NUMERICAL TREATMENT OF NEUTRAL FRACTIONAL VOLterra INTEGRO-DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. This paper deals with the numerical solution of fractional integro-differential equations with infinite delay. We applied the continuous spline collocation methods to approximate the solution. A new technique for evaluating the Caputo fractional derivative of the spline polynomials based on the Lagrange coefficients is obtained. Some numerical examples are provided to test the convergence of the method.

Keywords: fractional integro-differential equations, infinite delay, spline collocation method.

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1. Introduction

Consider the fractional integro-differential equation with infinite delay

\begin{equation}
\begin{cases}
D^\alpha_t y(t) = f(t, y(t)) + \int_{-\infty}^{t} k(t, s, y(t), y(s)) ds, & t \in [0, T], \\
y(t) = \phi(t), & t \in (-\infty, 0].
\end{cases}
\end{equation}

It will be assumed that functions f, \phi and k are sufficiently smooth, moreover \( D^\alpha_t \) denotes the fractional differential operator of order \( \alpha \in (0, 1) \) in the sense of Caputo and is defined by

\begin{equation}
D^\alpha_t u(t) = \frac{1}{\Gamma(1 - \alpha)} \int_{0}^{t} \frac{y'(s)}{(t - s)^\alpha} ds.
\end{equation}

For more properties of the operators \( D^\alpha_t \), one can see [16].

In this paper, we study the numerical solution of an equivalent form of equation (1.1) which is given by the initial value problem

\begin{equation}
D^\alpha_t y(t) = f(t, y(t)) + (\psi y)(t) + \int_{0}^{t} k(t, s, y(t), y(s)) ds, \quad y(0) = \phi(0), \quad t \in [0, T],
\end{equation}
where

\[(\psi y)(t) = \int_{-\infty}^{0} k(t, s, y(t), \phi(s))ds, \quad t \in [0, T].\]

The existence and uniqueness of equation (1.3) have received reasonable attention in the last few years (cf. [3] and [5]).

Recently, it turns out that fractional derivatives and integrals can be valuable tools in the modelling of many phenomena in applied sciences, therefore they have been investigated by many researchers (cf, e.g., [1], [2], [4], [9] and references therein). In particular, fractional integro-differential equations appear in electromagnetic, acoustics, viscoelasticity, heat conduction in materials with memory, fluid dynamics, biological models, chemical kinetics and many other phenomena (cf, e.g., [12], [13], [14], [17] and references therein).

Numerical solution of integro-differential equations with infinite delay has been attracted by few mathematicians in the literature. Brunner [7] and [8] applied the collocation method to solve integro-differential with finite and infinite delays. In his papers he mentioned two important applications; the Volterra’s population equation and integro-differential equations of polymer rheology. Jaradat et al. [11] used the homotopy analysis method to solve the population growth model. In this paper we extend the spline collocation method to fractional order integro-differential with infinite delay. To our best of knowledge, equation (1.1) has received little attention in the literature (cf, e.g., [15] and references therein).

This paper is organized as follows: In Section 2, we define the continuous spline space. In Section 3, a system of equations based on the collocation method is derived. Numerical examples are given in Section 4.

2. Polynomial spline space

For \(N \in \mathbb{N}\), let

\[(2.1) \quad Z_N = \{t_0, t_1, \ldots, t_N : 0 = t_0 < t_1 < \ldots < t_N = T\}\]

be a partition of the interval \([0, T]\), given by the grid points

\[(2.2) \quad t_n = nh, \text{ with } h = \frac{T}{N} \text{ and } n = 0, 1, \ldots, N.\]

Let

\[\sigma_n = [t_n, t_{n+1}], \quad (n = 0, \ldots, N - 1).\]

For given integer \(m \geq 1\), let \(S_m^{(0)}(Z_N)\) be the spline space of continuous polynomials on the grid (2.1) and (2.2)

\[S_m^{(0)}(Z_N) = \{u(t) : u(t) \big|_{t \in \sigma_n} = u_n(t) \in \pi_m \text{ on } \sigma_n(n = 0, \ldots, N - 1)\}\]

with

\[u_{n-1}(t_n) = u_n(t_n) \quad t_n \in Z_N - \{0, T\},\]
where \( \pi_m \) denotes the set of all real polynomials of degree not exceeding \( m \). The dimension of \( S_m(0)(Z_N) \) is given by

\[
\dim S_m(0)(Z_N) = mN + 1.
\]

