MODULES WHOSE PRIMARY-LIKE SPECTRA WITH THE ZARISKI-LIKE TOPOLOGY ARE NOETHERIAN SPACES

Hosein Fazaeli Moghimi[1]
Fatemeh Rashedi

Department of Mathematics
University of Birjand
P.O. Box 97175-615, Birjand
Iran
e-mails: hfazaeli@birjand.ac.ir
fatemehrashedi@birjand.ac.ir

Abstract. Let R be a commutative ring with identity and M be a unital R-module. The primary-like spectrum $Spec_L(M)$ is the collection of all primary-like submodules Q of M such that M/Q is a primeful R-module. The Zariski-like topology on $Spec_L(M)$, denoted $\mathcal{T}$, is described by taking the set $\eta = \{\nu(N) \mid N \text{ is a submodule of } M\}$ as the set of closed sets of $Spec_L(M)$, where $\nu(N) = \{Q \in Spec_L(M) \mid \sqrt{(N : M)} \subseteq \sqrt{(Q : M)}\}$. We establish necessary and sufficient conditions for topological space $(Spec_L(M), \mathcal{T})$ to be a Noetherian space. We show that if M is a finitely generated R-module and $|Spec_L(M)| < \infty$, then the combinatorial dimension of $(Spec_L(M), \mathcal{T})$ and the Krull dimension of $R/Ann(M)$ are equal. In particular, for the Noetherian space $(Spec_L(M), \mathcal{T})$ of zero combinatorial dimension the set of irreducible components is finite, and its elements have the form $\nu(pM)$ for some minimal prime ideal $p \supseteq Ann(M)$.

Keywords: primary-like spectrum, Z-Radical of a submodule, Noetherian spectrum, combinatorial dimension.

2000 Mathematics Subject Classification: 13C13, 13C99, 54B99.

1. Introduction

Throughout this article, R is a commutative ring with $1 \neq 0$ and M is a unitary R-module. For any ideal I of R containing $Ann(M)$, $\overline{I}$ and $\overline{R}$ will denote $I/Ann(M)$ and $R/Ann(M)$, respectively. Also the spectrum $Spec(R)$ of a ring R, consists of all prime ideals of R, will be considered as a topological space in which the closed sets are of the form $V(I) = \{p \in Spec(R) \mid I \subseteq p\}$, where I is an ideal of R (see, for example, [4], [9]).

[1]Corresponding author

Definition 1.1. For M as an R-module and P, Q, N its submodules, we define the following:

(1) The colon ideal of M into N is $(N : M) = \{r \in R \mid rM \subseteq N\} = Ann(M/N)$. Dually, the colon submodule of M into an ideal I of R is $(N : I) = \{m \in M \mid Im \subseteq N\}$. In the case that $I = Rr$, we write $(N : r)$.

(2) P is prime if $P \neq M$, and whenever $rm \in P$, $r \in R$ and $m \in M$, then $m \in P$ or $r \in (P : M)$ [20];

(3) The prime spectrum (or simply, the spectrum) of M, denoted by $Spec(M)$, is the set of all prime submodules of M. Also if p is a prime ideal of R, we let $Spec_p(M) = \{P \in Spec(M) \mid (P : M) = p\}$ [20];

(4) The radical of N, denoted by $\operatorname{rad} N$, is the intersection of all prime submodules of M which contain N, unless no such primes exist, in which case $\operatorname{rad} N = M$ [23]. The radical of an ideal I of R will be denoted by $\sqrt{I}$;

(5) Q is primary-like if $Q \neq M$ and $rm \in Q$ implies that $r \in (Q : M)$ or $m \in \operatorname{rad} Q$ [12];

(6) M is a primeful R-module if either $M = (0)$ or $Spec_p(M) \neq \emptyset$. N satisfies the primeful property if M/N is a primeful R-module. In this case, $\sqrt{(N : M)} = (\operatorname{rad} N : M)$ [18];

(7) If Q is a primary-like submodule satisfying the primeful property, then $(Q : M)$ is a primary ideal [12, Lemma 2.1]. In this case, Q is p-primary-like, where $p = \sqrt{(Q : M)} = (\operatorname{rad} Q : M)$;

(8) The primary-like spectrum of M denoted by $Spec_L(M)$ is the set of all primary-like submodules of M satisfying the primeful property [12]. If p is a prime ideal of R, we set $\mathcal{X}_p = \{Q \in Spec_L(M) \mid \sqrt{(Q : M)} = p\}$;

(9) M is a multiplication module if for every submodule N of M, there exists an ideal I of R such $N = IM$. In this case, we can take $I = (N : M)$ [11].

In the literature, there are many different generalizations of the Zariski topology on the prime spectrum of a ring to modules ([2], [7], [10], [19], [22]. In the following lemma, we introduce one of them.

Lemma 1.2. *Let M be an R-module. Then, for submodules N, N' and $\{N_i \mid i \in I\}$ of M, we have*

(1) $\nu(0) = Spec_L(M)$ *and* $\nu(M) = \emptyset$.

(2) $\bigcap_{i \in I} \nu(N_i) = \nu\left(\sum_{i \in I} (N_i : M)M\right)$.

(3) $\nu(N) \cup \nu(N') = \nu(N \cap N')$.

Proof. (1) and (3) are straightforward.

(2) follows from the following implications:

$$\begin{aligned}
Q \in \bigcap_{i\in I} \nu(N_i) &\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{(N_i:M)}\ \forall i \in I \\
&\Rightarrow \sqrt{(Q:M)} \supseteq (N_i:M)\ \ \forall i \in I \\
&\Rightarrow \sqrt{(Q:M)} \supseteq \sum_{i\in I}(N_i:M) \\
&\Rightarrow \sqrt{(Q:M)}M \supseteq (\sum_{i\in I}(N_i:M))M \\
&\Rightarrow (\sqrt{(Q:M)}M:M) \supseteq \left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right) \\
&\Rightarrow ((\operatorname{rad} Q:M)M:M) \supseteq \left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right) \\
&\Rightarrow (\operatorname{rad} Q:M) \supseteq \left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right) \\
&\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{\left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right)} \\
&\Rightarrow Q \in \nu\left(\left(\sum_{i\in I}(N_i:M)\right)M\right).
\end{aligned}$$

For the reverse inclusion, we have

$$\begin{aligned}
Q \in \nu\left(\left(\sum_{i\in I}(N_i:M)\right)M\right) &\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{\left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right)} \\
&\Rightarrow \sqrt{(Q:M)} \supseteq \left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right) \\
&\Rightarrow \sqrt{(Q:M)} \supseteq ((N_i:M)M:M)\ \ \forall i \in I \\
&\Rightarrow \sqrt{(Q:M)} \supseteq (N_i:M)\ \ \forall i \in I \\
&\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{(N_i:M)}\ \ \forall i \in I \\
&\Rightarrow Q \in \bigcap_{i\in I} \nu(N_i).
\end{aligned}$$

■

Let $\eta(M)$ denotes the collection of all subsets $\nu(N)$ of $Spec_L(M)$. Lemma 1.2 shows that $\eta(M)$ satisfies the axioms for the closed subsets of a topological space on $Spec_L(M)$, called Zariski-like topology and denoted by $\mathcal{T}$.

