-set.
3. Let C and B be two finite subsets of $\mathbb{D}$ and $C \subseteq B$. Then G_B is a subgroup of G_C.

Definition 2.3.

1. Consider a finite subset $C \subseteq \mathbb{D}$. A *G_C-nominal set* is a G_C-set every element of which has some finite support. In other words, for all $x \in X$, there exists a finite $E \subseteq \mathbb{D}$ such that
$$\mathrm{Fix}_{G_C} E \subseteq \mathrm{fix}_{G_C} x.$$
If $C = \emptyset$ then $G_C = G_\emptyset = G$ and a *G_C-nominal set* is called a G-nominal set.
2. Let X be a G_C-nominal set and Y a G_B-nominal set. A map $f : X \to Y$, for which there is a finite set $E \supseteq C \cup B$ such that for every $\pi \in G_E$ and $x \in X$ we have $f(\pi x) = \pi f(x)$, is called a *G_E-nominal map* or simply a *nominal map.* Notice that since $E \supseteq C \cup B$, by Remark 2.2, we get that X and Y are G_E-sets.
3. The class of G-nominal sets together with G-maps between them form a category, denoted by G-**Nom**.

Let $\mathcal{P}_\mathrm{f}(\mathbb{D})$ be the set of all finite subsets of $\mathbb{D}$. In the following lemma it is shown that the class of all G_C-nominal sets, $C \in \mathcal{P}_\mathrm{f}(\mathbb{D})$ and the maps between them (see Definition 2.3(2)) form a category, denoted by $G_{\mathcal{P}_\mathrm{f}(\mathbb{D})}$-**Nom**. Also, the category G-**Nom** is a full subcategory of $G_{\mathcal{P}_\mathrm{f}(\mathbb{D})}$-**Nom**.

Lemma 2.4. *The class $\{G_C$-nominal sets $\}_{C\in\mathcal{P}_\mathrm{f}(\mathbb{D})}$ and the morphisms between them form a category.*

Proof. Let A be a finite subset of $\mathbb{D}$ and X a G_A-nominal set. Then, clearly the identity map $id_X : X \to X$ is a G_A-nominal map. Now, let Y be a G_B-nominal set and Z a G_C-nominal set. Suppose $f : X \longrightarrow Y$ is a G_{D_1}-nominal map and $g : Y \longrightarrow Z$ a G_{D_2}-nominal map where D_1 and D_2 are finite subsets which contain $A \cup B$ and $B \cup C$, respectively. Now, let $E = D_1 \cup D_2$. Then, E is a finite set and for all $\pi \in G_E$, we have
$$\begin{aligned} \pi(gf)(x) &= g(\pi f(x)) \\ &= g(f(\pi x)), \end{aligned}$$
where the first equality is because $G_E \subseteq G_{D_2}$ and the second one is because $G_E \subseteq G_{D_1}$ (see Remark 2.2(3)). ∎

Example 2.5.

1. As we said above, the set $\mathbb{D}$ is itself a G-set. In this case, for every $d \in \mathbb{D}$, $\operatorname{supp} d = \{d\}$.

2. The set $\mathrm{Perm_f}(\mathbb{D})$, with the action by conjugation $(\pi, \pi') \rightsquigarrow \pi\pi'\pi^{-1}$ and $\operatorname{supp} \pi = \{d \in \mathbb{D} | \ \pi d \neq d\}$, is a G-nominal set.

3. The set $\mathbb{D} \setminus \{d\}$ where $d \in \mathbb{D}$, is a $G_{\{d\}}$-nominal set.

4. Note that, if X is a G-set, then the power set $\mathcal{P}(X) = \{Y | \ Y \subseteq X\}$ is also a G-set with the natural G-action

$$* : G \times \mathcal{P}(X) \to \mathcal{P}(X), \ \pi * Y \doteq \pi Y = \{\pi y | \ y \in Y\}.$$

Now, if X is a G-nominal set, we denote by $\mathcal{P}_{\mathrm{fs}}(X)$ the set of all subsets Y of X which are finitely G-supported relative to the action $*$. Thus, $Y \in \mathcal{P}_{\mathrm{fs}}(X)$ if and only if

$$(\exists C \subseteq \mathbb{D}), \text{ if } \pi \in G_C \text{ then } \pi * Y = \pi Y = Y,$$

which is again equivalent to Y being a G_C-set.

