EXISTENCE SOLUTION FOR WEIGHTED $p(x)$-LAPLACIAN EQUATION

S.R. Mousaviankhatir
M. Alimohammady[1]

Department of Mathematics
University of Mazandaran
P.O. Box 47416-95447, Babolsar
Iran

H. Jafari

Department of Mathematics
University of Mazandaran
P.O. Box 47416-95447, Babolsar
Iran
and
Department of Mathematical Sciences
University of South Africa
UNISA 0003
South Africa

Abstract. This paper deals with the existence solution for the following type of boundary value problems:

$$\begin{cases} \Delta\left(|x|^{p(x)}\,|\Delta u|^{p(x)-2}\,\Delta u\right) = \lambda\,|u|^{q(x)-2}\,u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\Re^N$. It is established for a negative λ, there exists at least one weak solution. Our approach relies on the variable exponent theory of generalized Lebesgue-Sobolev spaces and a variant of the Mountain Pass theorem.

Keywords: $p(x)$-biharmonic, variable exponent Lebesgue space, variable exponent Sobolev space.
2010 Mathematics Subject Classification: 46E35, 26A45, 28A12.

1. Introduction

The literature of various mathematical problems with variable exponent have received a lot of attention in recent years [1], [12]. Fourth order equations appears

[1]Corresponding author. E-mail: amohsen@umz.ac.ir

in many contexts. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells [5]. In addition, this type of equation can describe the static from change of beam or the sport of rigid body. There are many authors have pointed out that type of non linearity furnishes models to study travelling waves in suspension bridges [2], [8].

Given a bounded smooth domain $\Omega \subseteq \Re^N$, we recall some definitions and basic properties of the variable exponent Lebesgue space and Sobolev space $L^{p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$, where $p(x) : \bar{\Omega} \to (1, \infty)$ is a continuous function. For further information in this regards we refer to [7] and [10]. On the other hand, regarding applications of variable exponent Lebesgue and Sobolev spaces to PDEs we refer to [6] while for some physical motivations of such problems we remember the contributions of Rajagopal and Ruzicka [11], Ruzicka [12] and Zhikov[14]. For any continuous function $h : \overline{\Omega} \to (1, \infty)$, set

$$h^+ = \sup_{x \in \Omega} h\,(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h\,(x).$$

Given $p\,(x) \in C\left(\overline{\Omega}, (1, \infty)\right)$, the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{ \begin{array}{c} u;\ u \text{ is measurable real-valued function on } \Omega \\ \text{such that } \displaystyle\int_\Omega |u(x)|^{p(x)} dx < \infty \end{array} \right\}.$$

$L^{p(x)}(\Omega)$ endowed with the Luxemburg norm

$$|u|_{p(x)} = inf \left\{ \mu > 0; \int_\Omega \left| \frac{u\,(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\} \tag{1.1}$$

is a reflexive Banach space [4].

It is well know that if $p_1\,(x) \leq p_2\,(x)$ almost everywhere in Ω then there exists a continuous embedding $L^{p_2(x)}\,(\Omega) \hookrightarrow L^{p_1(x)}\,(\Omega)$, whose norm does not exceed $|\Omega| + 1$. We denote by $L^{q(x)}\,(\Omega)$ the conjugate space of $L^{p(x)}\,(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}\,(\Omega)$ and $v \in L^{q(x)}\,(\Omega)$ the Holder type inequality valid [4]

$$\left| \int_\Omega uvdx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)}\, |v|_{q(x)} \leq 2\, |u|_{p(x)}\, |v|_{q(x)}\,. \tag{1.2}$$

An important role in manipulating for generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(x)}\,(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}\,(\Omega) \to \Re$ defined by

$$\rho_{p(x)}\,(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If (u_n) and u are a sequence and an element respectively in $L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold [4]:

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{1.3}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{1.4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0, \tag{1.5}$$

$$|u_n|_{p(x)} \to \infty \Leftrightarrow \rho_{p(x)}(u_n) \to \infty. \tag{1.6}$$

As usual $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \|\nabla u\|_{p(x)}.$$

Set

$$C_+(\bar{\Omega}) = \left\{ p \in C(\bar{\Omega}) ; \inf_{x \in \bar{\Omega}} p(x) > 1 \right\}.$$

