PSEUDO SEMI B-FREDHOLM AND GENERALIZED DRAZIN INVERTIBLE OPERATORS THROUGH LOCALIZED SVEP

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Abstract. In this paper, we define and study the pseudo upper and lower semi B-Fredholm of bounded operators in a Banach space. In particular, we prove equality up to $S(T)$ between the left generalized Drazin spectrum and the pseudo upper semi B-Fredholm spectrum, $S(T)$ is the set where T fails to have the SVEP. Also, we prove that both of the left and the right generalized Drazin operators are invariant under additive commuting power finite rank perturbations and some perturbations for the pseudo upper and lower semi B-Fredholm operators are given. As applications, we investigate some classes of operators as the supercyclic and multiplier operators.

Keywords: pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm, left generalized Drazin, right generalized Drazin, Single-valued extension property.
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1. Introduction

Throughout, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X, we denote by T^*, $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $K(T)$, $H_0(T)$, $\rho(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma_p(T)$ and $\sigma(T)$, respectively the adjoint, the null space, the range, the hyper-range, the analytic core, the quasi-nilpotent part, the resolvent set, the approximate point spectrum, the surjectivity spectrum, the point spectrum and the spectrum of T.

Next, let $T \in \mathcal{B}(X)$, T has the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open neighborhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function

$f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T)$, then T and T^* have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum. In particular, T and T^* have the SVEP at every isolated point of the spectrum. We denote by $S(T)$ the open set of $\lambda \in \mathbb{C}$ where T fails to have SVEP at λ, and we say that T has SVEP if $S(T) = \emptyset$. It is easy to see that $S(T) \subset \sigma_p(T)$ ([1], [11]).

An operator $T \in \mathcal{B}(X)$ is said to be decomposable if for any open covering U_1, U_2 of the complex plane $\mathbb{C}$, there are two closed T-invariant subspaces X_1 and X_2 of X such that $X_1 + X_2 = X$ and $\sigma(T|X_k) \subset U_k$, $k = 1, 2$. Note that T is decomposable implies that T and T^* have the SVEP.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi-Fredholm) if $dimN(T) < \infty$ and $R(T)$ is closed (resp, $codimR(T) < \infty$). T is semi-Fredholm if it is a lower or upper semi-Fredholm. The index of a semi Fredholm operator T is defined by $ind(T) = dimN(T) - codimR(T)$.

T is a Fredholm operator if is lower and upper semi-Fredholm, and T is called a Weyl operator if it is a Fredholm of index zero.

The upper, lower and semi-Fredholm spectra of T are closed and defined by

$$\begin{aligned}
\sigma_{uF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not an upper semi-Fredholm operator}\} \\
\sigma_{lF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a lower semi-Fredholm operator}\} \\
\sigma_{sF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a semi-Fredholm operator}\}
\end{aligned}$$

The essential and Weyl spectra of T are closed and defined by

$$\begin{aligned}
\sigma_e(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a Fredholm operator}\} \\
\sigma_W(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a Weyl operator}\}
\end{aligned}$$

Now, consider a class of operators, introduced and studied by Berkani et al. in a series of papers which extends the class of semi-Fredholm operators [5]–[8]. For every $T \in \mathcal{B}(X)$ and a nonnegative integer n, let us denote by T_n the restriction of T to $R(T^n)$ viewed as a map from the space $R(T^n)$ into itself (we set $T_0 = T$).

An operator $T \in \mathcal{B}(X)$ is said to be upper (lower) semi B-Fredholm, if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n is an upper (lower) semi-Fredholm operator. Moreover, if T_n is a Fredholm operator, then T is called a B-Fredholm operator. A semi B-Fredholm operator is an upper or a lower semi B-Fredholm operator. It is easily seen that every nilpotent operator, as well as any idempotent bounded operator is B-Fredholm. The class of B-Fredholm operators contains the class of Fredholm operators as a proper subclass.

Let $T \in \mathcal{B}(X)$, according to [6, Proposition 2.6], T is a B-Fredholm operator if and only if there exists (X_1, X_2) a pair of T-invariant closed subspaces of X, such that $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$ where T_1 is Fredholm and T_2 is nilpotent. The upper, lower and B-Fredholm spectra are defined by

$$\begin{aligned}
\sigma_{uBF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not upper B-Fredholm}\} \\
\sigma_{lBF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not lower B-Fredholm}\} \\
\sigma_{BF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Fredholm}\}
\end{aligned}$$

Also $T \in \mathcal{B}(X)$ is a B-Weyl operator if there exists (X_1, X_2) a pair of T-invariant closed subspaces of X, such that $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$ where T_1 is Weyl operator and T_2 is nilpotent. The B-Weyl spectrum is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Weyl}\}.$$

Let $T \in \mathcal{B}(X)$, the ascent of T is defined by $a(T) = \min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$, if such p does not exists we let $a(T) = \infty$. Analogously the descent of T is $d(T) = \min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such q does not exists we let $d(T) = \infty$ [12]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$.

