FURTHER PROPERTIES OF THE GENERALIZATION OF PRIMAL SUPERIDEALS

Ameer Jaber[1]
Moh'D Yasein

Department of Mathematics
The Hashemite University
Zarqa 13115
Jordan
e-mails: ameerj@hu.edu.jo
myasein@hu.edu.jo

Abstract. In [7], the author studied several generalizations of primal superideals of a commutative super-ring R. This paper is devoted to study further properties of ϕ-primal superideals of R, where $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ is any function and $\Im(R)$ is the set of all proper superideals of R. In particular, if $J \in \Im(R)$ then there is a one-to-one correspondence between ϕ-P^I-primal superideals I of R with $J \subseteq \phi(I)$ and ϕ_J-P^I/J-primal superideals of R/J. Moreover, for a multiplicatively closed subset S of $h(R)$, if $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$ for any $I \in \Im(R)$, then there is a one-to-one correspondence between ϕ-P^I-primal superideals I of R and ϕ_S-P^I_S-primal superideals I_S of R_S with $P^I \cap S = \emptyset$.

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1. Introduction

Let R be any ring with unity, then R is called a super-ring if R is a $\mathbb{Z}_2$-graded ring such that if $a, b \in \mathbb{Z}_2$ then $R_a R_b \subseteq R_{a+b}$ where the subscripts are taken modulo 2. Let $h(R) = R_0 \cup R_1$ then $h(R)$ is the set of homogeneous elements in R and $1 \in R_0$.

Throughout, R will be a commutative super-ring with unity. By a proper superideal of R we mean a superideal I of R such that $I \neq R$. We will denote the set of all proper superideals of R by $\Im(R)$. If I and J are superideals of R, then the superideal $\{r \in R : rJ \subseteq I\}$ is denoted by $(I : J)$. Let $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be any function. We call a proper superideal I of R *ϕ-prime* (*prime*) if for $x, y \in h(R)$

[1]Corresponding author

with $xy \in I - \phi(I)$ ($xy \in I$) then $x \in I$ or $y \in I$. Since $I - \phi(I) = I - (\phi(I) \cap I)$, there is no loss of generality to assume that $\phi(I) \subseteq I$ for every proper superideal I of R. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, we say that an element $a \in h(R)$ is *ϕ-prime* to I (*prime* to I), if whenever $ra \in I - \phi(I)$ ($ra \in I$), where $r \in h(R)$, then $r \in I$. That is $a \in h(R)$ is ϕ-prime to I (*prime* to I), if $h((I : a)) - h((\phi(I) : a)) \subseteq h(I)$ (if $h((I : a)) \subseteq h(I)$).

Let $\nu(I)$ be the set of all homogeneous elements in R that are not prime to I. We define I to be *primal* if the set

$$P = (\nu(I))_0 + (\nu(I))_1 \cup \{0\}$$

forms a superideal in R. In this case we say that I is a P-primal superideal of R, and P is the adjoint superideal of I.

Let $\nu_\phi(I)$ be the set of all homogeneous elements in R that are not ϕ-prime to I. We define I to be *ϕ-primal* if the set

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\emptyset \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\emptyset \end{cases}$$

forms a superideal in R. In this case we say that I is a ϕ-P-primal superideal of R, and P is the adjoint superideal of I.

In [7], the author gave a generalization of primal superideals of R and he studied some properties of that generalization, he also studied the relation between ϕ-primary and ϕ-primal superideals. Moreover, the author introduced some conditions under which ϕ-primal superideals are primal.

In the next example we give some famous functions $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ and their corresponding ϕ-primal superideals.

Example 1.1.