For any $p(x) \in C_+(\bar{\Omega})$, denote by $p_k^*(x) = \frac{Np(x)}{N-kp(x)}$ if $p(x) < \frac{N}{k}$ and $p_k^*(x) = +\infty$ if $p(x) \geq \frac{N}{k}$. Define the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k \right\}.$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1{}^{\alpha_1} \dots \partial x_N{}^{\alpha_N}}$ with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{k,p(x)}(\Omega)$ endowed with norm $\|u\| = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p(x)}$ is a separable reflexive Banach space. For $p, r \in C_+(\bar{\Omega})$ in which $r(x) < p_k^*(x)$ for all $x \in \bar{\Omega}$, there is a continuous and compact embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$ with respect to the norm $\|u\|$. For more information one can refer to [7]-[9].

2. Existence solution

In this section, we denote by $D_0^{2,p(x)}(\Omega)$ the closure of $C_c^2(\Omega)$ endowed with the norm

$$\|u\| = \left| |x| \, |\Delta u| \right|_{p(x)}$$

$\left(D_0^{2,p(x)}(\Omega), \|.\|\right)$ is a reflexive Banach space. Assume that $q : \bar{\Omega} \to (1, \infty)$ is a continuous function satisfying $1 < q^- \leq q^+ < \frac{2Np^-}{2N+p^-}$. We investigate the existence of solutions for the following problem

$$\begin{cases} \Delta\left(|x|^{p(x)} |\Delta u(x)|^{p(x)-2} \Delta u(x)\right) = \lambda |u(x)|^{q(x)-2} u(x) & \text{for } x \in \Omega, \\ u(x) = \Delta u = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{2.1}$$

where λ is a negative constant. The above equation called weighted $p(x)$-Laplacian equation. If

$$\int_\Omega |x|^{p(x)} |\Delta u|^{p(x)-2} \Delta u \Delta v dx - \int_\Omega |u|^{q(x)-2} uv dx = 0, \qquad \forall v \in D_0^{2,p(x)}(\Omega),$$

then $u \in D_0^{2,p(x)}(\Omega)$ is a weak solution of problem (2.1).

Now, we show the following existence result for (2.1).

Theorem 1. *For each $\lambda < 0$, (2.1) has a nontrivial weak solution.*

Proof. For each $\lambda < 0$, we consider the energy functional associated with problem (2.1), $J_\lambda : D_0^{2.p(x)}(\Omega) \to \Re$ by

$$J_\lambda(u) = \int_\Omega \frac{|x|^{p(x)}}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx$$

for each $u \in D_0^{2,p(x)}(\Omega)$. Standard arguments show that $J_\lambda \in C^1\left(D_0^{2,p(x)}(\Omega), \Re\right)$ and its derivative is given by

$$\left\langle \acute{J}_\lambda(u), v\right\rangle = \int_\Omega |x|^{p(x)} |\Delta u|^{p(x)-2} \Delta u \Delta v dx - \lambda \int_\Omega |u|^{q(x)-2} uv dx,$$

for all $u \in D_0^{2,p(x)}(\Omega)$. We infer that u is a solution of problem (2.1) if and only if it is a critical point of J_λ. Consequently, we concentrate our efforts on finding critical points for J_λ. In this context we prove the following assertions:

(a) J_λ is weakly lower semi-continuous.

(b) J_λ is bounded from below and coercive.

(c) There exists $\psi \in D_0^{2,p(x)}(\Omega) - \{0\}$ such that $J_\lambda(\Psi) < 0$.

The arguments to prove $(a), (b)$ and (c) are detailed as below.

(a) First, we prove that the functional $\Lambda : D_0^{2,p(x)}(\Omega) \to \Re$ defined by

$$\Lambda(u) = \int_\Omega \frac{|x|^{p(x)}}{p(x)} |\Delta u|^{p(x)} dx,$$

is convex. Indeed, since the function

$$[0, \infty) \ni t \to t^\theta,$$

is convex for any $\theta > 1$, so for each $x \in \Omega$

$$\left|\frac{\xi + \psi}{2}\right|^{p(x)} \le \left(\frac{|\xi| + |\psi|}{2}\right)^{p(x)} \le \frac{1}{2}|\xi|^{p(x)} + \frac{1}{2}|\Psi|^{p(x)}, \ \forall \xi, \psi \in \Re^N,$$

holds. Using this inequality,

$$\left|\frac{\Delta u+\Delta v}{2}\right|^{p(x)} \leq \left(\frac{|\Delta u|+|\Delta v|}{2}\right)^{p(x)} \leq \frac{1}{2}\left|\Delta u\right|^{p(x)}+\frac{1}{2}\left|\Delta v\right|^{p(x)},$$

$\forall u,v \in D_0^{2,p(x)}(\Omega)$.