### 3. Derivation of the collocation method

In every subinterval \( \sigma_n = [t_n, t_{n+1}] \), \( (n = 0, ..., N-1) \), we introduce \( m \) interpolation points (called collocation points) \( t_{n,1} < ... < t_{n,m} \), with

\[
t_{n,j} = t_n + c_j h : j = 1, ..., m; n = 0, ..., N - 1,
\]

where \( c_1, ..., c_m \) do not depend on \( n \) and \( N \) and satisfy

\[
0 \leq c_1 < ... < c_m \leq 1.
\]

Let

\[
X(N) = \bigcup_{n=0}^{N-1} X_n \quad \text{with} \quad X_n = \{t_{n,j} = t_n + c_j h : j = 1, ..., m \} \subset \sigma_n.
\]

The exact solution \( y \) of (1.3) will be approximated on \( I \) by an element \( u \in S_m(0)(Z_N) \) (called the collocation solution) satisfying on the set \( X(N) \)

\[
(3.1) \quad D_\alpha^u(t) = f(t, u(t)) + (\psi u)(t) + \int_0^t k(t, s, u(s), u(t)) ds, \ u(0) = \phi(0), \ t \in X(N),
\]

where

\[
(\psi u)(t) = \int_{-\infty}^0 k(t, s, u(s), \phi(s)) ds, \quad t \in X(N).
\]

Now, to evaluate the fractional derivative of the spline function \( D_\alpha^u \), we need the following lemma.

**Lemma 3.1.** Let \( \alpha \in (0, 1) \), then

\[
D_\alpha^u(t_{n,j}) = \sum_{i=0}^{n-1} \sum_{k=1}^m D_{i,k} Y_{i,k} + \sum_{k=1}^m D_{n,k}(c_j) Y_{n,k}
\]

where

\[
Y_{n,k} = u'_n(t_{n,k}),
\]

\[
D_{i,k} = \frac{h}{\Gamma(1-\alpha)} \int_0^1 \frac{L_k(v)}{(t_{n,j} - t_i - vh)^\alpha} dv,
\]

\[
D_{n,k}(c_j) = \frac{h}{\Gamma(1-\alpha)} \int_0^{c_j} \frac{L_k(v)}{(t_{n,j} - t_n - vh)^\alpha} dv,
\]

\[
L_k(v) = \prod_{l=1,l \neq k}^m (v - c_l)/(c_k - c_l).
\]
**Proof.** On the interval \( \sigma_n \), \( u'_n \) is a polynomial of degree \( m - 1 \), thus it can be written in the form

\[
(3.2) \quad u'_n(t_n + vh) = \sum_{k=1}^{m} L_k(v) Y_{n,k}
\]

where \( Y_{n,k} = u'_n(t_{n,k}) \) and \( L_k(v) = \prod_{i=1, i \neq k}^{m} (v - c_i)/(c_k - c_i) \).

If we substitute \( s = t_i + vh \) in formula (1.2), we obtain

\[
D^2_s u(t_{n,j}) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t_{n,i}} \frac{u'(s)}{(t_{n,j} - s)^\alpha} ds
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \left( \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{u'_i(s)}{(t_{n,j} - s)^\alpha} ds + \int_{t_n}^{t_{n,j}} \frac{u'_n(s)}{(t_{n,j} - s)^\alpha} ds \right)
\]

\[
= \frac{h}{\Gamma(1-\alpha)} \left( \sum_{i=0}^{n-1} \sum_{k=1}^{m} \left( \int_{0}^{c_j} \frac{L_k(v)}{(t_{n,j} - t_i - vh)^\alpha} dv \right) Y_{i,k} + \sum_{k=1}^{m} \left( \int_{0}^{c_j} \frac{L_k(v)}{(t_{n,j} - t_n - vh)^\alpha} dv \right) Y_{n,k} \right)
\]

\[
= \sum_{i=0}^{n-1} \sum_{k=1}^{m} D_{i,k} Y_{i,k} + \sum_{k=1}^{m} D_{n,k}(c_j) Y_{n,k}.
\]

Note that Lemma 3.1 provides a new technique for finding \( D^\alpha_s u \), for other techniques one can see [6] and [10].