$\phi_\emptyset$	$\phi_\emptyset(I) = \emptyset \ \ \forall\, I \in \mathfrak{I}(R)$	primal superideal.
ϕ_0	$\phi_0(I) = \{0\} \ \ \forall\, I \in \mathfrak{I}(R)$	weakly primal superideal.
ϕ_2	$\phi_2(I) = I^2 \ \ \forall\, I \in \mathfrak{I}(R)$	almost primal superideal.
ϕ_n	$\phi_n(I) = I^n \ \ \forall\, I \in \mathfrak{I}(R)$	n-almost primal superideal.
ϕ_ω	$\phi_\omega(I) = \cap_{n=1}^{\infty} I^n \ \ \forall\, I \in \mathfrak{I}(R)$	ω-primal superideal.

Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2$.

In this paper, we prove that if J is a ϕ-prime superideal of R then then there is a one-to-one correspondence between ϕ-P^I-primal superideals I of R with $J \subseteq \phi(I)$ and ϕ_J-P^I/J-primal superideals of R/J. Also, we prove that for a multiplicatively closed subset S of $h(R)$ and under the condition that $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$ for any $I \in \mathfrak{I}(R)$, where $\rho : R \to R_S$ is the canonical homomorphism, there is a one-to-one correspondence between ϕ-P^I-primal superideals I of R and ϕ_S-P^I_S-primal superideals I_S of R_S with $P^I \cap S = \emptyset$.

2. ϕ_J-P/J-primal superideals

We start this section by the following two examples to show that the concepts "primal superideals" and "ϕ-primal superideals" are different.

Example 2.1. Let $R = \mathbb{Z}_{24} + u\mathbb{Z}_{24}$, where $u^2 = 0$, be a commutative super-ring and assume that $\phi = \phi_0$. Let $I = 8\mathbb{Z}_{24} + u\mathbb{Z}_{24}$.

(1) Since $0 \neq \bar{2} \cdot \bar{4} \in I$ with $\bar{2}, \bar{4} \notin I$, then we get that $\bar{2}$ and $\bar{4}$ are not ϕ-prime to I. Easy computations imply that $\bar{2} + \bar{4} = \bar{6}$ is ϕ-prime to I. Thus we obtain that I is not a ϕ-primal superideal of R.

(2) Set $P = 2\mathbb{Z}_{24} + u\mathbb{Z}_{24}$. We show that I is a primal superideal of R. It is easy to check that every element of $h(P)$ is not prime to I. Conversely, assume that $\bar{a} \in h(R) - h(P)$, then $\bar{a} \in \mathbb{Z}_{24}$ with $gcd(a, 8) = 1$. If $\bar{a} \cdot \bar{n} \in I$ for some $\bar{n} \in \mathbb{Z}_{24}$, then 8 divides n; hence $\bar{n} \in I$. Therefore, $h(P)$ is exactly the set of elements in $h(R)$ which are not prime to I. Thus I is a primal superideal of R.

Example 2.2. Let $\phi = \phi_0$, and let $T(R)$ be the collection of all homogeneous zero divisors of R. If R is not a superdomain such that $Z(R) = T_0(R) + T_1(R)$ is not a superideal of R. Then the trivial superideal of R is a ϕ-primal superideal which is not primal.

According to Examples 2.1 and 2.2 a primal superideal of R need not to be ϕ-primal and a ϕ-primal superideal of R need not to be primal.

Let R be a commutative super-ring with unity and let J be a proper superideal of R. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. As a generalization of [2], we define $\phi_J : \mathfrak{I}(R/J) \to \mathfrak{I}(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I)+J)/J$ for every superideal $I \in \mathfrak{I}(R)$ with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi = \phi_\emptyset$).

We leave the trivial proof of the next lemma to the reader.