The descent and ascent spectra of $T \in \mathcal{B}(X)$ are defined by

$$\begin{aligned}\sigma_{des}(T) &= \{\lambda \in \mathbb{C}, T - \lambda I \text{ has not finite descent}\}\\ \sigma_{acc}(T) &= \{\lambda \in \mathbb{C}, T - \lambda I \text{ has not finite ascent}\}\end{aligned}$$

Let $T \in \mathcal{B}(X)$, T is said to be a Drazin invertible if there exists a positive integer k and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \ \ T^{k+1}S = T^k \ \text{ and } S^2T = S,$$

which is also equivalent to the fact that $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent. It is well known that T is Drazin invertible if it has a finite ascent and descent.

Define two sets $LD(X)$ and $RD(X)$ as [4], [15]:

$$\begin{aligned}LD(X) &= \{T \in \mathcal{B}(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed }\}\\ RD(X) &= \{T \in \mathcal{B}(X) : d(T) < \infty \text{ and } R(T^{d(T)+1}) \text{ is closed }\}\end{aligned}$$

An operator $T \in \mathcal{B}(X)$ is said to be left (resp. right) Drazin invertible if $T \in LD(X)$ (resp. $T \in RD(X)$). The left and right Drazin invertible spectra are defined by:

$$\begin{aligned}\sigma_{lD}(T) &= \{\lambda \in \mathbb{C}, \ T - \lambda I \notin LD(X)\}\\ \sigma_{rD}(T) &= \{\lambda \in \mathbb{C}, \ T - \lambda I \notin RD(X)\}\end{aligned}$$

and we have [4], [5]:

$$\sigma_D(T) = \sigma_{lD}(T) \cup \sigma_{rD}(T)$$

The concept of Drazin invertible operators has been generalized by Koliha [13]. In fact, $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc(\sigma(T))$ ($acc(\sigma(T))$ is the set of all points of accumulation of $\sigma(T)$), which is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasi-nilpotent. The generalized Drazin invertible spectrum defined by

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not generalized Drazin invertible}\}$$

In [14], the authors introduced and studied a new concept of left and right generalized Drazin inverse of bounded operators as a generalization of left and right

Drazin invertible operators. In fact, an operator $T \in \mathcal{B}(X)$ is said to be left generalized Drazin invertible if $H_0(T)$ is closed and complemented with a subspace M in X such that $T(M)$ is closed which equivalent to $T = T_1 \oplus T_2$ such that T_1 is left invertible and T_2 is quasi-nilpotent see [14, Proposition 3.2].

An operator $T \in \mathcal{B}(X)$ is said to be right generalized Drazin invertible if $K(T)$ is closed and complemented with a subspace N in X such that $N \subseteq H_0(T)$ which equivalent to $T = T_1 \oplus T_2$ such that T_1 is right invertible and T_2 is quasi-nilpotent see [14, Proposition 3.4]. The left and right generalized Drazin spectra of $T \in \mathcal{B}(X)$ are defined by:

$$\begin{aligned} \sigma_{lgD}(T) &= \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not left generalized Drazin}\} \\ \sigma_{rgD}(T) &= \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not right generalized Drazin}\} \end{aligned}$$

From [14], we have:

$$\sigma_{gD}(T) = \sigma_{lgD}(T) \cup \sigma_{rgD}(T)$$

$$\sigma_{rgD}(T) \subset \sigma_{rD}(T)$$

$$\sigma_{lgD}(T) \subset \sigma_{lD}(T)$$

$$\lambda \in \sigma_{lgD}(T) \Longleftrightarrow \lambda \in acc(\sigma_{ap}(T))$$

$$\lambda \in \sigma_{rgD}(T) \Longleftrightarrow \lambda \in acc(\sigma_{su}(T))$$

This paper is organized as follows. In Sections 2 and 3, we introduce and study the class of pseudo upper semi B-Fredholm and pseudo lower semi B-Fredholm, and we show that the pseudo upper semi B-Fredholm and pseudo lower semi B-Fredholm spectra, for a bounded linear operator on a Banach space, are compact in the complex plane. Also, we prove equality up to $S(T)$ between the left generalized Drazin spectrum and the pseudo upper semi B-Fredholm spectrum. Some applications are given in Section 4. In Section 5, we prove that the left and the right generalized Drazin spectra of an operator are invariant under additive commuting power finite rank perturbations. Some sufficient conditions are given to assure that the pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm and pseudo B-Fredholm are stable under additive commuting power finite rank and nilpotent perturbations.

