SOME INCLUSION PROPERTIES OF STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH HOHLOV OPERATOR. II

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Abstract. A new subclass $K(\alpha, \beta)$ involving Hohlov Operator is introduced and some inclusion relations and distortion bounds are obtained for $f \in K(\alpha, \beta)$.

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1. Introduction

Let $A$ be the class of functions $f$ normalized by

\begin{equation}
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\end{equation}

which are analytic in the open unit disk

$$
    U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.
$$

As usual, we denote by $S$ the subclass of $A$ consisting of functions which are also univalent in $U$. The well known subclasses of $S$ are the class of starlike functions ($ST$) and convex functions ($CV$). A function $f(z) \in S$ is starlike of order $\alpha(0 \leq \alpha < 1)$ denoted by $ST(\alpha)$, if $\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha$ and it is convex of order $\alpha(0 \leq \alpha < 1)$ denoted by $CV(\beta)$, if $\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta$. It is an established fact that $f \in CV(\alpha) \iff zf' \in ST(\alpha)$.

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For functions \( f \in \mathcal{A} \) given by (1.1) and \( g \in \mathcal{A} \) given by \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \), we define the Hadamard product (or convolution) of \( f \) and \( g \) by

\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathbb{U}).
\]

Let \( \mathcal{T} \) denote the subclass of \( \mathcal{A} \) consisting of functions of the form

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0; \ z \in \mathbb{U}).
\]

The class \( \mathcal{T} \) was introduced by Silverman [10]. We denote by \( \mathcal{T}^*(\alpha) \) and \( \mathcal{C}(\alpha) \) denote the class of functions of the form (1.3) which are, respectively, starlike of order \( \alpha \) and convex of order \( \alpha \) with \( 0 < \alpha < 1 \).

The Gaussian hypergeometric function \( F(a, b; c, z) \) given by

\[
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in \mathbb{U})
\]

where, \( a, b, c \) are complex numbers such that \( c \neq 0, -1, -2, -3, \ldots \), \( (a)_0 = 1 \) for \( a \neq 0 \) and for each positive integer \( n \), \( (a)_n = a(a+1)(a+2)\ldots(a+n-1) \) is the Pochhammer symbol, and is the solution of the homogenous hypergeometric differential equation

\[
z(1-z)w''(z) + [c - (a + b + 1)z]w'(z) - abw(z) = 0
\]

has rich applications in various fields such as conformal mappings, quasi conformal theory, continued fractions and so on. The Gauss Summation theorem

\[
F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - a)\Gamma(c - b)} \text{ for } \text{Re}(c - a - b) > 0
\]

and the function \( F(a, b; c; 1) \) is bounded if \( \text{Re}(c - a - b) > 0 \) and has a pole at \( z = 1 \) if \( \text{Re}(c - a - b) \leq 0 \).

For \( f \in \mathcal{A} \), we recall the operator \( I_{a,b,c}(f) \) of Hohlov [5] which maps \( \mathcal{A} \) into itself defined by means of Hadamard product as

\[
I_{a,b,c}(f)(z) = zF(a, b; c; z) \ast f(z)
\]

Therefore, for a function \( f \) defined by (1.1), we have

\[
I_{a,b,c}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n.
\]

\[
\Phi(n) = \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \quad (a, b > 0; \ n \geq 2).
\]
A function $f \in A$ is said to be in the class $\mathcal{R}(A, B)$, $(\tau \in \mathbb{C}\setminus\{0\}, -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$
\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1 \quad (z \in \mathbb{U}).
$$

The class $\mathcal{R}(A, B)$ was introduced earlier by Dixit and Pal [3]. If we put

$$
\tau = 1, \quad A = \beta \quad \text{and} \quad B = -\beta \quad (0 < \beta \leq 1),
$$

we obtain the class of functions $f \in A$ satisfying the inequality

$$
\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in \mathbb{U}; 0 < \beta \leq 1)
$$

which was studied by (among others) Padmanabhan [8] and Caplinger and Causey [2], (see also [12]). We recall the following lemma relevant for our discussions.