Lemma 2.3. *Let R be a commutative super-ring with unity and let J be a proper superideal of R. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. If P is a ϕ-prime superideal of R containing J. Then P/J is a ϕ_J-prime superideal of R/J.* ∎

Lemma 2.4. *Let R be a commutative super-ring with unity and let J be a proper superideal of R, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. Let P be a superideal of R containing J. If P/J is a ϕ_J-prime superideal of R/J with $J \subseteq \phi(P)$, then P is a ϕ-prime superideal of R.*

Proof. Let a, b be homogeneous elements in R with $ab \in P - \phi(P)$. Then $ab \in J + P$ and $ab \notin J + \phi(P) = \phi(P)$. Thus, $ab \in (J + P) - (J + \phi(P))$, so $\overline{ab} \in (J + P)/J - (J + \phi(P))/J$ which implies that $\overline{ab} \in P/J - \phi_J(P/J)$, that is $\bar{a} \in P/J$ or $\bar{b} \in P/J$ so $a \in P$ or $b \in P$. Therefore, P is a ϕ-prime superideal of R. ∎

In the next result and under the condition that $J \subseteq \phi(I)$ we prove that I is a ϕ-primal superideal of R if and only if I/J is a ϕ_J-primal superideal of R/J.

Theorem 2.5. *Let R be a commutative super-ring with unity and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. Let I be a proper superideal of R, and let J be a superideal of R with $J \subseteq \phi(I)$. Then I is a ϕ-P-primal superideal of R if and only if I/J is a ϕ_J-P/J-primal superideal of R/J.*

Proof. Suppose that I is ϕ-P-primal superideal of R with $J \subseteq I$. Then, by [7, Theorem 2.5], P is a ϕ-prime superideal of R containing J. Therefore, by Lemma 2.3, P/J is a ϕ_J-prime superideal of R/J. We show that I/J is ϕ_J-P/J-primal superideal of R/J. That is we have to prove that

$$P/J = \begin{cases} [(\nu_{\phi_J}(I/J))_0 + (\nu_{\phi_J}(I/J))_1 \cup \{0\}] + \phi_J(I/J) & : \quad \text{if } \phi_J \neq \phi_\emptyset \\ (\nu_{\phi_J}(I/J))_0 + (\nu_{\phi_J}(I/J))_1 & : \quad \text{if } \phi_J = \phi_\emptyset \end{cases}.$$

Let $\bar{a} \in h(P/J)$ then $a \in h(P)$ is not ϕ-prime to I. That is there exists r in $h(R) - h(I)$ with $ra \in I - \phi(I)$. If $a \in J + \phi(I) = \phi(I)$ then $\bar{a} \in \phi_J(I/J)$. So we may assume that $a \notin J + \phi(I) = \phi(I)$. Therefore, $\bar{r}\bar{a} \in I/J - (J + \phi(I))/J = I/J - \phi_J(I/J)$ and because $\bar{r} \notin I/J$ we get that $\bar{a} \in \nu_{\phi_J}(I/J)$.

Now, assume that $\bar{b}$ is a homogeneous element in R/J such that $\bar{b} \in \nu_{\phi_J}(I/J)$. Then, there exists a homogeneous element $\bar{r}$ in $R/J - I/J$ such that $\bar{r}\bar{b} \in I/J - \phi_J(I/J)$, so $rb \in I - \phi(I)$ with $r \notin I$. Thus, b is not ϕ-prime to I which implies that $b \in P$, and hence $\bar{b} \in P/J$. Therefore,

$$P/J = \begin{cases} [(\nu_{\phi_J}(I/J))_0 + (\nu_{\phi_J}(I/J))_1 \cup \{0\}] + \phi_J(I/J) & : \quad \text{if } \phi_J \neq \phi_\emptyset \\ (\nu_{\phi_J}(I/J))_0 + (\nu_{\phi_J}(I/J))_1 & : \quad \text{if } \phi_J = \phi_\emptyset \end{cases}$$

and so I/J is ϕ_J-P/J-primal superideal of R/J.