Definition 1.3. For a topological space X we define the following:

(1) X is Noetherian provided that the open (respectively, closed) subsets of X satisfy the ascending (respectively, descending) chain condition[9, § 4.2];

(2) X is irreducible if the intersection of two non-empty open subsets of X is non-empty [9, §4.1];

(3) We consider strictly decreasing (or strictly increasing) chain $Y_0, Y_1, \cdots, Y_r$ of length r of irreducible closed subsets Y_i of X. The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of X and denoted by $dimX$. For the empty set, the combinatorial dimension of $\emptyset$ is defined to be -1 [17].

Definition 1.4. In this paper we define and use the following notions:

(1) Let $\eta^*(M)$ denotes the collection of all subsets

$$\nu^*(N) = \{Q \in Spec_L(M) \mid \text{rad}\, N \subseteq Q\},$$

of $Spec_L(M)$. It is easily seen that $\eta^*(M)$ contains the empty set and $Spec_L(M)$, and $\eta^*(M)$ is closed under arbitrary intersections. We shall say that M is a top-like module, if $\eta^*(M)$ is closed under finite unions, for in this case $\eta^*(M)$ induces a topology $\mathcal{T}^*$ on $Spec_L(M)$.

(2) Z-radical (resp. Z*-radical) of a submodule N of M, denoted by $\sqrt[Z]{N}$ (resp. $\sqrt[Z^*]{N}$), to be the intersection of all members of $\nu(N)$ (resp. $\nu^*(N)$);

(3) A submodule N of M is a Z-radical (resp. Z*-radical) submodule if $\sqrt[Z]{N} = N$ (resp. $\sqrt[Z^*]{N} = N$);

(4) If $Spec_L(M) \neq \emptyset$, the mapping $\phi : Spec_L(M) \to Spec(\overline{R})$ such that $\phi(Q) = \overline{\sqrt{(Q : M)}}$ for every $Q \in Spec_L(M)$, is called the natural map of $Spec_L(M)$

(5) A finitely generated module M is quasi-Laskerian if every submodule of M is the intersection of a finite number of primary-like submodules. A ring R is a Laskerian ring if R is a quasi-Laskerian R-module [14];

(6) An R-module M is a ZFG-module if for every submodule N of M we have $\sqrt[Z]{N} = \sqrt[Z]{IM}$ for some finitely generated ideal I of R;

(7) An R-module M is a FIC-module if every closed subset of $Spec_L(M)$ relative to the Zariski-like topology $\mathcal{T}$ has a finite number of irreducible components.

In recent years, the study of modules whose spectra have a Zariski-topology has grown in various directions. From an algebraic view, the varieties of submodules (closed sets related to a Zariki topology) forms a semimodule which also is called a Zariski-space (see for example [13], [21], [24]). Some of authors have investigated the interplay between algebraic properties of a module one hand and the topological properties of its spectrum on the other hand (see for example [1], [2], [3], [7], [8], [10], [17], [19], [22], [25]). In the present work, we study modules whose primary-like spectrums equipped with the Zariski-like topology are Noetherian spaces. For this purpose we need to obtain results about Z-Radical and Z*-Radical of submodules that are essential for the later sections of this article. Let M be an R-module. We see that if $(Spec_L(M), \mathcal{T})$ is a Noetherian space, then every Z-radical submodule of M satisfies ACC (Theorem 3.2). Moreover, if M is a top-like R-module and $(Spec_L(M), \mathcal{T}^*)$ is a Noetherian space, then every Z*-radical submodule of M satisfies ACC (Theorem 3.2). We also show that if M is a multiplication R-module and the natural map ϕ is surjective, then M is a ZFG-module if and only if $(Spec_L(M), \mathcal{T})$ is a Noetherian space (Theorem 3.14). Finally, it is proved that if M is a finitely generated R-module with $|Spec_L(M)| < \infty$, then the combinatorial dimension of $Spec_L(M)$ and the Krull dimension of $\overline{R}$ are equal (Theorem 4.6).

2. Z-radical and Z*-radical of submodules

We start this section with some elementary facts about $\nu(N)$ and $\nu^*(N)$.

Lemma 2.1. *Let I be an ideal of R. Let N, N' and $\{N_j \mid j \in J\}$ be submodules of an R-module M. Then the following hold.*

(1) *If $N \subseteq N'$, then $\nu(N') \subseteq \nu(N)$ and $\nu^*(N') \subseteq \nu^*(N)$.*

(2) $\nu^*(N) \subseteq \nu(N)$.

(3) $\nu(IM) = \nu(\sqrt{I}M) = \nu^*(IM) = \nu^*(\sqrt{I}M)$.

(4) $\nu(N) = \nu((N : M)M) = \nu(\sqrt{(N : M)}M) = \nu^*((N : M)M)$
$= \nu^*(\sqrt{(N : M)}M)$.

(5) *If $\sqrt{(N : M)} = \sqrt{(N' : M)}$, then $\nu(N) = \nu(N')$. The converse is also true if both N and N' are primary-like.*

(6) $\nu^*(N) = \nu^*(\operatorname{rad} N)$.

Proof. (1) Clear.

(2) Assume $Q \in \nu^*(N)$. Hence $N \subseteq \operatorname{rad} Q$. Thus $\sqrt{(N : M)} \subseteq \sqrt{(Q : M)}$, i.e., $Q \in \nu(N)$.

(3) We have $\nu^*(\sqrt{I}M) \subseteq \nu^*(IM) \subseteq \nu(IM)$, by (1) and (2). Now we show that $\nu(IM) \subseteq \nu^*(\sqrt{I}M)$. Suppose $Q \in \nu(IM)$. Thus $(\operatorname{rad} Q : M) \supseteq \sqrt{(IM : M)} \supseteq \sqrt{I}$. It follows that $\operatorname{rad} Q \supseteq \sqrt{I}M$, i.e., $Q \in \nu^*(\sqrt{I}M)$ and so $\nu(IM) \subseteq \nu^*(\sqrt{I}M)$. Hence we have $\nu^*(\sqrt{I}M) \subseteq \nu^*(IM) \subseteq \nu(IM) \subseteq \nu^*(\sqrt{I}M)$. Thus the assertion holds when I is replaced by $\sqrt{I}$.