2. Preliminaries

More recently, B-Fredholm and B-Weyl operators were generalized to pseudo B-Fredholm and pseudo B-Weyl [9], [18]. Precisely, T is a pseudo B-Fredholm operator if there exists (X_1, X_2) a pair of T-invariant closed subspaces of X, such that $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$ where $T_1 = T_{|X_1}$ is a Fredholm operator and $T_2 = T_{|X_2}$ is a quasi-nilpotent operator. The pseudo B-Fredholm spectrum is defined by

$$\sigma_{pBF}(T) = \{\lambda \in \mathbb{C} :\ T - \lambda I \text{ is not pseudo B-Fredholm}\}.$$

An operator T is a pseudo B-Weyl operator if there exists (X_1, X_2) a pair of T-invariant closed subspaces of X, such that $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$, where $T_1 = T_{|X_1}$ is a Weyl operator and $T_2 = T_{|X_2}$ is a quasi-nilpotent operator. The pseudo B-Weyl spectrum is defined by

$$\sigma_{pBW}(T) = \{\lambda \in \mathbb{C} : \; T - \lambda I \text{ is not pseudo B-Weyl}\}.$$

$\sigma_{pBW}(T)$ and $\sigma_{pBF}(T)$ is not necessarily non empty. For example, the quasi-nilpotent operator has empty pseudo B-Weyl and B-Fredholm spectrum. Evidentl, $\sigma_{pBF}(T) \subset \sigma_{pBW}(T) \subset \sigma(T)$.

In the following, we define the pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm and pseudo semi B-Fredholm of a bounded operator as a generalization of semi B-Fredholm and we give some fundamental results concerning these operators.

Definition 2.1 An operator $T \in \mathcal{B}(X)$ is said to be pseudo upper semi B-Fredholm if there exists two T-invariant closed subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_{|X_1}$ is upper semi-Fredholm operator and $T_{|X_2}$ is quasi-nilpotent.

Definition 2.2 An operator $T \in \mathcal{B}(X)$ is said to be pseudo lower semi B-Fredholm if there exists two T-invariant closed subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_{|X_1}$ is lower semi-Fredholm operator and $T_{|X_2}$ is quasi-nilpotent.

Definition 2.3 We say that $T \in \mathcal{B}(X)$ is pseudo semi B-Fredholm if T is pseudo lower semi B-Fredholm or pseudo upper semi B-Fredholm.

It is clear that T is a pseudo B-Fredholm operator if and only if T is a pseudo lower semi B-Fredholm operator and pseudo upper semi B-Fredholm operator.

The pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm and pseudo semi B-Fredholm spectra are defined by

$$\begin{aligned}
\sigma_{pBuF}(T) &= \{\lambda \in \mathbb{C} : \; T - \lambda I \text{ is not a pseudo upper semi B-Fredholm}\} \\
\sigma_{pBlF}(T) &= \{\lambda \in \mathbb{C} : \; T - \lambda I \text{ is not a pseudo lower semi B-Fredholm}\} \\
\sigma_{pBsF}(T) &= \{\lambda \in \mathbb{C} : \; T - \lambda I \text{ is not a pseudo semi B-Fredholm}\}
\end{aligned}$$

Therefore,

$$\sigma_{pBF}(T) = \sigma_{pBuF}(T) \cup \sigma_{pBlF}(T)$$

and

$$\sigma_{pBsF}(T) = \sigma_{pBuF}(T) \cap \sigma_{pBlF}(T)$$

It is easy to see that T is pseudo upper semi B-Fredholm if and only if T^* is pseudo lower semi B-Fredholm. Then:

$$\sigma_{pBuF}(T) = \sigma_{pBlF}(T^*) \text{ and } \sigma_{pBF}(T) = \sigma_{pBF}(T^*)$$

$\sigma_{pBsF}(T)$, $\sigma_{pBuF}(T)$ and $\sigma_{pBlF}(T)$ are not necessarily non empty. For example, the quasi-nilpotent operator has empty pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm and pseudo semi B-Fredholm spectrum.

