ON REVERSE WEIGHTED ARITHMETIC-GEOMETRIC MEAN INEQUALITIES FOR TWO POSITIVE OPERATORS

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Abstract. Let A, B be positive operators on a Hilbert space with $0 < m \le A, B \le M$. Then for every positive unital linear map Φ,

$$\begin{aligned} (A\nabla_\mu B)^2 &\le \left[\frac{(M+m)^2}{4Mm}\right]^2 (A\sharp_\mu B)^2, && 0 \le \mu \le 1,\\ \Phi^2(A\nabla_\mu B) &\le \left[\frac{(M+m)^2}{4Mm}\right]^2 \Phi^2(A\sharp_\mu B), && 0 \le \mu \le 1,\\ \Phi^2(A\nabla_\mu B) &\le \left[\frac{(M+m)^2}{4Mm}\right]^2 [\Phi(A)\sharp_\mu \Phi(B)]^2, && 0 \le \mu \le 1. \end{aligned}$$

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1. Introduction

Throughout this paper, let M, m be scalars, I be the identity operator and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The operator norm is denoted by $\|\cdot\|$. We write $A \ge 0$ if the operator A is positive. If $A - B \ge 0$, then we say that $A \ge B$. For $A, B > 0$, we use the following notation:

$$A\nabla_\mu B = (1-\mu)A + \mu B, \quad A\sharp_\mu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu A^{\frac{1}{2}}, \quad \text{where } 0 \le \mu \le 1.$$

When $\mu = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity for $A\nabla_{\frac{1}{2}}B$ and $A\sharp_{\frac{1}{2}}B$, respectively, see Kubo and Ando [1].

Let A and B be positive operators on a Hilbert space with $0 < m \le A$, $B \le M$. Tominaga [2] showed that the following reverse AM-GM inequality with Specht ratio:

$$A\nabla_\mu B \le S(h)A\sharp_\mu B, \tag{1.1}$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ for $h = \frac{M}{m} \geq 1$. Indeed,

$$S(h) \leq \frac{(M+m)^2}{4Mm} \leq S^2(h) \ (h \geq 1) \tag{1.2}$$

was observed by Lin [3, (3.3)].

Inequality (1.1) can be regarded as a counterpart to operator AM-GM inequality which says

$$A\nabla_\mu B \geq A\sharp_\mu B. \tag{1.3}$$

By (1.1) and (1.2), we have the following inequalities:

$$A\nabla_\mu B \leq \frac{(M+m)^2}{4Mm} A\sharp_\mu B, \tag{1.4}$$

$$\Phi(A\nabla_\mu B) \leq \frac{(M+m)^2}{4Mm} \Phi(A\sharp_\mu B). \tag{1.5}$$

It is well known that for two positive operators A, B,

$$A \geq B \nRightarrow A^2 \geq B^2.$$

Can inequalities (1.4) and (1.5) be squared? This is a main motivation for the present paper.

In this paper, the main results are that inequalities (1.4) and (1.5) can be squared, which we will present in the next section.

2. Main results

We begin this section with the following lemmas.

Lemma 2.1. [4] *Let $A, B > 0$. Then the following norm inequality holds:*

$$\|AB\| \leq \frac{1}{4} \|A+B\|^2 . \tag{2.1}$$

Lemma 2.2. [5] *Let A be a positive operator on a Hilbert space. Then for every positive unital linear map Φ,*

$$\Phi(A^{-1}) \geq \Phi^{-1}(A). \tag{2.2}$$

Theorem 2.3. *Let $0 < m \leq A, B \leq M$. Then*

$$(A\nabla_\mu B)^2 \leq \left[\frac{(M+m)^2}{4Mm}\right]^2 (A\sharp_\mu B)^2, \quad 0 \leq \mu \leq 1, \tag{2.3}$$

or equivalently

$$\left\|(A\nabla_\mu B)(A\sharp_\mu B)^{-1}\right\| \le \frac{(M+m)^2}{4Mm}, \quad 0 \le \mu \le 1. \tag{2.4}$$

Proof. By Lemma 2.1, inequality (2.4) is true if

$$A\nabla_\mu B + Mm(A\sharp_\mu B)^{-1} \le M + m.$$

By $(A\sharp_\mu B)^{-1} = A^{-1}\sharp_\mu B^{-1}$ and inequality (1.3), we have

$$\begin{aligned} A\nabla_\mu B + Mm(A\sharp_\mu B)^{-1} &\le A\nabla_\mu B + MmA^{-1}\nabla_\mu B^{-1} \\ &= (1-\mu)A + \mu B + Mm[(1-\mu)A^{-1} + \mu B^{-1}] \\ &\le M + m, \end{aligned}$$

where the last inequality is by [6, (2.3)]. This proves (2.4). ∎

Theorem 2.4. *Let $0 < m \le A, B \le M$. Then for every positive unital linear map Φ,*

$$\Phi^2(A\nabla_\mu B) \le \left[\frac{(M+m)^2}{4Mm}\right]^2 \Phi^2(A\sharp_\mu B), \qquad 0 \le \mu \le 1, \tag{2.5}$$

and

$$\Phi^2(A\nabla_\mu B) \le \left[\frac{(M+m)^2}{4Mm}\right]^2 [\Phi(A)\sharp_\mu \Phi(B)]^2, \quad 0 \le \mu \le 1. \tag{2.6}$$

Proof. Inequality (2.5) is equivalent to

$$\left\|\Phi(A\nabla_\mu B)\Phi^{-1}(A\sharp_\mu B)\right\| \le \frac{(M+m)^2}{4Mm}. \tag{2.7}$$

Compute

$$\begin{aligned} &\left\|\Phi(A\nabla_\mu B)Mm\Phi^{-1}(A\sharp_\mu B)\right\| \\ &\le \frac{1}{4}\left\|\Phi(A\nabla_\mu B) + Mm\Phi^{-1}(A\sharp_\mu B)\right\|^2 && \text{(by (2.1))} \\ &\le \frac{1}{4}\left\|\Phi(A\nabla_\mu B) + Mm\Phi[(A\sharp_\mu B)^{-1}]\right\|^2 && \text{(by (2.2))} \\ &= \frac{1}{4}\left\|\Phi(A\nabla_\mu B) + Mm\Phi[(A^{-1}\sharp_\mu B^{-1})]\right\|^2 \\ &\le \frac{1}{4}\left\|\Phi(A\nabla_\mu B) + Mm\Phi[(A^{-1}\nabla_\mu B^{-1})]\right\|^2 && \text{(by (1.3))} \\ &\le \frac{1}{4}(M+m)^2. && \text{(by [6, (2.3)])} \end{aligned}$$

That is,

$$\left\|\Phi(A\nabla_{\mu}B)\Phi^{-1}(A\sharp_{\mu}B)\right\| \leq \frac{(M+m)^2}{4Mm}.$$

Thus, (2.7) holds. The proof of (2.6) is similar, we omit the details. ∎

Remark 2.5. When $\mu = \frac{1}{2}$, by (2.5) and (2.6) we obtain [3, (2.1)] and [3, (2.2)], respectively. Thus, (2.5) and (2.6) are generalizations of [3, (2.1)] and [3, (2.2)], respectively.

References

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