Conversely, suppose that I/J is ϕ_J-P/J-primal superideal of R/J with the adjoint superideal P/J. We show that I is a ϕ-P-primal superideal of R. Now, by [7, Theorem 2.5], P/J is a ϕ_J-prime superideal of R/J with $J \subseteq P$, so by Lemma 2.4, P is a ϕ-prime superideal of R. To finish the proof we need to show that

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\emptyset \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\emptyset \end{cases}.$$

Clearly, $\phi(I) \subseteq I \subseteq P$. Let $a \in \nu_\phi(I)$, then there exists a homogeneous element $r \in R - I$ with $ra \in I - \phi(I)$. Since $ra \notin \phi(I) = J + \phi(I)$ we get that $\bar{r}\bar{a} \in I/J - (J + \phi(I))/J = I/J - \phi_J(I/J)$ and $\bar{r} \notin I/J$. So, $\bar{a} \in P/J$ and hence $a \in P$. Now, let $a \in h(P)$. Suppose that $\bar{a} \in I/J$, then $a \in I$. If $a \in \phi(I)$, then done. If $a \notin \phi(I)$, then $a \in I - \phi(I)$ and so, a is not ϕ-prime to I, hence $a \in \nu_\phi(I)$. Thus,

$$a \in \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\emptyset \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\emptyset \end{cases}.$$

Therefore, we may assume that $\bar{a} \notin I/J$, so there exists $\bar{r} \in h(R/J) - h(I/J)$ with $\bar{r}\bar{a} \in I/J - \phi_J(I/J)$, and so $\bar{r}\bar{a} \notin (J + \phi(I))/J$ that is $ra \notin J + \phi(I) = \phi(I)$. Therefore, $ra \in I - \phi(I)$ with $r \notin I$ that is $a \in \nu_\phi(I)$. ∎

By using Theorem 2.5, we have the following result.

Theorem 2.6. *Let R be a commutative super-ring with unity, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. let J be a superideal of R. Then there is one-to-one correspondence between the ϕ-P^I-primal superideals I of R containing J with $J \subseteq \phi(I)$ and ϕ_J-P^I/J-primal superideals of R/J.* ∎

3. Multiplicatively closed subsets

Let R be a commutative super-ring with unity, and let S be a multiplicatively closed subset of $h(R)$. Consider the canonical homomorphism $\rho : R \to R_S$ which is defined by $r \mapsto \frac{r}{1}$ for all $r \in h(R)$. Then ρ is a homogenous superhomomorphism of degree 0.

Now let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. We define $\phi_S : \mathfrak{I}(R_S) \to \mathfrak{I}(R_S) \cup \{\emptyset\}$ by $\phi_S(J) = (\phi(\rho^{-1}(J)))_S$ for every $J \in \mathfrak{I}(R_S)$. Note that $\phi_S(J) \subseteq J$, since for $J \in \mathfrak{I}(R_S)$, we get that $\phi(\rho^{-1}(J)) \subseteq \rho^{-1}(J)$ implies $\phi_S(J) \subseteq (\rho^{-1}(J))_S \subseteq J$.

Example 3.1. Let $R = \mathbb{Z}_6 + u\mathbb{Z}_6$ with $u^2 = 0$. Let $S = \{1, 2, 4\}$, $\phi = \phi_2$. Then S is a multiplicatively closed subset of $h(R)$. If $P = \{0\}$ then one can easily check that P is ϕ_2-P-primal superideal of R. Moreover, $P_S = (\phi(P))_S = \{0\}$, hence $\phi_S(P_S) = P_S$ since $\rho^{-1}(P_S) = \{0, 3, 3u\}$, where $\rho : R \to R_S$ is the canonical homomorphism. Therefore, P_S is ϕ_S-P_S-primal superideal in R_S.

We start by proving the following properties about ϕ-prime superideals of R, where $\rho : R \to R_S$ is the canonical homomorphism.