Example 1 Let T_1 be defined on $l^2(\mathbb{N})$ by

$$T_0(x_1, x_2, ...) = (x_1, 0, x_2, 0, x_3, 0, ...)$$

T_0 is injective with closed range of infinite-codimension. Consider the operator T_2 defined on $l^2(\mathbb{N})$ as

$$T_2(x_1, x_2, ...) = (x_2/2, x_3/3, ...).$$

T_2 is a quasi-nilpotent operator. We have $T = T_0 \oplus T_2$ is a pseudo upper semi B-Fredholm operator. Note that $0 \in \sigma_{pBuF}(T_0)$, but $0 \notin \sigma_{pBlF}(T_0)$.

Example 2 Let T_1 be defined on $l^2(\mathbb{N})$ by:

$$T_1(x_1, x_2, ...) = (x_2, x_3, ...)$$

is surjective, but not injective, then is a lower semi-Fredholm operator. Let $T = T_1 \oplus T_2$, T_2 be as in Example 1. Then T is a pseudo lower semi B-Fredholm operator.

3. The class of pseudo semi B-Fredholm operators

Denote the open disc centered at λ_0 with radius ε in $\mathbb{C}$ by $D(\lambda_0, \epsilon)$ and

$$D^*(\lambda_0, \epsilon) = D(\lambda_0, \epsilon) \backslash \{\lambda_0\}.$$

The following theorem establishes that if T is a pseudo semi B-Fredholm operator, then $\lambda I - T$ is semi Fredholm in an open punctured neighborhood of 0.

Theorem 3.1 *Let $T \in \mathcal{B}(X)$ a pseudo semi B-Fredholm, then there exists a constant $\epsilon > 0$ such that $\lambda I - T$ is semi Fredholm for all $\lambda \in \mathbb{D}^*(0, \epsilon)$.*

Proof. If T is a pseudo semi B-Fredholm operator, then there exists two closed T-invariant subspaces X_1 and X_2 such that $X = X_1 \oplus X_2; T_{|X_1}$ is semi Fredholm, $T_{|X_2}$ is quasi-nilpotent and $T = T_{|X_1} \oplus T_{|X_2}$.

If $X_1 = \{0\}$, then T is quasi-nilpotent, then for all $\lambda \neq 0$ $\lambda I - T$ is invertible, hence $T - \lambda I$ is semi Fredholm for all $\lambda \neq 0$.

If $X_1 \neq \{0\}$, thus $T_{|X_1}$ is semi Fredholm, then there exists $\epsilon > 0$ such that $(T - \lambda I)_{|X_1}$ is semi Fredholm for all $\lambda \in D(0, \epsilon)$.

As $T_{|X_2}$ is quasi-nilpotent, then $(T - \lambda I)_{|X_2}$ is invertible for all $\lambda \neq 0$, hence $(T - \lambda I)_{|X_2}$ is semi Fredholm. Therefore, $(T - \lambda I)_{|X_1}$ is semi Fredholm for all $\lambda \in \mathbb{D}(0, \epsilon)$ and $(T - \lambda I)_{|X_2}$ is semi Fredholm for all $\lambda \neq 0$, hence $(T - \lambda I)$ is semi Fredholm for all $\lambda \in \mathbb{D}^*(0, \epsilon)$. ∎

From Theorem 3.1, we derive the following corollary.

Corollary 3.1 *Let $T \in \mathcal{B}(X)$, then $\sigma_{pBuF}(T), \sigma_{pBlF}(T), \sigma_{pBsF}(T)$ are compact subsets of $\mathbb{C}$.*

Moreover, $\sigma_{uF}(T) \setminus \sigma_{pBuF}(T)$, $\sigma_{lF}(T) \setminus \sigma_{pBlF}(T)$, $\sigma_{sF}(T) \setminus \sigma_{pBsF}(T)$ consist of at most countably many isolated points.