Multiplying with $\frac{|x|^{p(x)}}{p(x)}$ and integrating over Ω we obtain:

$$\Lambda\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Lambda(u)+\frac{1}{2}\Lambda(v), \forall u,v \in D_0^{2,p(x)}(\Omega).$$

Next, we show that Λ is weakly lower semi continuous on $D_0^{2,p(x)}(\Omega)$. Taking into account that Λ is convex, by Corollary 3.8 in [3] it is enough to show that Λ is strongly lower semi-continuous on $D_0^{2,p(x)}(\Omega)$. We fix $u \in D_0^{2,p(x)}(\Omega)$, $\epsilon > 0$ and $v \in D_0^{2,p(x)}(\Omega)$. Since Λ is convex and holder type inequality (1.2) holds true,

$$\begin{aligned}
\Lambda(v) &\geq \Lambda(u)+\langle \Lambda'(u), v-u\rangle \\
&\geq \Lambda(u)-\int_\Omega |\Delta u|^{p(x)-1}\left|\Delta(v-u)\right| dx \\
&\geq \Lambda(u)-D_1\left||\Delta u|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}\left|\Delta(u-v)\right|_{p(x)} \\
&\geq \Lambda(u)-D_2\left\|u-v\right\|_{p(x)} \\
&\geq \Lambda(u)-\epsilon,
\end{aligned}$$

for all $v \in D_0^{2,p(x)}(\Omega)$ with

$$|u-v|_{p(x)} < \frac{\epsilon}{||\Delta u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}}}.$$

We have denoted by D_1, D_2 two positive constants. It follows that Λ is strongly lower semi continuous and convex, so (by corollary 3.8 in [3]) Λ is weakly lower semi continuous. Finally, if $\{u_n\} \subset D_0^{2,p(x)}(\Omega)$ is a sequence weakly converges to u in $D_0^{2,p(x)}(\Omega)$, then $\{u_n\}$ converges strongly to u in $L^{q(x)}(\Omega)$. Thus, J_λ is weakly lower semi continuous and proof of (a) is complete.

(b) It is clear that for any $u \in D_0^{2,p(x)}(\Omega)$

$$\begin{aligned}
J_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega |x|^{p(x)}\left|\Delta u\right|^{p(x)} dx-\frac{\lambda}{q^-}\int_\Omega |u|^{q(x)} dx, \\
&\geq \frac{1}{p^+}\int_\Omega |x|^{p(x)}\left|\Delta u\right|^{p(x)} dx-\frac{\lambda}{q^-}\left(|u|_{q(x)}^{q^-}+|u|_{q(x)}^{q^-}\right).
\end{aligned}$$

Since $1 < q^- \leq q \leq q^+ < \frac{2Np^-}{2N+p^-} < p^-$, by Theorem 2 in [9] there is a continuous compact embedding of $D_0^{2,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ such that,

$$\exists c > 0 : |u|_{q(x)} \leq c\,\|u\|\,, \forall u \in D_0^{2,p(x)}(\Omega)\,.$$

If $\|u\| > 1$, using this inequality,

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{k\lambda}{q^-}\left(\|u\|^{q^-} + \|u\|^{q^+}\right),$$

where k is a positive constant. For $\lambda < 0$, $\lim\limits_{\|u\|\to\infty} J_\lambda(u) = \infty$, means that J_λ is coercive. It is clear that for any $u \in D_0^{2,p(x)}(\Omega)$

$$J_\lambda(u) \geq \frac{1}{p^+}\min\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\} - \frac{k\lambda}{q^-}\left(\|u\|^{q^-} + \|u\|^{q^+}\right).$$

We deduce that J_λ is bounded from below.