(4) It suffices to show that $\nu(N) = \nu((N : M)M)$ by (3). $Q \in \nu(N)$ if and only if

$\sqrt{(Q:M)} \supseteq \sqrt{(N:M)} = \sqrt{((N:M)M:M)}$ if and only if $Q \in \nu((N:M)M)$. Thus $\nu(N) = \nu((N:M)M)$.

(5) is clear.

(6) $\nu^*(\operatorname{rad} N) \subseteq \nu^*(N)$ by (1). Suppose $Q \in \nu^*(N)$. Hence $N \subseteq \operatorname{rad} Q$. Thus $\operatorname{rad} N \subseteq \operatorname{rad} Q$ and so $Q \in \nu^*(\operatorname{rad} N)$. ∎

Proposition 2.2. *Let I be an ideal of R and M an R-module. If N, N' are submodules of M, then the following hold.*

(1) $\sqrt[z]{N} \subseteq \sqrt[z^*]{N}$.

(2) $\sqrt[z]{IM} = \sqrt[z]{\sqrt{I}M} = \sqrt[z^*]{IM} = \sqrt[z^*]{\sqrt{I}M}$.

(3) $\sqrt[z]{N} = \sqrt[z]{(N:M)M} = \sqrt[z]{\sqrt{(N:M)}M} = \sqrt[z^*]{(N:M)M)}$
$= \sqrt[z^*]{\sqrt{(N:M)}M}$.

Proof. (1) By Lemma 2.1(2), $\nu^*(N) \subseteq \nu(N)$. Thus $\sqrt[z]{N} \subseteq \sqrt[z^*]{N}$.

(2) Use Lemma 2.1(3).

(3) It is clear by Lemma 2.1(4). ∎

From now on, we use $\mathcal{X}$ to denote $Spec_L(M)$ for short. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$. We will denote the closure of $\mathcal{Y}$ in $\mathcal{X}$ by $\overline{\mathcal{Y}}$ and the intersection of all elements in $\mathcal{Y}$ by $\gamma(\mathcal{Y})$ (note that if $\mathcal{Y} = \emptyset$, then $\gamma(\mathcal{Y}) = M$). It is easy to verify that, if $\mathcal{Y}_1, \mathcal{Y}_2 \subseteq \mathcal{X}$, then $\gamma(\mathcal{Y}_1 \cup \mathcal{Y}_2) = \gamma(\mathcal{Y}_1) \cap \gamma(\mathcal{Y}_2)$.

Lemma 2.3. *Let M be an R-module, $|\mathcal{X}| < \infty$ and $\mathcal{Y} \subseteq \mathcal{X}$. Then $\nu(\gamma(\mathcal{Y})) = \overline{\mathcal{Y}}$. In particular, $\mathcal{Y}$ is closed if and only if $\nu(\gamma(\mathcal{Y})) = \mathcal{Y}$.*

Proof. Suppose $Q \in \mathcal{Y}$. Hence $\gamma(\mathcal{Y}) \subseteq Q$. Therefore $\sqrt{(Q:M)} \supseteq \sqrt{(\gamma(\mathcal{Y}):M)}$. Thus $Q \in \nu(\gamma(\mathcal{Y}))$ and so $\mathcal{Y} \subseteq \nu(\gamma(\mathcal{Y}))$. Next, let $\nu(N)$ be any closed subset of $\mathcal{X}$ containing $\mathcal{Y}$. Then $\sqrt{(Q:M)} \supseteq \sqrt{(N:M)}$ for every $Q \in \mathcal{Y}$ so that $\sqrt{(\gamma(\mathcal{Y}):M)} \supseteq \sqrt{(N:M)}$. Hence, for every $Q' \in \nu(\gamma(\mathcal{Y}))$; $\sqrt{(Q':M)} \supseteq \sqrt{(\gamma(\mathcal{Y}):M)} \supseteq \sqrt{(N:M)}$. Then $\nu(\gamma(\mathcal{Y})) \subseteq \nu(N)$. Thus $\nu(\gamma(\mathcal{Y}))$ is the smallest closed subset of $\mathcal{X}$ containing $\mathcal{Y}$ and so $\nu(\gamma(\mathcal{Y})) = \overline{\mathcal{Y}}$. ∎

Lemma 2.4. *Let M be an R-module and N be a submodule of M. If $|\mathcal{X}| < \infty$, then $\nu(\gamma(\nu(N))) = \nu(\sqrt[z]{N}) = \nu(N)$.*

Proof. It is clear by Lemma 2.3. ∎

In the following proposition, we list some more properties of $\sqrt[z]{N}$ and $\sqrt[z^*]{N}$ for a submodule N of M.

Proposition 2.5. *Let N, N' be submodules of an R-module M. Then the following hold.*

(1) *If* $\nu(N) \subseteq \nu(N')$, *then* $\sqrt[Z]{N'} \subseteq \sqrt[Z]{N}$. *The converse is true if* $N' \subseteq \sqrt[Z]{N'}$.

(2) *If* $Q \in \nu(N)$, *then* $\sqrt[Z]{N} \subseteq Q$.

(3) *If* $|\mathcal{X}| < \infty$, *then* $\sqrt[Z]{\sqrt[Z]{N}} = \sqrt[Z]{N}$.

(4) $\sqrt[Z]{N \cap N'} = \sqrt[Z]{N} \cap \sqrt[Z]{N'}$.

(5) *If* $\nu^*(N) \subseteq \nu^*(N')$, *then* $\sqrt[Z^*]{N'} \subseteq \sqrt[Z^*]{N}$. *The converse is true if* $N' \subseteq \sqrt[Z^*]{N'}$.

(6) *If* $Q \in \nu^*(N)$, *then* $\sqrt[Z^*]{N} \subseteq Q$.

(7) $\sqrt[Z^*]{N} = \sqrt[Z^*]{\operatorname{rad} N}$.

Proof. (1) Suppose $\nu(N) \subseteq \nu(N')$. Hence $\gamma(\nu(N')) \subseteq \gamma(\nu(N))$ and so $\sqrt[Z]{N'} \subseteq \sqrt[Z]{N}$. Conversely, assume that $Q \in \nu(N)$. Since $N' \subseteq \sqrt[Z]{N'} \subseteq \sqrt[Z]{N}$, then $Q \in \nu(N')$.

(2) Suppose $Q \in \nu(N)$. Hence $\sqrt{(N : M)} \subseteq \sqrt{(Q : M)}$. Thus $\sqrt[Z]{N} \subseteq Q$.

(3) $\nu(\sqrt[Z]{N}) = \nu(N)$, by Lemma 2.4. Therefore $\gamma(\nu(\sqrt[Z]{N})) = \gamma(\nu(N))$. Thus $\sqrt[Z]{\sqrt[Z]{N}} = \sqrt[Z]{N}$.