Since $\sigma_{pBuF}(T) \subset \sigma_{uBF}(T) \subset \sigma_{uF}(T)$ and $\sigma_{pBlF}(T) \subset \sigma_{lBF}(T) \subset \sigma_{lF}(T)$ the following corollary hold:

Corollary 3.2 *Let $T \in \mathcal{B}(X)$, then $\sigma_{uBF}(T) \setminus \sigma_{pBuF}(T)$, $\sigma_{lBF}(T) \setminus \sigma_{pBlF}(T)$ consist of at most countably many isolated points.*

Recall that $T \in \mathcal{B}(X)$ is said to be Kato operator or semi-regular if $R(T)$ is closed and $N(T) \subseteq R^{\infty}(T)$. Denote by $\rho_K(T) : \rho_K(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is Kato}\}$ the Kato resolvent and $\sigma_K(T) = \mathbb{C} \backslash \rho_K(T)$ the Kato spectrum of T. An operator $T \in \mathcal{B}(X)$ admit a generalized Kato decomposition, abbreviated as GKD if there exists two T-invariant closed subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_{\restriction X_1}$ is semi-regular (or a Kato) operator and $T_{\restriction X_2}$ is quasi-nilpotent. It is easy to see that every pseudo semi B-Fredholm is a pseudo Fredholm. According to [10, Theorem 2.2], the following proposition hold.

Proposition 3.1 *Let $T \in \mathcal{B}(X)$ a pseudo semi B-Fredholm operator. Then there exists a constant $\epsilon > 0$ such that $T - \lambda I$ is a Kato operator, for all $\lambda \in \mathbb{D}^*(0, \epsilon)$.*

As a consequence of the preceding Proposition, we have:

Corollary 3.3 *Let $T \in \mathcal{B}(X)$, $\sigma_K(T) \setminus \sigma_{pBsF}(T)$ consist of at most countably many isolated points.*

Define $pBuW(X)$ and $pBlW(X)$:

> $T \in pBuW(X)$ if there exists two T-invariant closed subspaces X_1 and X_2 of X where $X = X_1 \oplus X_2$ and $T_{\restriction X_1}$ is upper semi Fredholm of $ind(T_{\restriction X_1}) \leq 0$ and $T_{\restriction X_2}$ is quasi-nilpotent.
>
> $T \in pBlW(X)$ if there exists two T-invariant closed subspaces X_1 and X_2 of X where $X = X_1 \oplus X_2$ and $T_{\restriction X_1}$ is lower semi Fredholm of $ind(T_{\restriction X_1}) \geq 0$ and $T_{\restriction X_2}$ is quasi-nilpotent. The corresponding spectra of these sets are defined by:
>
> $$\sigma_{pBuW}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \notin pBuW(X)\}$$
> $$\sigma_{pBlW}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \notin pBlW(X)\}$$
>
> Then $\sigma_{pBuF}(T) \subseteq \sigma_{pBuW}(T) \subseteq \sigma_{lgD}(T)$ and $\sigma_{pBlF}(T) \subseteq \sigma_{pBlW}(T) \subseteq \sigma_{rgD}(T)$.

Remark 1 We have $\sigma_{pBuF}(T) \subset \sigma_{lgD}(T)$, this inclusion is proper. Indeed, let L be the unilateral left shift operator defined on the Hilbert $l^2(\mathbb{N})$:

$$L(x_1, x_2, ...) = (x_2, x_3, ...).$$

Since $\sigma_{ap}(L) = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$, then

$$\sigma_{lgD}(L) = acc(\sigma_{ap}(L)) = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$$

Observe that L is an upper semi-Fredholm operator, then $0 \notin \sigma_{pBuF}(L)$. This shows that the inclusion $\sigma_{pBuF}(T) \subset \sigma_{lgD}(T)$ is proper. Then it is naturel to ask about the defect set $\sigma_{lgD}(T) \backslash \sigma_{pBuF}(T)$, where T is a bounded operator.

In the following theorem, we get a characterization of this defect set.

Theorem 3.2 *Let $T \in \mathcal{B}(X)$. Then:*

$$\sigma_{lgD}(T) = \sigma_{pBuF}(T) \cup S(T) = \sigma_{pBuW}(T) \cup S(T)$$

Proof. $S(T) \subseteq \sigma_{lgD}(T)$ and $\sigma_{pBuF}(T) \subseteq \sigma_{lgD}(T)$, hence

$$\sigma_{pBuF}(T) \cup S(T) \subseteq \sigma_{lgD}(T).$$

Indeed, let $\lambda \notin \sigma_{lgD}(T)$. Then $T - \lambda I$ is a left generalized Drazin invertible, and then $H_0(T - \lambda I)$ is closed, by [3, Theorem 1.7] T has the SVEP at λ, hence $S(T) \subseteq \sigma_{lgD}(T)$.