Theorem 3.2. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-prime superideal of R with $I \cap S = \emptyset$, then I_S is a ϕ_S-prime superideal of R_S.*

Proof. Let $\frac{x}{s}, \frac{y}{t}$ be homogeneous elements in R_S with $(\frac{x}{s})(\frac{y}{t}) \in I_S - \phi_S(I_S)$, then for some $u \in S$ $xyu \in I - \phi(I)$, so $x \in I$ or $yu \in I$ and thus $\frac{x}{s} \in I_S$ or $\frac{y}{t} \in I_S$, hence I_S is a ϕ_S-prime superideal of R_S. ∎

Theorem 3.3. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let P be a ϕ-prime superideal of R with $h(P) \cap S = \emptyset$. If $\rho^{-1}((\phi(P))_S) \subseteq P$, then $\rho^{-1}(P_S) = P$.*

Proof. It is easy to see that $P \subseteq \rho^{-1}(P_S)$.

Conversely, let x be a homogeneous element in $\rho^{-1}(P_S)$, then for some $s \in S$, $xs \in P$. If $xs \notin \phi(P)$ then $xs \in P - \phi(P)$ and $s \notin P$ so $x \in P$. Therefore we may assume that $xs \in \phi(P)$, so x is a homogeneous element in $\rho^{-1}((\phi(P))_S)$. Thus,

$$\rho^{-1}(P_S) \subseteq P \cup \rho^{-1}((\phi(P))_S),$$

and since $\rho^{-1}((\phi(P))_S) \subseteq P$ then we have that $\rho^{-1}(P_S) = P$. ∎

Lemma 3.4. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. If $a/s \in h(I_S) - h(\phi_S(I_S))$ then $a \in I$.*

Proof. Let $a/s \in h(I_S) - h(\phi_S(I_S))$, so $a/s \notin \phi_S(I_S)$, and hence $a/1 \notin \phi_S(I_S)$, thus $a/1 \notin (\phi(I))_S$ since $(\phi(I))_S \subseteq \phi_S(I_S)$. Now, $\frac{a}{1} = \frac{r}{u}$ for some $r \in h(I)$ and $u \in S$, so, $r = au \in I$ and $au \notin \phi(I)$ because if $au \in \phi(I)$ then $a/1 \in (\phi(I))_S$, a contradiction. Thus $au \in I - \phi(I)$, if $a \notin I$ then u is not a ϕ-prime to I which mplies that $u \in h(P) \cap S$, a contradiction. Therefore, $a \in I$. ∎

Lemma 3.5. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. Then $h(\rho^{-1}(I_S)) - h(\rho^{-1}(\phi_S(I_S))) \subseteq h(I) - h(\phi(I))$.

Proof. Let a be a homogeneous element in $\rho^{-1}(I_S)$ such that $a \notin h(\rho^{-1}(\phi_S(I_S)))$, then $a/1 \in h(I_S) - h(\phi_S(I_S))$ and by Lemma 3.4, $a \in I$. If $a \in \phi(I)$ then $a/1 \in (\phi(I))_S \subseteq \phi_S(I_S)$ implies that $a \in h(\rho^{-1}(\phi_S(I_S)))$ a contradiction. Therefore, $a \in h(I) - h(\phi(I))$. ∎

Lemma 3.6. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. Then $\nu_{\phi_S}(I_S) \subseteq (\nu_\phi(I))_S$.*

Proof. If $a/s \in \nu_{\phi_S}(I_S)$, then we show in the proof of Lemma 3.4 that $au \in I - \phi(I) \subseteq \nu_\phi(I)$ for some u in S, hence $ua/us = a/s \in (\nu_\phi(I))_S$. ∎

Corollary 3.7. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. If $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$ then $\nu_{\phi_S}(I_S) = (\nu_\phi(I))_S$.*

Proof. In Lemma 3.6, we proved that $\nu_{\phi_S}(I_S) \subseteq (\nu_\phi(I))_S$.