(4) $\sqrt[Z]{N \cap N'} = \gamma(\nu(N \cap N')) = \gamma(\nu(N) \cup \nu(N')) = \gamma(\nu(N)) \cap \gamma(\nu(N')) = \sqrt[Z]{N} \cap \sqrt[Z]{N'}$, by Lemma 1.2(3).

(5) Suppose $\nu^*(N') \subseteq \nu^*(N)$. Hence $\gamma(\nu^*(N)) \subseteq \gamma(\nu^*(N'))$ and so $\sqrt[Z^*]{N} \subseteq \sqrt[Z^*]{N'}$. Conversely, assume that $Q \in \nu^*(N)$. Since $N' \subseteq \sqrt[Z^*]{N'} \subseteq \sqrt[Z^*]{N}$, then $Q \in \nu^*(N')$.

(6) Assume $Q \in \nu^*(N)$, i.e., $N \subseteq \operatorname{rad} Q$. Thus $\sqrt[Z^*]{N} \subseteq Q$.

(7) By Lemma 2.1(6) $\nu^*(N) = \nu^*(\operatorname{rad} N)$. Hence $\gamma(\nu^*(N)) = \gamma(\nu^*(\operatorname{rad} N))$. Thus $\sqrt[Z^*]{N} = \sqrt[Z^*]{\operatorname{rad} N}$. ∎

Proposition 2.6. *Let* N *be a submodule of an* R*-module* M. *If* $N \subseteq \sqrt[Z]{N}$, *then* $\nu(N) = \nu^*(N)$. *In particular,* $\sqrt[Z]{N} = \sqrt[Z^*]{N}$.

Proof. $\nu^*(N) \subseteq \nu(N)$ by Lemma 2.1(2). Suppose $Q \in \nu(N)$. Hence $\sqrt[Z]{N} \subseteq Q$ by Proposition 2.5(2). Thus $N \subseteq Q \subseteq \operatorname{rad} Q$ and so $Q \in \nu^*(N)$. ∎

Let N be a submodule of M. Unlike the prime radical case the following example shows that $N \nsubseteq \sqrt[Z]{N}$ may be occurred in general.

Example 2.7. Let V be a vector space over a field F. Then $Spec_L(V) = Spec(V)$ = the set of all proper vector subspaces of V. Suppose W is a non-zero subspace of V. Hence $\sqrt[Z]{W} = 0$. Thus $W \nsubseteq \sqrt[Z]{W}$.

Proposition 2.8. *Let* M *be a finitely generated* R*-module. Then the following hold.*

(1) $\sqrt[Z]{N} \neq M$ *if and only if* $\nu(N) \neq \emptyset$ *if and only if* $N \neq M$.

(2) $\sqrt[Z^*]{N} \neq M$ *if and only if* $\nu^*(N) \neq \emptyset$ *if and only if* $N \neq M$.

Proof. (1) Assume $N \neq M$. Then $(N : M) \neq R$ and so $(N : M) \subseteq p$ for some prime ideal p of R. Since M is finitely generated, M is primeful by [18, Proposition 3.8]. Hence there exists $Q \in Spec(M) \subseteq \mathcal{X}$ such that $\sqrt{(N : M)} \subseteq \sqrt{(Q : M)}$. Thus $Q \in \nu(N)$ and so $\nu(N) \neq \emptyset$. Now suppose $\nu(N) \neq \emptyset$ and $Q \in \nu(N)$. Hence $\sqrt[Z]{N} \subseteq Q \neq M$ by Proposition 2.5(2). If $\sqrt[Z]{N} \neq M$, then $N \neq M$.

(2) Suppose $\sqrt[Z^*]{N} \neq M$. Hence $N \neq M$. Now assume $N \neq M$. Then $(N : M) \neq R$ and so $(N : M) \subseteq p$ for some prime ideal p of R. Since M is primeful, there exists $Q \in Spec(M) \subseteq \mathcal{X}$ such that $N \subseteq \operatorname{rad} Q$. Hence $Q \in \nu^*(N)$. Thus $\nu^*(N) \neq \emptyset$. If $\nu^*(N) \neq \emptyset$ and $Q \in \nu^*(N)$. Hence $\sqrt[Z^*]{N} \subseteq Q \neq M$ by Proposition 2.5(6). ■

Proposition 2.9. *Let M be a multiplication R-module. Then the following hold.*

(1) $\sqrt[Z]{N} = \sqrt[Z^*]{N}$.

(2) *If $|\mathcal{X}| < \infty$, then* $\sqrt[Z]{\sqrt[Z^*]{N}} = \sqrt[Z^*]{\sqrt[Z]{N}} = \sqrt[Z^*]{N} = \sqrt[Z]{N}$.

Proof. (1) Since M is multiplication, $\nu(N) = \nu^*(N)$ and so $\sqrt[Z]{N} = \sqrt[Z^*]{N}$.
(2) It is clear by (1) and Proposition 2.5(3). ■

Proposition 2.10. *Let M be an R-module and $Q \in \mathcal{X}_p$ for some prime ideal p of R. Then $\sqrt[Z^*]{Q} = \sqrt[Z^*]{Q + pM}$. Furthermore, if M is a multiplication module, then $\sqrt[Z]{Q} = \sqrt[Z]{Q + pM}$.*

Proof. Since $Q \subseteq Q + pM$, then $\sqrt[Z^*]{Q} \subseteq \sqrt[Z^*]{Q + pM}$. Now, assume $Q_i \in \mathcal{X}_{p_i}$ such that $Q \subseteq \operatorname{rad} Q_i$, $(i \in I)$. Hence $\sqrt{(Q : M)} \subseteq (\operatorname{rad} Q_i : M)$. Therefore $pM \subseteq p_i M$. So $Q + pM \subseteq \operatorname{rad} Q_i + p_i M \subseteq \operatorname{rad} Q_i$. Thus $\sqrt[Z^*]{Q + pM} \subseteq \sqrt[Z^*]{Q}$. Suppose M is a multiplication module. Thus $\sqrt[Z]{Q} = \sqrt[Z]{Q + pM}$ by (1) and Proposition 2.9. ■

3. Noetherian primary-like spectrum

Recall that a topological space X is a Noetherian space provided that the open (resp. closed) subsets of X satisfy the ascending (resp. descending) chain condition.