Conversely, let $\lambda \notin \sigma_{pBuF}(T) \cup S(T)$, then $T - \lambda I$ is a pseudo upper semi B-Fredholm, then there exists two closed subspaces T-invariant X_1 and X_2 such that $X = X_1 \oplus X_2$, $(T - \lambda I)_{|X_1}$ is upper semi Fredholm and $(T - \lambda I)_{|X_2}$ is quasi-nilpotent. Since T has SVEP at λ then $T_{|X_1}$ and $T_{|X_2}$ have the SVEP at λ. Since $(T - \lambda I)_{|X_1}$ is upper semi Fredholm, then $(T - \lambda I)_{|X_1}$ admits a (GKD), since $T_{|X_1}$ has SVEP at λ, by [10, Theorem 3.5], we have $\lambda \notin acc(\sigma_{ap}(T_{|X_1})) = \sigma_{lgD}(T_{|X_1})$, then $(T - \lambda I)_{|X_1}$ is left generalized Drazin invertible, hence $X_1 = X_1^{'} \oplus X_1^{''}$ with $(T - \lambda I)_{|X_1^{'}}$ is left invertible and $(T - \lambda I)_{|X_1^{''}}$ is quasi-nilpotent, thus

$$X = X_1^{'} \oplus X_1^{''} \oplus X_2$$

with $(T - \lambda I)_{|X_1^{'}}$ is left invertible and $(T - \lambda I)_{|X_1^{''} \oplus X_2}$ is quasi-nilpotent, therefore, $T - \lambda I$ is left generalized Drazin invertible. ■

By duality, we get a similar result for the right Drazin invertible spectrum.

Theorem 3.3 *Let $T \in \mathcal{B}(X)$. Then:*

$$\sigma_{rgD}(T) = \sigma_{pBlF}(T) \cup S(T^*) = \sigma_{pBlW}(T) \cup S(T^*)$$

Proof. $\sigma_{rgD}(T) = \sigma_{lgD}(T^*) = \sigma_{pBuF}(T^*) \cup S(T^*) = \sigma_{plBF}(T) \cup S(T^*)$, then

$$\sigma_{rgD}(T) = \sigma_{pBlF}(T) \cup S(T^*)$$ ■

Remark 2 Theorems 3.2 and 3.3 extend [5, Theorem 2.1] and [5, Theorem 2.2].

From the preceding theorems, we get the following corollaries.

Corollary 3.4 *Let $T \in \mathcal{B}(X)$. Then*

$$\sigma_{gD}(T) = \sigma_{pBF}(T) \cup S(T) \cup S(T^*)$$

Corollary 3.5 *Let $T \in \mathcal{B}(X)$.*

If T has the SVEP then:

(1) $$\sigma_{lgD}(T) = \sigma_{pBuF}(T) = \sigma_{pBuW}(T)$$

If T^* has the SVEP then:

$$\sigma_{rgD}(T) = \sigma_{pBlF}(T) = \sigma_{pBlW}(T) \tag{2}$$

If both T and T^* have the SVEP, then all the spectra in (1) *and* (2) *coincide and are equal to the pseudo B-Fredholm, pseudo B-Weyl and generalized Drazin spectra.*

Example 3 Let T be the unilateral weighted shift on $l^2(\mathbb{N})$ defined by:

$$Te_n = \begin{cases} 0, & \text{if } n = p! \text{ for some } p \in \mathbb{N} \\ e_{n+1} & \text{otherwise.} \end{cases}$$

The adjoint operator of T is:

$$T^*e_n = \begin{cases} 0 & \text{if } n = 0 \textit{ or } n = p! + 1 \text{ for some } p \in \mathbb{N} \\ e_{n-1} & \text{otherwise.} \end{cases}$$

We have $\sigma(T) = \overline{D(0,1)}$ the unit closed disc. The point spectrum of T and T^* are

$$\sigma_p(T) = \sigma_p(T^*) = \{0\}.$$

Hence T and T^* have the SVEP. Then $\sigma_{ap}(T) = \sigma_{su}(T) = \sigma(T)$, hence

$$\sigma_{gD}(T) = \sigma_{lgD}(T) = \sigma_{rgD}(T) = \overline{D(0,1)}$$

From Corollary 3.5, we have

$$\sigma_{pBuF}(T) = \sigma_{pBlF}(T) = \sigma_{pBF}(T) = \sigma_{gD}(T) = \sigma_{lgD}(T) = \sigma_{rgD}(T) = \overline{D(0,1)}$$