Clearly, $\mathcal{P}_{\mathrm{fs}}(X)$ is a G-set, under the restriction of the action $*$ to it and, since every element of this G-set is finitely G-supported, it is itself a G-nominal set. Thus, finitely supported subsets of X are the elements of $\mathcal{P}_{\mathrm{fs}}(X)$.

4. In particular, the set $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$, of finite subsets of $\mathbb{D}$, is a G-nominal set with the above action $\pi * C = \{\pi \cdot c = \pi(c) | \ c \in C\}$ and $\operatorname{supp} C = C$, for $C \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$.

Definition 2.6.

1. Let Y be a G_C-nominal set and $X \subseteq Y$. If there is some finite set $E \supseteq C$ such that for all $\pi \in G_E$ we have $\pi X = X$ then we say that X is a *sub-G_E-nominal set of Y*. Notice that G_E-nominal map $X \hookrightarrow Y$, is called *G_E-inclusion nominal map* and denoted by i. Clearly, every G-subset X of a G-nominal set Y is a G-nominal set, called a *sub-G-nominal set* of Y, where $\operatorname{supp}_Y y = \operatorname{supp}_X y$, for $y \in X$.

2. Let X be a G-nominal set. A subset $Y \subseteq X$ is *uniformly G-supported* if there exists a finite subset $C \subseteq \mathbb{D}$ which supports each $x \in Y$.

3. An element of a G-set X which is G-supported by the empty set $\emptyset$ is a zero element.

Example 2.7. Suppose X is a G_C-nominal set and Y a G_B-nominal set. Consider a G_E-nominal map $f : X \to Y$, where $E \supseteq B \cup C$. Let $\ker f = \{(x, x') \in X \times X \mid f(x) = f(x')\}$. Then, $\ker f$ is a sub-G_E-nominal set of X. This is because, for all $\pi \in G_E$ we have $f(\pi x) = \pi f(x) = \pi f(x') = f(\pi x')$.

Lemma 2.8. *The nominal set X is indecomposable if and only if it is single-orbit.*

Proof. Let $G \leq \mathrm{Perm}\mathbb{D}$ and X be an indecomposable nominal set and $x \in X$. Then $Gx \subseteq X$. We show $X = Gx$. Suppose $y \in X \setminus (Gx)$. Thus for all $\pi \in G$, we have $\pi y \in X \setminus (Gx)$. Since if there is π in G such that $\pi y \in Gx$, then $y \in Gx$, which is impossible. Therefore, $X = Gx \cup (X \setminus Gx)$, which contradicts by the assumption that X is indecomposable. The converse is clear. ∎

Remark 2.9.

1. Let X be a G-set and $x \in X$. If there exists a finite set G-supporting x, then there exists a least finite one, with respect to the subset inclusion (see [17]). This G-support is called "the" *G-support* of x and is denoted by $\mathrm{supp}_X x$, or simply by $\mathrm{supp}\, x$. In fact,
$$\mathrm{supp}_X\, x = \bigcap \{C \subseteq \mathbb{D} |\ C \text{ is a finite support of } x\}.$$
2. If X and Y are nominal sets, so is their cartesian product $X \times Y$ as a G-sets and with
$$\mathrm{supp}_{X\times Y}\,(x, y) = \mathrm{supp}_X\, x \cup \mathrm{supp}_Y\, y.$$
3. Every G-nominal set is also isomorphic to the coproduct of the single-orbit sub-G-nominal sets $O \in X/G$. That is, $X \cong \coprod_{O\in X/G} O$ (see [17]).

3. Injectivity in G-Nom

It is well-known (see, for example, Exercise 1.9 of [14]) that a G-set is injective if and only if it has a zero element. We show that the same holds for the injectivity of G-nominal sets with respecct to monomorphisms (see Theorem 3.3). In this section, we also study the injectivity of G-nominal sets with respect to monomorphims into single-orbit G-nominal and monomorphisms of single-orbit G-nominal sets as their domains and show that every G-nominal set is injective with respect to these classes of monomorphisms(see Theorem 3.9 and Lemma 3.6).