Theorem 3.1. *Let M be an R-module and $(\mathcal{X}, \mathcal{T})$ be a Noetherian space. Then every Z-radical submodule of M satisfies ACC.*

Proof. Suppose $(\mathcal{X}, \mathcal{T})$ is a Noetherian space. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of Z-radical submodules of M. Then $\nu(N_1) \supseteq \nu(N_2) \supseteq \cdots$ is a descending chain of closed sets $\nu(N_i)$ of $\mathcal{X}$. Hence there exists a positive integer k such that $\nu(N_m) = \nu(N_k)$ for every $m \geq k$. Thus $\sqrt[Z]{N_m} = \sqrt[Z]{N_k}$ and so $N_m = N_k$ for every $m \geq k$. ■

Theorem 3.2. *Let M be a top-like R-module. If $(\mathcal{X}, \mathcal{T}^*)$ is a Noetherian space, then every Z^*-radical submodule of M satisfies ACC.*

Proof. The proof is similar to that of Theorem 3.1. ∎

Proposition 3.3. *Let M be a multiplication R-module. Then $(\mathcal{X},\mathcal{T})$ is a Noetherian space if and only if $(\mathcal{X},\mathcal{T}^*)$ is a Noetherian space.*

Proof. It follows from the fact that $\nu(N)=\nu^*(N)$ for every submodule N of M. ∎

Recall that if $\mathcal{X}\neq\emptyset$, the mapping $\phi:\mathcal{X}\to Spec(\overline{R})$ such that $\phi(Q)=\overline{\sqrt{(Q:M)}}$ for every $Q\in\mathcal{X}$ is called the natural map of $\mathcal{X}$.

Proposition 3.4. *Let M be an R-module. Then the following hold.*

1. *$\phi^{-1}(V^{\overline{R}}(\overline{I}))=\nu(IM)$, for every ideal $I\in V(Ann(M))$. Therefore the map ϕ is continuous for the Zariski-like topology on $\mathcal{X}$.*
2. *If the map ϕ is surjective, then $\phi(\nu(N))=V^{\overline{R}}(\overline{(N:M)})$ and $\phi(\mathcal{X}-\nu(N))=Spec(\overline{R})-V^{\overline{R}}(\overline{(N:M)})$ for every submodule N of M, i.e. the map ϕ is both closed and open.*

Proof. (1) Obvious.

(2) As we have seen in (1), ϕ is a continuous map such that $\phi^{-1}(V^{\overline{R}}(\overline{I}))=\nu(IM)$ for every ideal I of R containing $Ann(M)$. Hence, for every submodule N of M we have $\phi^{-1}(V^{\overline{R}}(\overline{(N:M)}))=\nu((N:M)M)=\nu(N)$. It follows that $\phi(\nu(N))=\phi\circ\phi^{-1}(V^{\overline{R}}(\overline{(N:M)}))=V^{\overline{R}}(\overline{(N:M)})$ as ϕ is surjective. Similarly, we have $\phi(\mathcal{X}-\nu(N))=\phi(\phi^{-1}(Spec(\overline{R})-\phi^{-1}(V^{\overline{R}}(\overline{(N:M)}))))=Spec(\overline{R})-V^{\overline{R}}(\overline{(N:M)})$. ∎

Theorem 3.5. *Let M be a finitely generated R-module. Then $(\mathcal{X},\mathcal{T})$ is a Noetherian space if and only if $Spec(\overline{R})$ is a Noetherian space equipped with the Zariski topology.*

Proof. Suppose $(\mathcal{X},\mathcal{T})$ is a Noetherian space. Assume $V(\overline{I}_1)\supseteq V(\overline{I}_2)\supseteq\cdots$ is a descending chain of closed sets in $Spec(\overline{R})$. Hence by Proposition 3.4(1), $\phi^{-1}(V(\overline{I}_1))\supseteq\phi^{-1}(V(\overline{I}_2))\supseteq\cdots$ is a descending chain of closed sets in $\mathcal{X}$. By hypothesis, there exists an i such that $\phi^{-1}(V(\overline{I}_i))=\phi^{-1}(V(\overline{I}_{i+1}))$. Thus $V(\overline{I}_i)=V(\overline{I}_{i+1})$ because ϕ is surjective. Therefore $Spec(\overline{R})$ is a Noetherian space. Conversely, Assume $\nu(N_1)\supseteq\nu(N_2)\supseteq\cdots$ is a descending chain of closed sets in $\mathcal{X}$. Therefore $\phi(\nu(N_1))\supseteq\phi(\nu(N_2))\supseteq\cdots$ is a descending chain of closed sets in $Spec(\overline{R})$. Hence there exists an i such that $\phi(\nu(N_i))=\phi(\nu(N_{i+1}))$. It implies that $V(\overline{(N_i:M)})=V(\overline{(N_{i+1}:M)})$ by Proposition 3.4(2). Therefore we have $V(\sqrt{(N_i:M)})=V(\sqrt{(N_{i+1}:M)})$ and so $\nu(N_i)=\nu(N_{i+1})$. ∎

Lemma 3.6. *Let M be an R-module. If Q is a primary-like submodule of M and N is a submodule of M such that* $\operatorname{rad} Q\cap N=\operatorname{rad}(Q\cap N)$, *then $N\subseteq Q$ or $Q\cap N$ is a primary-like submodule of N.*

Proof. Suppose $N\nsubseteq Q$ and for $n\in N$, $rn\in Q\cap N$ such that $r\notin(Q\cap N:N)$. It implies that $rn\in Q$ and $r\notin(Q:M)$. Since Q is a primary-like submodule of M, we have $n \in \operatorname{rad} Q \cap N$, and so by our assumption $n \in \operatorname{rad}(Q \cap N)$. Thus $Q \cap N$ is a primary-like submodule of N. ∎

Let M be a finitely generated R-module. We recall that M is a quasi-Laskerian if every submodule of M is the intersection of a finite number of primary-like submodules.

Proposition 3.7. *Let M be a quasi-Laskerian R-module. If N is a finitely generated submodule of M such that for every primary-like submodule Q of M we have* $\operatorname{rad} Q \cap N = \operatorname{rad}(Q \cap N)$*, then N and $\frac{M}{N}$ are quasi-Laskerian.*

Proof. Suppose that N' is a submodule of N. Since M is quasi-Laskerian, there exist primary-like submodules Q_i $(1 \leq i \leq n)$ of M such that $N' = \bigcap_{i=1}^{n} Q_i$. Hence $Q_i \cap N$ is a primary-like submodule of N by Lemma 3.6. Thus N is quasi-Laskerian. The scend part follows from [12, Corollary 3.5]. ∎

Proposition 3.8. *Let M be a quasi-Laskerian R-module. Then $Spec(\overline{R})$ is a Noetherian space equipped with the Zariski topology.*

Proof. Since M is finitely generated, $\overline{R} \cong M$. Hence $\overline{R}$ is a Laskerian ring. Thus $Spec(\overline{R})$ is a Noetherian space by [14, Theorem 4]. ∎

Corollary 3.9. *Let M be a quasi-Laskerian R-module. Then $(\mathcal{X}, \mathcal{T})$ is a Noetherian space.*