Now, from equation (3.2), we get

\[
(3.3) \quad u_n(t_n + vh) = y_n + h \sum_{k=1}^{m} a_k(v) Y_{n,k}
\]

where \( y_n = u_n(t_n) = u_{n-1}(t_n), \) \( y_0 = y(0) = \phi(0), \) and \( a_k(v) = \int_{0}^{v} L_k(z) dz \).

Applying these representation in the collocation equation (3.1) and using Lemma 3.1, we obtain the following nonlinear system

\[
\sum_{i=0}^{n-1} \sum_{k=1}^{m} D_{i,k} Y_{i,k} + \sum_{k=1}^{m} D_{n,k}(c_j) Y_{n,k} = f(t_{n,j}, W_{n,j}) + (\psi u)(t_{n,j})
\]

\[
+ h \sum_{i=0}^{n-1} \int_{0}^{c_j} k(t_{n,j}, t_i + vh, W_{n,j}, y_i + h \sum_{k=1}^{m} a_k(v) Y_{i,k}) dv
\]

\[
+ h \int_{0}^{c_j} k(t_{n,j}, t_n + vh, W_{n,j}, y_n + h \sum_{k=1}^{m} a_k(v) Y_{n,k}) dv,
\]

\[
(3.4)
\]
where

\[ W_{n,j} = u_n(t_{n,j}) = y_n(t_n) + h \sum_{k=1}^{m} a_k(c_j)Y_{n,k} \]

and

\[ (\psi u)(t_{n,j}) = \int_{-\infty}^{t_{n,j}} k(t_{n,j}, s, W_{n,j}, \phi(s))ds. \]

For each \( n = 0, 1, \ldots, N-1 \), (3.4) represents a nonlinear system with the variables

\[ Y_n = \begin{pmatrix} Y_{n,1} \\ \vdots \\ Y_{n,m} \end{pmatrix}, \]

where \( k, j = 1, 2, \ldots, m \). Once the vectors \( Y_n \) are known, the collocation solution \( u \in S_m^{(0)}(Z_n) \) that is given by (3.3) is completely determined.

4. Numerical illustration

We solve the nonlinear system (3.4) in the space \( S_2^{(0)}(Z_N) \) for a couple of examples:

1) Consider the following fractional integro-differential equation with infinite delay

\[ \begin{cases} D_{\ast}^{0.25} y(t) = f(t, y(t)) + \int_{-\infty}^{t} e^{4t+s} y(s)ds, & t \in [0, 1], \\ y(t) = t^t, & t \in (-\infty, 0], \end{cases} \]

where \( f \) have been chosen in such a way that the exact solution of (4.1) is \( y(t) = t^3 \).

2) Consider the following fractional integro-differential equation with infinite delay

\[ \begin{cases} D_{\ast}^{0.5} y(t) = f(t, y(t)) + \int_{-\infty}^{t} (t+s)y(s)ds, & t \in [0, 1], \\ y(t) = e^t, & t \in (-\infty, 0], \end{cases} \]

where

\[ f(t, y(t)) = \sqrt{2} e^{2t} \text{erf} (\sqrt{2t}) - te^{2t} - \frac{4te^{2t} - e^{2t} - 2t - 3}{4} - te^{-2t} y(t), \]

and the exact solution of (4.2) is \( y(t) = e^{2t} \). Here \( \text{erf} \) is the error function, and is defined by

\[ \text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_{0}^{t} e^{-\tau^2} d\tau. \]

In both examples we choose the collocation parameters to be the Gauss points \( (c_1 = \frac{\sqrt{3}+1}{2\sqrt{3}}, \ c_2 = \frac{\sqrt{3}-1}{2\sqrt{3}}) \) and the Radau II points \( (c_1 = \frac{1}{3} \text{ and } c_2 = 1) \).

The absolute error \( |y(t_n) - u(t_n)| \) where \( u \in S_2^{(0)}(Z_N) \) at certain values of \( t \in [0, 1] \) is listed in Table 4.1 and Table 4.2.
It is clear that in both examples, the spline collocation method is convergent to the exact solution and the method is much better if we choose the collocation parameters to be the Gauss points. This is because the \( m \)-point interpolatory quadrature formula has the highest degree of precision \( 2m - 1 \) on the interval \([0, 1]\).

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References


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