First, we recall some facts needed in this section. In the category of G-nominal sets, monomorphisms are injective maps (see [17]). Pushouts are calculated as in the category G-**Set**. In fact, consider the following pushout situation in the category of G-nominal sets,

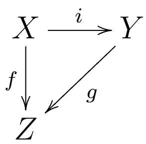
$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\big\downarrow & & \\ B & & \end{array} \qquad \text{(I)}$$

where A, B, C are G-nominal sets and f, g are equivariant maps. Notice that the coproduct of two G-nominal sets is a G-nominal set, so $B \sqcup C \in$G-**Nom** (see Section 2.2 of [17]). Now, take $O = (B \sqcup C)/\sim$ where $\sim$ is the congruence relation on $B \sqcup C$ generated by all pairs $(\tau_B f(a), \tau_C g(a))$, $a \in A$, $\tau_B : B \to B \sqcup C$ and $\tau_C : C \to B \sqcup C$. Since $B \sqcup C \in$ G-**Nom** and $\sim$ is a congruence on $B \sqcup C$, O is also a G-nominal set. And therefore, as a G-**Set** (see [14, 9]), O is the pushout of the diagram (I).

Also, recall the following:

Definition 3.1.

1. For a subclass $\mathcal{M}$ of monomorphisms in a category $\mathfrak{C}$, an object $Z \in \mathfrak{C}$ is called $\mathcal{M}$-*injective* if for each $\mathcal{M}$-*morphism* $i : X \to Y$ and any morphism $f : X \to Z$ there exists a morphism $g : Y \to Z$ such that $gi = f$:



If $\mathcal{M}$ is the class of monomorphisms, then $\mathcal{M}$-injective objects are simply called *injective* objects.

2. An object Z of a category $\mathfrak{C}$ is called $\mathcal{M}$-*absolute retract* if it is a retract of each of its $\mathcal{M}$-extensions; that is, for each $\mathcal{M}$-morphism $f : Z \to Y$ there exists a morphism $h : Y \to Z$ such that $hf = id_Z$, in which case h is said to be a *retraction.*

In the following, we recall a fact from the literature about G-sets (see [11], [14]).

Remark 3.2. Every injective G-nominal set contains a zero element (see [14]).

Theorem 3.3.

(i) *Every G-nominal set is injective if and only if it contains a zero element.*

(ii) *A single-orbit G-nominal set is injective if and only if it is trivial* (*singleton*)*.*

(iii) *A uniformly supported G-nominal set is injective if and only if it is discrete.*

(iv) *Injectivity is equivalent to absolute retractness.*

Proof. (i) Suppose Z is a G-nominal set with zero θ. Also, suppose X is a sub-G-nominal set of a nominal set Y. Consider inclusion map $i : X \hookrightarrow Y$ and equivariant map $f : X \to Z$. We show there is an equivariant map $g : Y \to Z$ such that $gi = f$. Define

$$g(y) = \begin{cases} f(y) & \text{if} \quad y \in X \\ \theta & \text{if} \quad y \notin X \end{cases}$$

Notice that $Y \setminus X$ is a sub-G-nominal set of Y. To see this, let $y \in (Y \setminus X)$, and $\pi \in G$. Then $\pi y \in Y \setminus X$. If there is $\pi \in G$ such that $\pi y \in Y$ then

$y = \pi^{-1}(\pi y) \in Y$, which is impossible. Therefore g is a required equivariant map and $gi = f$. The converse holds by Remark 3.2.

(ii) Let X be a single-orbit G-nominal set. Then $X = Gx$, where $x \in X$. Now, by (i), X is injective if and only if X has a zero element θ. Therefore X is injective if and only if $X = G\theta = \{\theta\}$.

(iii) Suppose X is an injective uniformly supported G-nominal set. By (i), X has a zero θ. Since X is uniformly supported, the support of all elements of X is empty. Thus X is discrete. The converse is clear.