Proof. By Proposition 3.8, $Spec(\overline{R})$ is a Noetherian space. Thus $(\mathcal{X}, \mathcal{T})$ is a Noetherian space by Theorem 3.5. ∎

Theorem 3.10. *Let M be a finitely generated R-module. Then M is quasi-Laskerian if and only if*

(1) *$Spec(\overline{R})$ is a Noetherian space equipped with the Zariski topology.*

(2) *For every proper submodule N of M, there is a minimal prime ideal p of $\sqrt{(N:M)}$ and an element $r \in R \backslash p$ for which the submodule $(N:r)$ is p-primary-like.*

Proof. Assume M is quasi-Laskerian. Therefore $Spec(\overline{R})$ is a Noetherian space by Proposition 3.8. Suppose N is a proper submodule of M. Hence there exist primary-like submodules $Q_1, \cdots, Q_n$ such that $N = Q_1 \cap \cdots \cap Q_n$. Let p be minimal prime of $\sqrt{(N:M)}$. Then $p = \sqrt{(Q_i:M)}$ for one of the primary-like submodules. Assume $p = \sqrt{(Q_1:M)}$. Let r be an element in $(\sqrt{(Q_2:M)} \cap \cdots \cap \sqrt{(Q_n:M)}) \backslash p$; replace r, if necessary, by a power so that $r \in (Q_2 \cap \cdots \cap Q_n) : M)$. Thus $(N:r) = Q_1$. Conversely, suppose M satisfies (1) and (2) and N is a submodule of M. Assume p is a minimal prime of $\sqrt{(N:M)}$ and $r \in R \backslash p$ so that $(N:r) = Q_1$ is p-primary-like. Then $N = Q_1 \cap N_1$, where $N_1 = N + rM$. Since $N \subseteq N_1$, then $\sqrt{(N:M)} \subseteq \sqrt{(N_1:M)}$. Applying the process repeatedly yields two sequences of submodules $Q_1, \cdots, Q_k$ and $N_1, \cdots, N_k$, where $N_{j-1} = Q_j \cap N_j$ and Q_j is primary-like for $1 \leq j \leq k$. Also $\overline{\sqrt{(N_1:M)}} \subseteq \cdots \subseteq \overline{\sqrt{(N_k:M)}}$ is a chain of prime ideals of $\overline{R}$. Since $Spec(\overline{R})$ is a Noetherian space, the sequence of radicals terminates; but it can terminate at $\overline{\sqrt{(N_k : M)}}$ only if $N_k = M$. Thus $N = Q_1 \cap \cdots \cap Q_k$ and so M is quasi-Laskerian. ∎

Proposition 3.11. *Let M be an R-module. Then the set*

$$\mathcal{B} = \{\mathcal{X}_r = \mathcal{X} - \nu(rM) \mid r \in R\}$$

forms a base for the topology $\mathcal{T}$ on $\mathcal{X}$.

Proof. If $\mathcal{X} = \emptyset$, then $\mathcal{B} = \emptyset$ and the proposition is trivially true. Hence we assume that $\mathcal{X} \neq \emptyset$ and let $\mathcal{U}$ be any open set in $\mathcal{X}$. Hence $\mathcal{U} = \mathcal{X} - \nu(IM)$ for some ideal I of R. Note that

$$\nu(IM) = \nu\left(\sum_{a_i \in I} a_i M\right) = \nu\left(\sum_{a_i \in I} (a_i M : M) M\right) = \bigcap_{a_i \in I} \nu(a_i M)$$

by Lemma 1.2. Hence $\mathcal{U} = \mathcal{X} - \bigcap_{a_i \in I} \nu(a_i M) = \bigcup_{a_i \in I} \mathcal{X}_{a_i}$. This proves that $\mathcal{B}$ is a base for the topology $\mathcal{T}$ on $\mathcal{X}$. ∎

Proposition 3.12. *Let M be an R-module and the natural map ϕ be surjective. Then $\mathcal{X}_r$ is a quasi-compact subset of $\mathcal{X}$.*

Proof. For any open covering of $\mathcal{X}_r$, there is a family $\{r_\lambda \in R : \lambda \in \Lambda\}$ of elements of R such that $\mathcal{X}_r \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$ by Proposition 3.11. Since the map ϕ is surjective,

$$D_{\overline{r}} = \phi(\mathcal{X}_r) \subseteq \bigcup_{\lambda \in \Lambda} \phi(\mathcal{X}_{r_\lambda}) \subseteq \bigcup_{\lambda \in \Lambda} D_{\overline{r}_\lambda}.$$

It follows that there exists a finite subset Λ' of Λ such that $D_{\overline{r}} \subseteq \bigcup_{\lambda \in \Lambda'} D_{\overline{r}_\lambda}$ as $D_{\overline{r}}$ is quasi-compact and so

$$\mathcal{X}_r = \phi^{-1}(D_{\overline{r}}) \subseteq \bigcup_{\lambda \in \Lambda'} \mathcal{X}_{r_\lambda}.$$

Thus $\mathcal{X}_r$ is quasi-compact. ∎

As noted earlier, an R-module M is a ZFG-module if for every submodule N of M we have $\sqrt[Z]{N} = \sqrt[Z]{IM}$ for some finitely generated ideal I of R. It is easy to see that every multiplication module over a Noetherian ring is a ZFG-module.

Proposition 3.13. *Let N be a submodule of an R-module M and $\sqrt{(N : M)} = \sqrt{I}$ for some finitely generated ideal I of R. Then N is a ZFG-submodule of M.*

Proof. Suppose $\sqrt{(N : M)} = \sqrt{I}$ for some finitely generated ideal I of R. Hence by Proposition 2.2(2,3) we have

$$\sqrt[Z]{N} = \sqrt[Z]{\sqrt{(N : M)}M} = \sqrt[Z]{\sqrt{I}M} = \sqrt[Z]{IM}.$$

Thus N is a ZFG-submodule of M. ∎

Theorem 3.14. *Let M be a multiplication R-module and the natural map ϕ be surjective. Then M is a ZFG-module if and only if $(\mathcal{X}, \mathcal{T})$ is a Noetherian space.*

Proof. Assume that N is a submodule of M. Therefore $\sqrt[Z]{N} = \sqrt[Z]{IM}$ for some finitely generated ideal $I = \sum_{i=1}^{n} Rr_i$ of R if and only if $\nu(N) = \nu(\sum_{i=1}^{n} r_iM) = \nu(\sum_{i=1}^{n}(r_iM : M)M) = \bigcap_{i=1}^{n} \nu(r_iM)$, by Lemma 1.2(2) and Lemma 2.4 if and only if $\mathcal{U} = \mathcal{X} - \nu(N) = \mathcal{X} - \bigcap_{i=1}^{n} \nu(r_iM) = \bigcup_{i=1}^{n}(\mathcal{X} - \nu(r_iM))$ if and only if $\mathcal{U}$ is quasi-compact by proposition 3.12 if and only if $(\mathcal{X}, \mathcal{T})$ is a Noetherian space by [9, P. 123, Proposition 9]. ∎

Corollary 3.15. *Let R be a ring. Then R-module R is a ZFG-module if and only if $Spec_L(R)$ is a Noetherian space.*

Proof. It is clear by Theorem 3.14. ∎

4. Noethrian spectrum, the number of irreducible components and the length of their chains

Recall that a topological space X is irreducible if the intersection of two non-empty open subsets of X is non-empty. Every subset of a topological space consisting of a single point is irreducible and a subset Y of a topological space X is irreducible if and only if its closure is irreducible [9, §4.1]. A maximal irreducible subset Y of X is called an irreducible component of X and it is always closed.