**Lemma 1.1** [3] If $f \in \mathcal{R}(A, B)$ is of form (1.1), then

$$
|a_n| \leq (A - B)\frac{\tau}{n}, \quad n \in \mathbb{N} \setminus \{1\}.
$$

The result is sharp for the function

$$
f(z) = \int_0^z \left( 1 + \frac{(A - B)\tau z^{n-1}}{1 + Bz^{n-1}} \right) dz, \quad (n \geq 2; \ z \in \mathbb{U}).
$$

In this paper, we consider the following subclass of $S$ due to Kamali et al. [7] as given below:

For some $\alpha (0 \leq \alpha < 1)$ and $\lambda (0 \leq \lambda < 1)$, we let $K(\lambda, \alpha)$ be a new subclass of $S$ consisting of functions of the form (1.1) satisfying the analytic criteria

$$
\text{Re} \left( \frac{z^3 f'''(z) + (2\lambda + 1)z^2 f''(z) + zf'(z)}{\lambda z^2 f''(z) + zf'(z)} \right) > \alpha, \ z \in \mathbb{U}.
$$

We recall the following lemma due to Kamali et al. [7] to prove the main results.

**Lemma 1.2** A function $f \in T$ belongs to the class $K(\lambda, \alpha)$ if and only if

$$
\sum_{n=2}^{\infty} n(n - \alpha)(1 + n\lambda - \lambda)|a_n| \leq 1 - \alpha.
$$

**Lemma 1.3** [10] A function $f$ of the form (1.3) is in $T^*(\alpha)$ if and only if

$$
\sum_{n=2}^{\infty} (n - \alpha)a_n \leq 1 - \alpha \quad (0 \leq \alpha < 1)
$$

and is in $C(\alpha)$ if and only if

$$
\sum_{n=2}^{\infty} n(n - \alpha)a_n \leq 1 - \alpha \quad (0 \leq \alpha < 1).
$$
Motivated by the earlier works on hypergeometric functions studied recently in [9], [11]–[14], we will study the action of the hypergeometric function on the class $\mathbf{K}(\lambda, \alpha)$.

2. Main results

**Theorem 2.1** [14] Let $a, b \in \mathbb{C} \setminus \{0\}$, and $c$ be a real number. If $f \in ST$ and the inequality

$$
(2.11) \quad \frac{|a||b|(1 + |a|)(1 + |b|)(2 + |a|)(2 + |b|)(3 + |a|)(3 + |b|)}{c(1 + c)(2 + c)(3 + c)} \quad F_1(4 + |a|, 4 + |b|; 4 + c, 1)
$$

$$
+ [1 - \lambda(\alpha - 9)] \frac{|a||b|(1 + |a|)(1 + |b|)(2 + |a|)(2 + |b|)}{c(1 + c)(2 + c)} \quad F_1(3 + |a|, 3 + |b|; 3 + c, 1)
$$

$$
+ [6 - \lambda(5\alpha - 19) - \alpha] \frac{|a||b|(1 + |a|)(1 + |b|)}{c(1 + c)} \quad F_1(2 + |a|, 2 + |b|; 2 + c, 1)
$$

$$
+ (1 - \alpha) \frac{|ab|}{c} \quad F_1(1 + |a|, 1 + |b|; 1 + c, 1) \leq 1 - \alpha
$$

is satisfied, then $I_{a, b, c}(f) \in \mathbf{K}(\lambda, \alpha)$.

**Theorem 2.2** [14] Let $a, b \in \mathbb{C} \setminus \{0\}$ and let $c$ be a real number. If $f \in \mathcal{CV}$ and the inequality

$$
(2.12) \quad \frac{|a||b|(1 + |a|)(1 + |b|)(2 + |a|)(2 + |b|)}{c(1 + c)(2 + c)} \quad F_1(3 + |a|, 3 + |b|; 3 + c, 4; 1)
$$

$$
+ [1 - \lambda(\alpha - 5)] \frac{|a||b|(1 + |a|)(1 + |b|)}{c(1 + c)} \quad F_1(2 + |a|, 2 + |b|; 2 + c, 3; 1)
$$

$$
+ [3 - 2\lambda(\alpha - 2) - \alpha] \frac{|ab|}{c} \quad F_1(1 + |a|, 1 + |b|; 1 + c, 2; 1)
$$

$$
+ (1 - \alpha) \quad F_1(|a|, |b|; c; 1; 1) \leq 2(1 - \alpha)
$$

is satisfied, then $I_{a, b, c}(f) \in \mathbf{K}(\lambda, \alpha)$.