(iv) See Remark 1.6 and Proposition 1.2 of [11]. ∎

Lemma 3.4. *For every equivariant map $f : X \longrightarrow Y$ from a single-orbit G-nominal set X to a G-nominal set Y, $f(X)$ is a single-orbit G-nominal set.*

Proof. Note that $f(X)$ is a sub-G-nominal set of X. Now, let $y_1, y_2 \in f(X)$. Then there are $x_1, x_2 \in X$ such that $y_1 = f(x_1)$, and $y_2 = f(x_2)$. Since X is a single-orbit G-nominal set and $x_1, x_2 \in X$, there exists $\pi \in G$ such that $x_1 = \pi x_2$. Hence $y_1 = f(x_1) = f(\pi x_2) = \pi f(x_2) = \pi y_2$, which means that $f(X)$ is single-orbit. ∎

Lemma 3.5. *The only sub-G-nominal sets of a single-orbit G-nominal set X are X and the empty set.*

Proof. Let Y be a single-orbit G-nominal set. Then $Y = Gy$, for $y \in Y$. Suppose X is a non-empty sub-G-nominal set of Y. We show $Y \subseteq X$. Let $z \in Y$. Then there exists $\pi \in G$ such that $z = \pi\, y$. Let $x \in X$. Since $X \subseteq Y$, we get $x \in Y$ and there exists $\pi' \in G$ such that $x = \pi'\, y$. Now, we have $z = \pi\, y = \pi\,(\pi'^{-1}\, x) = (\pi\pi'^{-1})\, x$ and therefore $z \in X$. Thus $X = Y$. ∎

Lemma 3.6. *Every G-nominal set is injective with respect to monomorphisms into single-orbit G-nominal sets.*

Proof. Let Z be a G-nominal set, Y a single-orbit G-nominal set, and X a sub-G-nominal set of Y. Consider the inclusion map $i : X \hookrightarrow Y$. Also, consider the equivariant map $f : X \to Z$. We prove that there exists an equivariant map $g : Y \to Z$ such that $gi = f$. First, notice that since Y is a single-orbit G-nominal set, by Lemma 3.5, we get $X = Y$. Hence taking $g = f$, we get the required (needed) equivariant map. ∎

Lemma 3.7. *Let $f : X \to Y$ be an equivariant map from a single-orbit G-nominal set X to a G-nominal set Y and Z be a sub-G-nominal set of Y. Then* $\mathrm{Imf} \subseteq Z$ *or* $Z \cap \mathrm{Imf} = \emptyset$.

Proof. Let $Z \cap \mathrm{Imf} \neq \emptyset$. Then we show $\mathrm{Imf} \subseteq Z$. Suppose $y \in \mathrm{Imf}$. So there exists $x' \in X$ such that $y = f(x')$. Since $Z \cap \mathrm{Imf} \neq \emptyset$, there is $z \in Z$ and $x \in X$ such that $z = f(x)$. By the assumption, X is a single-orbit G-nominal set and $x, x' \in X$. So there exists $\pi \in G$ such that $\pi x = x'$. Now, $y = f(x') = f(\pi x) = \pi f(x) = \pi z$. Hence $y \in Z$, since Z is a sub-G-nominal set of Y and $z \in Z$. ∎

Lemma 3.8. *Suppose X is a single-orbit sub-G-nominal set of the coproduct of single-orbit G-nominal sets Y_i. Then there exists a unique Y_i such that $X = Y_i$.*

Proof. Suppose $Y = \coprod_{i\in I} Y_i$, and $j : X \hookrightarrow Y$ is an inclusion. We show that there is a unique i_0 such that $X = Y_{i_0}$. Suppose $X \subseteq \bigcup_{j=1}^{l} Y_{i_j}$, where $l \in I$. So, $X = \bigcup_{j=1}^{l}(X \cap Y_{i_j})$, which is impossible. Since X is single-orbit and, by Lemma 2.8, X is an indecomposable G-nominal set. Therefore there is a unique i_0 in I such that $X \subseteq Y_{i_0}$. Now, by Lemma 3.5, $X = Y_{i_0}$. ∎

Theorem 3.9. *Every G-nominal set is injective with respect to monomorphisms with single-orbit G-nominal sets as their domains.*