4. Applications

A bounded linear operator T is called supercyclic provided there is some $x \in X$ such that the set $\{\lambda T^n,\ \lambda \in \mathbb{C},\ n = 0, 1, 2, ...\}$ is dense in X. It is well known that, if T is supercyclic, then

$$\sigma_p(T^*) = \{0\} \quad \text{or} \quad \sigma_p(T^*) = \{\alpha\}$$

for some nonzero $\alpha \in \mathbb{C}$. Since an operator with countable point spectrum has SVEP, then we have the following:

Proposition 4.2 *Let $T \in \mathcal{B}(X)$, a supercyclic operator. Then:*

$$\sigma_{rgD}(T) = \sigma_{pBlF}(T)$$

Since every hyponormal operator T on a Hilbert space has the single valued extension property, we have

Proposition 4.3 *Let T a hyponormal operator on a Hilbert space. Then:*

$$\sigma_{lgD}(T) = \sigma_{pBuF}(T)$$

In particular, if T is auto-adjoint then

$$\sigma_{gD}(T) = \sigma_{pBF}(T).$$

Let A be a semi-simple commutative Banach algebra.

The mapping $T : A \longrightarrow A$ is said to be a multiplier of A if $T(x)y = xT(y)$ for all $x, y \in A$.

It is well known each multiplier on A is a continuous linear operator and that the set of all multiplier on A is a unital closed commutative subalgebra of $B(A)$ [11, Proposition 4.1.1]. Also the semi-simplicity of A implies that every multiplier has the SVEP (see [11, Proposition 2.2.1]). According to Corollary 3.5, we have

Proposition 4.4 *Let T be a multiplier on semi-simple commutative Banach algebra A, then the following assertions are equivalent*

(1) *T is pseudo upper semi B-Fredholm.*

(2) *T is left generalized Drazin invertible.*

Now, if assume, in additiona,l that A is regular and Tauberian (see [11, Definition 4.9.7]), then every multiplier T^* has SVEP. Hence, we have the following Proposition.

Proposition 4.5 *Let T be a multiplier on semi-simple regular and Tauberian commutative Banach algebra A, then the following assertions are equivalent*

(1) *T is pseudo B-Fredholm .*

(2) *T is generalized Drazin invertible.*

Let G a locally compact abelian group, with group operation $+$ and Haar measure μ, let $L^1(G)$ consist of all $\mathbb{C}$-valued functions on G integrable with respect to Haar measure and $M(G)$ the Banach algebra of regular complex Borel measures on G. We recall that $L^1(G)$ is a regular semi-simple Tauberian commutative Banach algebra. Then we have the following:

Corollary 4.6 *Let G be a locally compact abelian group, $\mu \in M(G)$. Then every convolution operator $T_\mu : L^1(G) \longrightarrow L^1(G)$, $T_\mu(k) = \mu \star k$ is pseudo B-Fredholm if and only if is generalized Drazin invertible.*

Remark 3 Proposition 4.5 and Corollary 4.6 generalize [5, Proposition 3.4] and [5, Corollary 3.3]. These results also generalize some results in [8].

5. Perturbation

Let $\mathcal{F}(X)$ denote the ideal of finite rank operators on X. In the following, we show that both $\sigma_{lgD}(T)$ and $\sigma_{rgD}(T)$ are stable under additive commuting power finite rank operator.

Proposition 5.6 *Suppose that $F \in \mathcal{B}(X)$ satisfies $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ and that $T \in \mathcal{B}(X)$ commutes with F. Then we have*

$$\sigma_{lgD}(T) = \sigma_{lgD}(T+F) \text{ and } \sigma_{rgD}(T) = \sigma_{rgD}(T+F)$$

Proof. According to [19, Theorem 2.2], we have $acc(\sigma_{ap}(T)) = acc(\sigma_{ap}(T+F))$. Then $\lambda \in \sigma_{lgD}(T)$ if and only if $\lambda \in acc(\sigma_{ap}(T))$ if and only if $\lambda \in acc(\sigma_{ap}(T+F))$ if and only if $\lambda \in \sigma_{lgD}(T+F)$. So $\sigma_{lgD}(T+F) = \sigma_{lgD}(T)$.
By duality, we have $\sigma_{rgD}(T) = \sigma_{rgD}(T+F)$. ∎

As a consequence of Proposition 5.6, we have the following corollary.