Proposition 4.1. *Let M be an R-module and the natural map ϕ be surjective. If $|\mathcal{X}| < \infty$ and $\mathcal{Y} \subseteq \mathcal{X}$, then $\mathcal{Y}$ is an irreducible closed subset of $\mathcal{X}$ if and only if $\mathcal{Y} = \nu(Q)$ for some $Q \in \mathcal{X}$.*

Proof. Suppose $\mathcal{Y} = \nu(Q)$. Since $\{Q\}$ is an irreducible subset of $\mathcal{X}$, by [9, P. 13, Exercise 20] $\overline{\{Q\}}$ is an irreducible subset of $\mathcal{X}$. Thus $\mathcal{Y} = \nu(Q) = \overline{\{Q\}}$ is an irreducible closed subset of $\mathcal{X}$. Conversely, if $\mathcal{Y}$ is an irreducible closed subset of $\mathcal{X}$, then $\mathcal{Y} = \nu(N)$ for some submodule N of M such that $\sqrt{(\gamma(\nu(N)) : M)} = \sqrt{(\gamma(\mathcal{Y}) : M)} = p$ is a prime ideal of R. Since ϕ is surjective, there exists a p-primary-like submodule $Q \in \mathcal{X}$ such that $\sqrt{(Q : M)} = p$. It follows that $p = \sqrt{(\gamma(\nu(N)) : M)} = \sqrt{(Q : M)}$. Hence $\nu(\gamma(\nu(N))) = \nu(Q)$ by Lemma 2.1(5). Thus $\mathcal{Y} = \nu(Q)$ by Lemma 2.3. Therefore $\nu(Q)$ is an irreducible subset of $\mathcal{X}$. ∎

Proposition 4.2. *Let M be an R module, $|\mathcal{X}| < \infty$ and the natural map ϕ be surjective. Then the correspondence $\nu(Q) \longmapsto \overline{\sqrt{(Q : M)}}$ is a bijection of the set of irreducible components of $\mathcal{X}$ and the set of minimal prime ideals of $\overline{R}$.*

Proof. It is easy to see that the correspondence is well-defined and an injection by Lemma 2.1(5). Suppose $\overline{p} \in Spec(\overline{R})$. Since ϕ is surjective, $\overline{p} = \overline{\sqrt{(Q : M)}}$ for some $Q \in \mathcal{X}$. Thus the correspondence is a surjection. Since each irreducible component of $\mathcal{X}$ is a maximal element of the set $\{\nu(Q) : Q \in \mathcal{X}\}$ by Proposition 4.1, the assertion hold. ∎

We recall that an R-module M is said to be a FIC-module if every closed subset of $\mathcal{X}$ has a finite number of irreducible components. A ring R is said to be a FIC-ring if and only if R-module R is FIC.

Theorem 4.3. *Let M be an R-module. Then the following hold.*

(1) *If M is finitely generated and $|\mathcal{X}| < \infty$, then M is FIC if and only if for every submodule N of M the ideal $\sqrt{(N : M)}$ is contained in a finite number minimal prime ideal of R.*

(2) *If $(\mathcal{X}, \mathcal{T})$ is a Noetherian space, then M is FIC.*

Proof. (1) is a direct result of Proposition 4.2.

(2) Since $(\mathcal{X}, \mathcal{T})$ is a Noetherian space, every closed subset of $\mathcal{X}$ is Noetherian by [9, P. 123, Proposition 8(i)]. Hence every closed subset of $\mathcal{X}$ has a finite number of irreducible components, by [9, P. 124, Proposition 10]. Thus M is FIC. ∎

Corollary 4.4. *The following are true.*

(1) *If $|Spec_L(R)| < \infty$, then R is FIC if and only if every ideal I of R, $\sqrt{I}$ is contained in a finite number minimal prime ideal.*

(2) *If $Spec_L(R)$ is a Noetherian space, then R is FIC.*

Proof. By Theorem 4.3 is clear. ∎

As it was mentioned, if X is a topological space, we consider strictly decreasing (or strictly increasing) chain $Y_0, Y_1, ..., Y_r$ of length r of irreducible closed subsets Y_i of X. The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of X and denoted by $dimX$. For the empty set, the combinatorial dimension of $\emptyset$ is defined to be -1.

Proposition 4.5. *Let M be a finitely generated R-module and $|\mathcal{X}| < \infty$. Then $\mathcal{X}$ has a chain of irreducible closed subsets of $\mathcal{X}$ of length r if and only if $\overline{R}$ has a chain of prime ideals of length r.*

Proof. Assume that $\mathcal{Y}_0 \subset \mathcal{Y}_1 \subset \cdots \subset \mathcal{Y}_r$ is a strictly increasing chain of irreducible closed subsets $\mathcal{Y}_i$ of $\mathcal{X}$ of length r. By Proposition 4.1, $\mathcal{Y}_i = \nu(Q_i)$ for some $Q_i \in \mathcal{X}$. Hence $\nu(Q_0) \subset \nu(Q_1) \subset \cdots \subset \nu(Q_r)$. Thus $\overline{\sqrt{(Q_0 : M)}} \supset \overline{\sqrt{(Q_1 : M)}} \supset \cdots \supset \overline{\sqrt{(Q_r : M)}}$ is a strictly decreasing chain of prime ideals of $\overline{R}$ of length r. Conversely, suppose $\overline{p}_0 \supset \overline{p}_1 \supset \cdots \supset \overline{p}_r$ is a strictly decreasing chain of prime ideals of $\overline{R}$ of length r. Since M is finitely generated, M is primeful by [18, Theorem 2.2]. Hence there exists $Q_i \in Spec(M) \subseteq \mathcal{X}$ such that $\sqrt{(Q_0 : M)} \supset \sqrt{(Q_1 : M)} \supset \cdots \supset \sqrt{(Q_r : M)}$. Thus $\nu(Q_0) \subset \nu(Q_1) \subset \cdots \subset \nu(Q_r)$ is a strictly increasing chain of irreducible closed subsets $\mathcal{X}$ of length r by proposition 4.1. ∎

For a ring R, the Krull dimension of R, $dim(R)$, equals the combinatorial dimension of $Spec(R)$ equipped with the Zariski topology.