Proof. Consider the following diagram,

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \\ Z & & \end{array}$$

where X is a single-orbit sub-G-nominal set of Y, Z is a G-nominal set and f is an equivariant map. We prove there exists an equivariant map $g : Y \to Z$ such that $gi = f$. By Remark 2.9(3), $Y \cong \coprod_i Y_i$, where Y_i's are single-orbit sub-G-nominal sets of Y. So the above diagram can be drawn as:

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{k} & \coprod_{i\in I} Y_i \\ {\scriptstyle f}\downarrow & & & & \\ Z & & & & \end{array}$$

where k is an isomorphism. Since k is a monomorphism, and monomorphisms are injective maps, so $(ki)(X) \cong X$. Therefore we have:

$$\begin{array}{ccccc} X & \xrightarrow{l} & (ki(X)) & \xrightarrow{j} & \coprod_{i\in I} Y_i \\ {\scriptstyle f}\downarrow & & & & \\ Z & & & & \end{array}$$

where l is an isomorphism, j is an inclusion and $ki = jl$. Also, X is a single-orbit sub-G-nominal set of Y and $k \circ i$ is an equivariant map, so, by Lemma 3.4, $k(i(X))$ is a single-orbit sub-G-nominal set of Y. Thus, by Lemma 3.8, $k(i(X)) = k(X)$ is equal to a Y_{i_0}, for some $i_0 \in I$. Now, we define $h : Y \to Z$ as follows:

$$h(y) = \begin{cases} (fl^{-1})(y) & \text{if} \quad y \in Y_{i_0} \\ \theta & \text{if} \quad y \in (\coprod_{i\in I} Y_i) \setminus Y_{i_0} \end{cases}$$

Notice that, since $(\coprod_{i\in I} Y_i) \setminus Y_{i_0}$ is a G-nominal set, h is an equivariant map and $hj = fl^{-1}$. Take $g = hk$. Thus g is an equivariant map and $gi = (hk)i = h(ki) = h(jl) = (fl^{-1})l = f$, as required. ∎

4. Injectivity in the category $G_{\mathcal{P}_f(\mathbb{D})}$-Nom

In this section, we study injectivity in the categroy $G_{\mathcal{P}_f(\mathbb{D})}$-**Nom**. The category of G-**Nom** is a full subcategory of $G_{\mathcal{P}_f(\mathbb{D})}$-**Nom**.

Lemma 4.1. *Monomorphisms in the category* $G_{\mathcal{P}_f(\mathbb{D})}$-**Nom** *are exactly injective morphisms.*

Proof. Suppose X is a G_C-nominal set and Y a G_B-nominal set. Also, suppose $f : X \longrightarrow Y$ is a monomorphism in the category $G_{\mathcal{P}_f(\mathbb{D})}$-**Nom**. We show f is an injective map.

Consider $\ker f = \{(x, x') \in X \times X \mid f(x) = f(x')\}$. By Example 2.7, $\ker f$ is a sub-G_E-nominal set of X, where E is a finite set which contains $C \cup B$. Consider the projection map $p_x : \ker f \to X$ and $p_{x'} : \ker f \to X$ where $p_x(x, x') = x$, and $p_{x'}(x, x') = x'$. It is clear that p_x and $p_{x'}$ are G_E-nominal maps. Let $x, x' \in \ker f$. Then $f(x) = f(x')$. Now, we have $f(p_x(x, x')) = f(x) = f(x') = f(p_{x'}(x, x'))$. Since f is a monomorphism, we get $x = p_x(x, x') = p_{x'}(x, x') = x'$. Therefore $\ker f = \Delta$, where $\Delta \doteq \{(x, x) \in X \times X\}$. So f is an injective map. ■

For the counterpart of this lemma in the category of G-**Nom**, where $G = \mathrm{Perm}_f(\mathbb{D})$, see [23].