**Theorem 2.3** Let $a, b \in \mathbb{C} \setminus \{0\}$ and let $c$ be a real number such that $c > |a| + |b| + 1$. If $f \in \mathcal{R}(A, B)$ and if the inequality

$$
(2.13) \quad \frac{|a||b|(1 + |a|)(1 + |b|)}{c(1 + c)} \quad F(2 + |a|, 2 + |b|; 2 + c; 1)
$$

$$
+ [1 - \lambda(\alpha - 2)] \frac{|ab|}{c} \quad F(|a|, |b|; 1 + c; 1) + (1 - \alpha) F(|a|, |b|; c; 1)
$$

$$
\leq (1 - \alpha) \left( \frac{1}{(A - B)|\tau|} + 1 \right)
$$

is satisfied, then $I_{a, b, c}(f) \in \mathbf{K}(\lambda, \alpha)$.  


Let \( f \) be of the form (1.1) belong to the class \( \mathcal{R}^r(A,B) \). By virtue of Lemma 1.1, it suffices to show that

\[
(2.14) \quad \sum_{n=2}^{\infty} n(n-\alpha)(1+n\lambda-\lambda) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n \right| \leq 1 - \alpha.
\]

Taking into account the inequality (1.9) and the relation \( |(a)_{n-1}| \leq |(a)|_{n-1} \), we deduce that

\[
\sum_{n=2}^{\infty} n(n-\alpha)(1+n\lambda-\lambda) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n \right| \\
\leq (A-B)|\tau| \left( \lambda \sum_{n=2}^{\infty} (n-1)(n-2) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \right) \\
+ [1-\lambda(\alpha-2)] \sum_{n=2}^{\infty} (n-1) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| + (1-\alpha) \sum_{n=2}^{\infty} \left( \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) \\
\leq (A-B)|\tau| \left( \lambda \sum_{n=2}^{\infty} \left( \frac{|a|_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) \right) \\
+ (1-\alpha) \sum_{n=2}^{\infty} \left( \frac{|a|_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) \\
= (A-B)|\tau| \left( \lambda \frac{|a|_2(b)_2}{(c)_2} \sum_{n=2}^{\infty} \frac{(2+|a|)_{n-3}(2+|b|)_{n-3}}{(2+c)_{n-3}(1)_{n-3}} \right) \\
+ [1-\lambda(\alpha-2)] \frac{|ab|}{c} \sum_{n=2}^{\infty} \frac{(1+|a|)_{n-2}(1+|b|)_{n-2}}{(1+c)_{n-2}(1)_{n-2}} + (1-\alpha) \sum_{n=2}^{\infty} \left( \frac{|a|_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) \\
= (A-B)|\tau| \left( \lambda \frac{|a|b(1+|a|)(1+|b|)}{c(1+c)} F(2+|a|,2+|b|,2+c;1) \right) \\
+ [1-\lambda(\alpha-2)] \frac{|ab|}{c} F(1+|a|,1+|b|,1+c;1)+(1-\alpha) \left( F(|a|,|b|,c;1)-1 \right),
\]

where we use the relation \( (a)_n = a(a+1) \).

The proof now follows by an application of the Gauss summation theorem and (1.5).

Next, we prove the following properties for the operator \( I_{a,b,c}(f) \), when a function \( f \) belongs to the class \( \mathbf{K}(\lambda,\alpha) \).

**Theorem 2.4** Let \( a,b > 0 \), \( c \geq \max\{0,a+b-1,1/2(ab+a+b-1)\} \) and let a function \( f \) of the form (1.3) be in \( \mathbf{K}(\lambda,\alpha) \). Then

\[
(2.15) \quad |z| - \frac{(1-\alpha)}{2(2-\alpha)(1+\lambda)} \frac{ab}{c} |z|^2 \leq |I_{a,b,c} f(z)| \leq |z| + \frac{(1-\alpha)}{2(2-\alpha)(1+\lambda)} \frac{ab}{c} |z|^2
\]
and

\[ 1 - \frac{(1 - \alpha)}{(2 - \alpha)(1 + \lambda)} \frac{ab}{c} \leq \frac{|(I_{a,b,c}f(z))'|}{|z|} \leq 1 + \frac{(1 - \alpha)}{(2 - \alpha)(1 + \lambda)} \frac{ab}{c} |z|. \]

The results are sharp.