Conversely, let $\frac{x}{s}$ be a homogeneous element in $(\nu_\phi(I))_S$, then $\frac{x}{s} = \frac{y}{t}$, where $y \in \nu_\phi(I)$. If $y \in I$, then $\frac{y}{t} \in I_S - (\phi(I))_S$ and so

$$y \in \rho^{-1}(I_S) - \rho^{-1}((\phi(I))_S) \subseteq I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S)).$$

Hence $\frac{y}{t} \in I_S - \phi_S(I_S) \subseteq \nu_{\phi_S}(I_S)$. Therefore we may assume that $y \notin I$. Since $y \in \nu_\phi(I)$ there exists a homogeneous element u in $R - I$ with $uy \in I - \phi(I)$, but $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$, so $(\frac{u}{1})(\frac{y}{t}) \in I_S - \phi_S(I_S)$ with $\frac{u}{1} \notin I_S$, thus $\frac{y}{t} \in \nu_{\phi_S}(I_S)$. ∎

We recall that if J is a superideal in R, then $J \subseteq \rho^{-1}(J_S)$ and therefore we may assume that $(\phi(J))_S \subseteq \phi_S(J_S)$. Under the condition that

$$I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$$

for all superideals I of R, we have the following proposition.

Proposition 3.8. *Let S be a multiplicatively closed subset of $h(R)$ with $1 \in S$, let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. Then I_S is a ϕ_S-P_S-primal superideal of R_S.*

Proof. By Theorem 3.2, P_S is a ϕ_S-prime superideal of R_S. To show that I_S is a ϕ_S-P_S-primal superideal of R_S, we must prove that

$$P_S = \begin{cases} [(\nu_{\phi_S}(I_S))_0 + (\nu_{\phi_S}(I_S))_1 \cup \{0\}] + \phi_S(I_S) & : \text{ if } \phi_S \neq \phi_\emptyset \\ (\nu_{\phi_S}(I_S))_0 + (\nu_{\phi_S}(I_S))_1 & : \text{ if } \phi_S = \phi_\emptyset \end{cases}.$$

Clearly, $\phi_S(I_S) \subseteq P_S$, let a/s be a homogenous element in $\nu_{\phi_S}(I_S)$, then there exists $r/u \in h(R_S) - h(I_S)$ with $(\frac{r}{u}) \cdot (\frac{a}{s}) \in I_S - \phi_S(I_S)$ so, by Lemma 3.5, $ra \in I - \phi(I)$ and $r \notin h(I)$, thus $a \in \nu_\phi(I) \subseteq P$ and hence $a/s \in P_S$.

Conversely, let $a/s \in h(P_S)$ such that $a/s \notin \phi_S(I_S)$. If $a/s \in I_S$, then $(1/1)(a/s) \in I_S - \phi_S(I_S)$, $(1/1) \notin I_S$, so a/s is not ϕ_S-prime to I_S, thus $a/s \in \nu_{\phi_S}(I_S)$. Therefore, we may assume that $a/s \notin I_S$, that is $ta \notin I$ for every $t \in S$. Since $a/s \in P_S$, then for some $t \in S$, $ta \in P - I$, and so $ta \in \nu_\phi(I)$ which implies that $a/s \in (\nu_\phi(I))_S$, and by Corollary 3.7, we have that $\nu_{\phi_S}(I_S) = (\nu_\phi(I))_S$ thus $a/s \in \nu_{\phi_S}(I_S)$. ■

Let R be a commutative super-ring with unity, and let S be a multiplicatively closed subset of $h(R)$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, then under the condition that

$$I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$$

for all proper superideals I of R, we have the following proposition.

Proposition 3.9. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let J be a ϕ_S-Q-primal superideal of R_S, then $\rho^{-1}(J)$ is ϕ-primal superideal of R with the adjoint superideal $\rho^{-1}(Q)$. Moreover, $J = (\rho^{-1}(J))_S$.*

Proof. To show that $\rho^{-1}(J)$ is ϕ-primal superideal of R with the adjoint superideal $\rho^{-1}(Q)$ we must show that

$$\rho^{-1}(Q) = \begin{cases} [(\nu_\phi(\rho^{-1}(J)))_0 + (\nu_\phi(\rho^{-1}(J)))_1 \cup \{0\}] + \phi(\rho^{-1}(J)) & : \text{ if } \phi \neq \phi_\emptyset \\ (\nu_\phi(\rho^{-1}(J)))_0 + (\nu_\phi(\rho^{-1}(J)))_1 & : \text{ if } \phi = \phi_\emptyset \end{cases}.$$