Corollary 5.7 *Suppose that $F \in \mathcal{B}(X)$ satisfies $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ and that $T \in \mathcal{B}(X)$ commutes with F. Then we have*

$$\sigma_{gD}(T) = \sigma_{gD}(T+F)$$

The following example illustrates that the approximate point spectrum $\sigma_{ap}(.)$ in general is not preserved under commuting finite rank perturbations.

Example 4 Let $A \in \mathcal{B}(l^2)$ defined by:

$$A(x_1, x_2, ...) = (0, x_1, x_2, ...).$$

Let $0 < \varepsilon < 1$, $F_\varepsilon \in \mathcal{B}(l^2)$ be a finite rank operator defined by:

$$F_\varepsilon(x_1, x_2, ...) = (-\varepsilon x_1, 0, 0, ...).$$

Let $T = A \oplus I$ and $F = 0 \oplus F_\varepsilon$. Then F is a finite rank operator and $TF = FT$. But $\sigma_{ap}(T) = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$, $\sigma_{ap}(T+F) = \{\lambda \in \mathbb{C}, |\lambda| = 1\} \cup \{1-\varepsilon\}$.

Using Corollary 3.5, Proposition 5.6 and Corollary 5.7, we can prove the following corollary.

Corollary 5.8 *Suppose that $F \in \mathcal{B}(X)$ satisfies $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ and that $T \in \mathcal{B}(X)$ commutes with F.*

(1) *If T has the SVEP, then $\sigma_{pBuF}(T) = \sigma_{pBuF}(T+F)$*

(2) *If T^* has the SVEP, then $\sigma_{pBlF}(T) = \sigma_{pBlF}(T+F)$*

(3) *If T and T^* have the SVEP, then $\sigma_{pBF}(T) = \sigma_{pBF}(T+F)$*

Let $T \in \mathcal{B}(X)$, Q a quasi-nilpotent such that $QT = TQ$, from [18, Proposition 2.9], we have $\sigma_{lgD}(T+Q) = \sigma_{lgD}(T)$ and $\sigma_{rgD}(T+Q) = \sigma_{rgD}(T)$.

Proposition 5.7 *Let $T \in \mathcal{B}(X)$, N a nilpotent operator commutes with T then:*

$$\sigma_{pBuF}(T+N) \cup S(T) = \sigma_{pBuF}(T) \cup S(T)$$

Proof. From Theorem 3.2, we have $\sigma_{lgD}(T) = \sigma_{pBuF}(T) \cup S(T)$, then

$$\sigma_{lgD}(T+N) = \sigma_{pBuF}(T+N) \cup S(T+N)$$

since $\sigma_{lgD}(T+N) = \sigma_{lgD}(T)$ and $S(T+N) = S(T)$. Hence

$$\sigma_{pBuF}(T+N) \cup S(T) = \sigma_{pBuF}(T) \cup S(T) \qquad \blacksquare$$

By duality, we have the following proposition.

Proposition 5.8 *Let $T \in \mathcal{B}(X)$, N a nilpotent operator commutes with T then:*

$$\sigma_{pBlF}(T+N) \cup S(T^*) = \sigma_{pBlF}(T) \cup S(T^*)$$

As a consequence of Proposition 5.7 and Proposition 5.8, the following corollaries hold.

Corollary 5.9 *Let $T \in \mathcal{B}(X)$, N a nilpotent operator commute with T then:*

$$\sigma_{pBF}(T+N) \cup S(T) \cup S(T^*) = \sigma_{pBF}(T) \cup S(T) \cup S(T^*)$$

Corollary 5.10 *Let $T \in \mathcal{B}(X)$, N a nilpotent operator commutes with T.*

(1) *If T has SVEP, then $\sigma_{pBuF}(T+N) = \sigma_{pBuF}(T)$*

(2) *If T^* has SVEP, then $\sigma_{pBlF}(T+N) = \sigma_{pBlF}(T)$*

(3) *If T and T^* have SVEP, then $\sigma_{pBF}(T+N) = \sigma_{pBF}(T)$*

Remark 4 Let $T \in \mathcal{B}(X)$, by the same argument of [18, Theorem 2.12] and [4, Theorem 4.1], we can prove that :

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{lgD}(T+F) \subseteq \sigma_{pBuW}(T)$$

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{rgD}(T+F) \subseteq \sigma_{pBlW}(T)$$

We would like to finish this work with the following questions.

Questions: Is it true that:

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{lgD}(T+F) = \sigma_{pBuW}(T)$$

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{rgD}(T+F) = \sigma_{pBlW}(T) \ ?$$

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