Lemma 4.2. *Isomorphisms in the category* $G_{\mathcal{P}_f(\mathbb{D})}$-**Nom** *are exactly bijective maps.*

Proof. Let X be a G_C-nominal set and Y a G_B-nominal set. Also, let $f : X \longrightarrow Y$ be an isomorphism in $G_{\mathcal{P}_f(\mathbb{D})}$-**Nom**. Then there exists a map $g : Y \longrightarrow X$ such that $gf = id_X$ and $fg = id_Y$. We show g is a morphism in $G_{\mathcal{P}_f(\mathbb{D})}$-**Nom**. Since f is a morphism in $G_{\mathcal{P}_f(\mathbb{D})}$-**Nom**, there exists a finite subset E containing $C \cup B$ such that for all $x \in X$ and $\pi \in G_E$ we have $f(\pi x) = \pi f(x)$. Suppose $\pi \in G_E$ and $y \in Y$. Since f is surjective, there exists an element $x \in X$ such that $f(x) = y$. Now, for every $\pi \in G_E$,

$$\begin{aligned} \pi g(y) &= \pi g(f(x)) \\ &= \pi id_X(x) \\ &= \pi x \\ &= id_X(\pi x) \\ &= (gf)(\pi x) \\ &= g(f(\pi x)) \\ &= g(\pi f(x)) \\ &= g(\pi y). \end{aligned}$$ ■

We recall the definition of a zero element in a G_E-nominal set Z. The element $\theta \in Z$ is a zero if for all $\pi \in G_E$, $\pi\theta = \theta$.

Theorem 4.3. *Let* $Z \in G_{\mathcal{P}_f(\mathbb{D})}$-**Nom**. *Then Z is injective with respect to monomorphisms if and only if it contains a zero.*

Proof. Let $Z \in G_{\mathcal{P}_f(\mathbb{D})}$-**Nom** with zero θ. Then there is a finite set $E \subseteq \mathbb{D}$, such that Z is a G_E-nominal set. Also, suppose Y is a G_B-nominal set and X a sub-G_C-nominal set of Y, where C and B are finite subsets of $\mathbb{D}$ and C contains B. Consider a G_C-inclusion nominal map $i : X \hookrightarrow Y$ and a morphism $f : X \to Z \in G_{\mathcal{P}_f(\mathbb{D})}$-**Nom**. We prove that there exists a morphism $g : Y \to Z \in G_{\mathcal{P}_f(\mathbb{D})}$-**Nom** such that $gi = f$. Since f is a morphism in $G_{\mathcal{P}_f(\mathbb{D})}$-**Nom**, there exists a finite set F containing $C \cup E$ such that for $x \in X$ and $\pi \in G_F$, we have $f(\pi x) = \pi f(x)$. Define $g : Y \to Z$ by

$$g(y) = \begin{cases} f(y) & \text{if} \quad y \in X \\ \theta & \text{if} \quad y \notin X \end{cases}$$

Notice that $F \supseteq E \cup C \supseteq C$. Let $y \in Y$ and $\pi \in G_F$. Then we show that g is a G_F-nominal map. Suppose $y \in X$. Since X is a sub-G_C-nominal set and $G_F \subseteq G_C$, we get X is a sub-G_F-nominal set. Hence

$$\begin{aligned} g(\pi y) &= f(\pi y) \\ &= \pi f(y) \\ &= \pi g(y). \end{aligned}$$

If $y \notin X$, then, since X is a sub-G_C-nominal set and $G_F \subseteq G_C$, for each $\pi \in G_F$ we get $\pi y \notin X$. Thus

$$g(\pi y) = \theta = \pi\, \theta = \pi g(y),$$

where the second equality is because θ is a zero element of Z and $G_F \subseteq G_C$. Also $gi(y) = g(y) = f(y)$, when $y \in Y$.

Conversely, Suppose Z is an injective G_E-nominal set. Consider $Z\dot{\cup}\{\theta\}$ where $\theta \notin Z$ and $\operatorname{supp}\theta = \emptyset$. So $Z\dot{\cup}\{\theta\}$ is a G_E-nominal set. Since Z is injective, there exists a G_E-nominal map $g : Z\dot{\cup}\{\theta\} \to Z$ such that $gi = id$. Now, for all $\pi \in G_E$, we have $\pi g(\theta) = g(\pi\theta) = g(\theta)$. This means $g(\theta)$ is a zero element of Z. ∎

Acknowledgement. The author thanks the kind hospitality of Malayer University, where this work was completed during one week stay there, which helped me to finish this paper. Also, my special thanks go to Professors M.Mehdi Ebrahimi and M. Mahmoudi for their suggestions. The author would like to thank the referee for his/her helpful comments, which made the paper better readable.

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Accepted: 08.06.2016