**Proof.** We note that

\[ I_{a,b,c}f(z) = \left( zF(a, b; c; z) * f \right)(z) = z - \sum_{n=2}^{\infty} \Phi(n)a_nz^n, \]

where

\[ \Phi(n) = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} (a, b > 0; n \geq 2) \]

and \(0 < \Phi(n + 1) \leq \Phi(n) (n \geq 2)\) under the assumption for \(c\). Since \(f \in \mathbb{K}(\lambda, \alpha)\), by Lemma 1.2, we have

\[ 2(2 - \alpha)(1 + \lambda) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n(n - \alpha)(1 + n\lambda - \lambda)a_n \leq 1 - \alpha. \]

Therefore, by using (2.17), we obtain

\[ |I_{a,b,c}(f)| \leq |z| + \sum_{n=2}^{\infty} \Phi(n)a_n|z|^n \]

\[ \leq |z| + \Phi(2)|z|^2 \sum_{n=2}^{\infty} a_n \]

\[ \leq |z| + \frac{(1 - \alpha)}{2(2 - \alpha)(1 + \lambda)} \frac{ab}{c} |z|^2 \]

and

\[ |I_{a,b,c}(f)| \geq |z| - \sum_{n=2}^{\infty} \Phi(n)a_n|z|^n \]

\[ \geq |z| - \Phi(2)|z|^2 \sum_{n=2}^{\infty} a_n \]

\[ \geq |z| - \frac{(1 - \alpha)}{2(2 - \alpha)(1 + \lambda)} \frac{ab}{c} |z|^2. \]

From (2.17), we note that

\[ \sum_{n=2}^{\infty} na_n \leq \frac{(1 - \alpha)}{(2 - \alpha)(1 + \lambda)}. \]

By using (2.18), we obtain (2.16). The results are sharp for the function \(f(z) = z - \frac{1 - \alpha}{2(2 - \alpha)(1 + \lambda)} z^2\). \(\blacksquare\)

Now, we find the order \(\beta (0 \leq \beta < 1)\) for which the operator \(I_{a,b,c}(f)\) belongs to the classes \(\mathcal{T}^*(\beta)\) and \(\mathcal{C}(\beta)\) when a function \(f\) belongs to the class \(\mathbb{K}(\lambda, \alpha)\).
**Theorem 2.5** Let $a, b > 0$, $\max\{2ab/3, a + b - 1, (1/2)(ab + a + b - 1)\} \leq c \leq ab$ and let a function $f$ of the form (1.3) be in $K(\lambda, \alpha)$. Then $I_{a,b,c}(f) \in T^*(\beta)$, where

\[(2.19) \quad \beta = 1 - \frac{\Phi(2)(1 - \alpha)}{2(2 - \alpha)(1 + \lambda) - \Phi(2)(1 - \alpha)}.\]

**Proof.** Let $f \in K(\lambda, \alpha)$. Consider the operator

\[I_{a,b,c} f(z) = z + \sum_{n=2}^{\infty} \Phi(n) a_n z^n,\]

where

\[\Phi(n) = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \quad (a, b > 0; \ n \geq 2).\]

Since $\Phi(n)$ is a decreasing function for $n$, by Lemma 1.3, we need to find $\beta$ ($0 \leq \beta < 1$) that

\[\Phi(2) \sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta} a_n \leq 1.\]

Since $f \in K(\lambda, \alpha)$, by Lemma 1.2, we have

\[\sum_{n=2}^{\infty} n(n - \alpha)(1 + n\lambda - \lambda)a_n \leq 1 - \alpha.\]

To complete the proof, it suffices to find $\beta$ such that

\[(2.20) \quad \frac{n - \beta}{1 - \beta} \Phi(2) \leq \frac{n(n - \alpha)(1 + n\lambda - \lambda)}{1 - \alpha}.\]

From (2.20), we obtain

\[\beta \leq \Psi(n),\]

where

\[(2.21) \quad \Psi(n) = \frac{n(n - \alpha)(1 + n\lambda - \lambda) - n\Phi(2)(1 - \alpha)}{n(n - \alpha)(1 + n\lambda - \lambda) - \Phi(2)(1 - \alpha)}.\]