But $\phi(\rho^{-1}(J)) \subseteq \rho^{-1}(J) \subseteq \rho^{-1}(Q)$. Now let a be a homogenous element in $\nu_\phi(\rho^{-1}(J))$, then $\frac{a}{1} \in (\nu_\phi(\rho^{-1}(J)))_S$, but by Corollary 3.7, $(\nu_\phi(\rho^{-1}(J)))_S = \nu_{\phi_S}(J)$, so $\frac{a}{1} \in \nu_{\phi_S}(J) \subseteq Q$ and hence $a \in \rho^{-1}(Q)$.

Conversely, let a be a homogeneous element in $\rho^{-1}(Q)$, then $a/1$ in Q. We may assume that $a \notin \phi(\rho^{-1}(J))$, so $a/1 \notin \phi_S(J)$. If $a/1 \in J$, then $(a/1) \in J - \phi_S(J)$ and since $\phi(\rho^{-1}(J)) \subseteq \rho^{-1}(\phi_S(J))$, we have that $a \in \rho^{-1}(J) - \rho^{-1}(\phi_S(J)) \subseteq \rho^{-1}(J) - \phi(\rho^{-1}(J))$, but $1 \notin \rho^{-1}(J)$, so $a \in \nu_\phi(\rho^{-1}(J))$. If $a/1 \notin J$, then $a/1 \in Q - J$ and so $a/1 \in \nu_{\phi_S}(J)$. Let $\frac{x}{s}$ be a homogeneous element in $R_S - J$ with $(\frac{a}{1})(\frac{x}{s}) \in J - \phi_S(J)$ then $ax \in \rho^{-1}(J) - \rho^{-1}(\phi_S(J)) \subseteq \rho^{-1}(J) - \phi(\rho^{-1}(J))$, since $\frac{ax}{1} \in J$ and $\frac{ax}{1} \notin \phi_S(J)$, because if $\frac{ax}{1} \in \phi_S(J)$, then $\frac{ax}{s} \in \phi_S(J)$, a contradiction. Thus we have that $ax \in \rho^{-1}(J) - \phi(\rho^{-1}(J))$ and $x \notin \rho^{-1}(J)$, since $\frac{x}{s} \notin J$. Therefore, $a \in \nu_\phi(\rho^{-1}(J))$ and hence $\rho^{-1}(J)$ is ϕ-primal superideal of R with the adjoint superideal $\rho^{-1}(Q)$. Moreover, by [7, Theorem 2.5], $\rho^{-1}(Q)$ is a ϕ-prime superideal of R.

Finally, we show that $J = (\rho^{-1}(J))_S$. Clearly, $J \subseteq (\rho^{-1}(J))_S$.

Conversely, let $\frac{x}{s}$ be a homogeneous element in $(\rho^{-1}(J))_S$, then $xt \in \rho^{-1}(J)$ for some $t \in S$. Thus, $\frac{xt}{1} \in \rho(\rho^{-1}(J)) = J$, and hence $(\frac{xt}{1})(\frac{1}{st}) = \frac{x}{s} \in J$. Therefore, $J = (\rho^{-1}(J))_S$. ∎

By using Propositions 3.8 and 3.9, we have the following result.

Theorem 3.10. *Let R be a commutative super-ring with unity. Let S be a multiplicatively closed subset of $h(R)$, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function with the condition that $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$ for any proper superideal I of R. Then there is one-to-one correspondence between the ϕ-P^I-primal superideals I of R and ϕ_S-P^I_S-primal superideals I_S of R_S, where P^I is a ϕ-prime superideal of R with $P^I \cap S = \emptyset$.* ∎

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