Theorem 4.6. *Let M be a finitely generated R-module and $|\mathcal{X}| < \infty$. Then the combinatorial dimension of $\mathcal{X}$ and the Krull dimension of $\overline{R}$ are equal.*

Proof. Use Proposition 4.5. ∎

Corollary 4.7. *Let M be a finitely generated R-module such that $\mathcal{X}$ has combinatorial dimension zero. Then the following hold.*

(1) *Every irreducible closed subset of $\mathcal{X}$ is an irreducible component.*

(2) *If $p \in V(Ann(M))$ and $|\mathcal{X}| < \infty$, then $\mathcal{X}_p = \nu(Q)$ for every $Q \in \mathcal{X}_p$.*

(3) *If $|\mathcal{X}| < \infty$ and $(\mathcal{X}, \mathcal{T})$ is a Noetherian space, then the set of irreducible components of $\mathcal{X}$ is $\{\nu(p_1M), ..., \nu(p_nM)\}$, where the p_i $(1 \leq i \leq n)$ are all the minimal prime containing $Ann(M)$.*

Proof. (1) is obvious.

(2) By Theorem 4.6, $dim(\mathcal{X}) = dim(\overline{R})$. Hence $p = \sqrt{(Q : M)}$ is a maximal ideal of R. If $Q' \in \nu(Q)$, then $\sqrt{(Q' : M)} = \sqrt{(Q : M)} = p$ and so $\nu(Q) \subseteq \mathcal{X}_p$. Now suppose $Q' \in \mathcal{X}_p$. Hence $\sqrt{(Q' : M)} = p = \sqrt{(Q : M)}$. Thus $Q' \in \nu(Q)$ and so $\mathcal{X}_p \subseteq \nu(Q)$.

(3) Since $(\mathcal{X}, \mathcal{T})$ is a Noetherian space with $dim(\mathcal{X}) = 0$, $\overline{R}$ has Noetherian spectrum and $dim(\overline{R}) = 0$ by Theorems 3.5 and 4.6. Hence $Spec(\overline{R})$ has only finitely many elements $\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_n$ each of which is both maximal and minimal prime ideal of $\overline{R}$ by [15, P. 41, Examples 1.4, c) and d)]. Since M is a finitely generated R-module, $(p_iM : M) = p_i$ is a maximal ideal of R. Hence $p_iM \in \mathcal{X}_{p_i}$. So $\nu(p_iM)$ is an irreducible component of $\mathcal{X}$ for every i by (1) and Proposition 4.1. Thus by Proposition 4.2, $\{\nu(p_1M), \cdots, \nu(p_nM)\}$ is the set of all irreducible components of $\mathcal{X}$. ∎

References

[1] Abbasi, A., Hassanzadeh-Lelekaami, D., *Modules and spectral spaces*, Comm. Algebra, 40 (2012), 4111-4129.

[2] Ansari-Toroghy, H., Ovlyaee-Sarmazdeh, R., *On the prime spectrum of a module and Zariski topologies*, Comm. Algebra, 38 (2010), 4461-4475.

[3] Ansari-Toroghy, H., Ovlyaee-Sarmazdeh, R., *On the prime spectrum of X-injective modules,* Comm. Algebra, 38 (2010), 2606-2621.

[4] Atiyah, M.F., McDonald, I.G., *Introduction to commutative algebra*, Addison Weisley Publishing Company, Inc., 1969.

[5] Azizi, A., *Prime submodules and flat modules*, Acta Mathematatica Sinica, English series Jan., 23 (2007), 147-152.

[6] Barnard, A., *Multiplication modules*, J. Algebra, 71 (1981), 174-178.

[7] Behboodi, M., Haddadi, M.R.M., *Classical Zariski topology of modules and spectral spaces.* I, Int. Elec. J. Algebra, 4 (2008), 104-130.

[8] Behboodi, M., Haddadi, M.R.M., *Classical Zariski topology of modules and spectral spaces.* II. Int. Elec. J. Algebra, 4 (2008), 131-148.

[9] Bourbaki, N., *Algebra commutative*, Hermann, Paris, 1972.

[10] Duraivel, T., *Topology on spectrum of modules*, J. Ramanujan Math. Soc., 9 (1994), 25-34.

[11] El-Bast, Z.A., Smith, P.F., *Multiplication modules,* Comm. Algebra, 16 (1988), 755-779.

[12] Fazaeli Moghimi, H., Rashedi, F., *Primary-like submodules satisfying the primeful property*, (to appear).

[13] Fazaeli Moghimi, H., Rashedi, F., *Zariski-like spaces of certain modules*, (to appear).

[14] Gilmer, R., Heinzer, W., *The laskerian property, power series rings and noetherian spectra,* Proc. Amer. Math. Soc., 79 (1980), 13-16.

[15] Kunz, E., *Introduction to commutative algebra and algebraic geometry*, Boston: Birkhuser, 1985.

[16] Low, G.M., Smith, P.F., *Multiplication modules and ideals*, Comm. Algebra, 18 (1990), 4353-4375.

[17] Lu, C.P., *Modules with noetherian spectrum*, Comm. Algebra, 38 (2010), 807-828.

[18] Lu, C.P., *A module whose prime spectrum has the surjective natural map,* Houston J. Math., 33 (2007), 125-143.

[19] Lu, C.P., *The Zariski topology on the prime spectrum of a module*, Houston J. Math., 25 (1999), 417-425.

[20] Lu, C.P., *Prime submodules of modules,* Comment. Math. Univ. St. Pauli, 33 (1984), 61-69.

[21] McCasland, R.L., Moore, M.E., Smith, P.F. *An introduction to Zariski spaces over Zariski topologies,* Rocky Mountain J. Math., 28 (1998), 1358- 1369.

[22] McCasland, R.L., Moore, M.E., Smith, P.F. *On the spectrum of a module over a commutative ring,* Comm. Algebra, 25 (1997), 79-103.

[23] Moore, M.E., Smith, S.J., *Prime and radical submodules of modules over commutative rings,* Comm. Alegbra, 30 (2002), 5073-5064.

[24] Nikseresht, A., Azizi, A., *Zariski spaces of modules*, J. Pure Appl. Algebra, 217 (2013), 1187-1194.

[25] Ohm, J., Pendleton, R.L., *Rings with Noetherian spectrum*, Duke Math. J., 35 (1968), 631-639.

Accepted: 16.05.2016