By the assumption of the theorem, it is easy to see that $\Psi(n)$ is an increasing function for $n$ ($n \geq 2$). Setting $n = 2$ in (2.21), we have

\[\beta = \frac{2(2 - \alpha)(1 + \lambda) - 2\Phi(2)(1 - \alpha)}{2(2 - \alpha)(1 + \lambda) - \Phi(2)(1 - \alpha)},\]

hence we get (2.19). Therefore we complete the proof of Theorem 2.5. \[\blacksquare\]

**Theorem 2.6** Let $a, b > 0$, $\max\{2ab/3, a + b - 1, (1/2)(ab + a + b - 1)\} \leq c \leq ab$ and let a function $f$ of the form (1.3) be in $K(\lambda, \alpha)$. Then $I_{a,b,c}(f) \in C^*(\beta)$, where

\[(2.22) \quad \beta = 1 - \frac{\Phi(2)(1 - \alpha)}{(2 - \alpha)(1 + \lambda) - \Phi(2)(1 - \alpha)}.\]
Proof. Let \( f \in K(\lambda, \alpha) \). Consider the operator

\[
I_{a,b,c}f(z) = z + \sum_{n=2}^{\infty} \Phi(n)a_n z^n,
\]

where

\[
\Phi(n) = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \quad (a, b > 0; \ n \geq 2).
\]

Since \( \Phi(n) \) is a decreasing function for \( n \), by Lemma 1.3, we need to find \( \beta \) (\( 0 \leq \beta < 1 \)) such that

\[
\Phi(2) \sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta} a_n \leq 1.
\]

Since \( f \in K(\lambda, \alpha) \), by Lemma 1.2, we have

\[
\sum_{n=2}^{\infty} n(n - \alpha)(1 + n\lambda - \lambda)a_n \leq 1 - \alpha.
\]

To complete the proof, it suffices to find \( \beta \) such that

\[
(2.23) \quad \frac{n - \beta}{1 - \beta} \Phi(2) \leq \frac{n(n - \alpha)(1 + n\lambda - \lambda)}{1 - \alpha}.
\]

From (2.23), we obtain

\[
\beta \leq \Upsilon(n),
\]

where

\[
(2.24) \quad \Upsilon(n) = \frac{(n - \alpha)(1 + n\lambda - \lambda) - n\Phi(2)(1 - \alpha)}{(n - \alpha)(1 + n\lambda - \lambda) - \Phi(2)(1 - \alpha)}.
\]

By the assumption of the theorem, it is easy to see that \( \Psi(n) \) is an increasing function for \( n \) (\( n \geq 2 \)). Setting \( n = 2 \) in (2.21), we have

\[
\beta = \frac{(2 - \alpha)(1 + \lambda) - 2\Phi(2)(1 - \alpha)}{(2 - \alpha)(1 + \lambda) - \Phi(2)(1 - \alpha)},
\]

hence we get (2.19). Therefore we complete the proof of Theorem 2.6.

3. Concluding remarks

If \( a = 1, b = 1 + \delta, c = 2 + \delta \) with \( \text{Re}(\delta) > -1 \), then the convolution operator \( I_{a,b,c}(f) \) turns into a Bernardi operator

\[
B_f(z) = [I_{a,b,c}(f)](z) = \frac{1 + \delta}{z^\delta} \int_0^1 t^{\delta-1} f(t) dt.
\]
Further, $I_{1,1,2}(f)$ and $I_{1,2,3}(f)$ are known as Alexander and Libera operators, respectively. Further, note that, when $|b| = 1$, we get $I_{a,1,c}(f) = L(a,c)f(z) = \left(z + \sum_{n=2}^{\infty} \frac{(a_{n-1} - c_{n-1})}{(c_{n-1})} z^n\right)^* f(z) = z + \sum_{n=2}^{\infty} \frac{(a_{n-1} - c_{n-1})}{(c_{n-1})} a_n z^n$, the Carlson-Shaffer operator and also for $a = b = c = 1$ the Ruscheweyh derivative operator

$$D^\delta f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\delta + n - 1}{n - 1}\right) a_n z^n,$$

hence one can deduce various interesting results for the function class defined by these operator as a corollary, we omit the details involved.

References


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