(c) Suppose that $\varphi \in C_c^2(\Omega)\,, \varphi \neq 0$. Then, for each $t \in (0.1)$,

$$\begin{aligned} J_\lambda(t\varphi) &= \int_\Omega \frac{|x|^{p(x)}\,t^{p(x)}}{p(x)}|\Delta\varphi|^{p(x)}\,dx - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}|\varphi|^{q(x)}\,dx \\ &\leq t^{p^-}\int_\Omega \frac{|x|^{p(x)}}{p(x)}|\Delta\varphi|^{p(x)}\,dx - \lambda t^{q^+}\int_\Omega \frac{1}{q(x)}|\varphi|^{q(x)}\,dx \\ &\leq \frac{t^{p^-}}{p^-}\max\left\{\|\varphi\|^{p^-}, \|\varphi\|^{p^+}\right\} - \frac{\lambda t^{q^+}}{q^+}\left(|\varphi|_{q(x)}^{q^+} + |\varphi|_{q(x)}^{q^-}\right). \end{aligned}$$

Set

$$L_1 = \frac{1}{p^+}\max\left\{|\varphi|^{p^-}, |\varphi|^{p^+}\right\}$$

and

$$L_2 = \frac{\lambda}{q^+}\left(|\varphi|_{q(x)}^{q^+} + |\varphi|_{q(x)}^{q^-}\right).$$

Then

$$\begin{aligned} J_\lambda(t\varphi) &\leq L_1 t^{p^-} - L_2 t^{q^+},\ q^+ < p^- \\ &= L_1 t^{p^-}\left(1 - \frac{L_2}{L_1}t^{q^+-p^-}\right). \end{aligned}$$

Since L_1 is a positive constant, from this inequality

$$\begin{aligned} J_\lambda < 0 &\iff 1 - \frac{L_2}{L_1}\,t^{q^+-p^-} < 0 \\ &\iff t < \left(\frac{L_2}{L_1}\right)^{\frac{1}{p^- - q^+}}. \end{aligned}$$

We infer that, for any $t \in (0, \min\{1, (\frac{L_2}{L_1})^{\frac{1}{p^- - q^+}}\})$,

$$J_\lambda (t\varphi) < 0.$$

These facts together with Theorem (1) of [13] implies the existence of $u_\lambda \in D_0^{2,p(x)}(\Omega)$ as a global minimum point of J_λ. Moreover, since (c) hold true it follows that $u_\lambda \neq 0$. ∎

References

[1] Acerbi, E., Mingion, G., *Regularrity results for a class of functional with non-standard growth*, Arch. Rational Mech. Anal., 156 (2001), 121-140.

[2] Bonanno, G., Bella, B.Di., *A boundary value problem for forth-order elastic beam equations*, J. Math. Anal. Appl., 343 (2) (2008), 1116-1176.

[3] Brezis, H., *Analise functional Theorie Methodes et Applications*, Masson Paris, 1992.

[4] Denisa, S.D., Mihailescu, M., Radulescu, V., *A Caffarrelli-Kohn-Nirenberg.type in equality with variable exponent and applications to PDEs*, Complex Variables and Elliptic Equations, 56 (2011), 659-669.

[5] Ferrero, A., Warnault, G., *On a solutions of second and forth order elliptic with power type nonlinearities*, Nonlinear Anal. T.M.A, 70 (2009), 2889-2902.

[6] Harjulehto, P., Hasto, P., Le, U.V., Nuortio, M., *Overview of differential equations with non-standard growth*, Nonlinear Anal., 72, (2010), 4551-4574.

[7] Kovacik, O., Rakosnik, J., *On spaces $L^{p(x)}$ $W^{1,p(x)}$*, Czechoslovak Math. J., 41 (1991), 592-618.

[8] Micheletti, A.M., Pistoia, A., *Multiplicity results for a fourth-order semilinear elliptic problem*, Nonlinear Anal., 31 (7) (1998), 895-908.

[9] Mihaillescu, M., Radulescu, V., *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. R. Soc. Lond. Ser. A, 462 (2006), 2625-2641.

[10] Musielak, J., *Orlicz spaces and modular spaces*, Lecture notes in mathematics, Springer, Berlin, vol. 1034, 1983.

[11] Rajagopal, K.R., Ruzicka, M.,

Mathematical modeling of electrorheological fluids, Mech. Thermodyn., 13 (2001), 59-78.

[12] Ruzicka, M., *Electrorheological fluids modeling and mathematical theory*, Springer-Verlag, Berlin, 2002.

[13] Struwe, M., *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems*, Spinger, Heidelberg, 1996.

[14] Zhikov, V., *Averaging of functional of the calculus of variations elasticity theory*, Math. USSR. IZV, 29 (1987), 33-66.

Accepted: 21.01.2016