NUMERICAL TREATMENT OF NEUTRAL FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

E.A. Rawashdeh

Department of Mathematics
Yarmouk University
Irbid
Jordan
e-mail: edris@yu.edu.jo

Abstract. This paper deals with the numerical solution of fractional integro-differential equations with infinite delay. We applied the continuous spline collocation methods to approximate the solution. A new technique for evaluating the Caputo fractional derivative of the spline polynomials based on the Lagrange coefficients is obtained. Some numerical examples are provided to test the convergence of the method.

Keywords: fractional integro-differential equations, infinite delay, spline collocation method.
2000 AMS Subject Classification: 26A33, 45J05, 65M70.

1. Introduction

Consider the fractional integro-differential equation with infinite delay

$$(1.1)\qquad \begin{cases} D_*^\alpha y(t) = f(t, y(t)) + \int_{-\infty}^t k(t, s, y(t), y(s))ds, & t \in [0, T], \\ y(t) = \phi(t), & t \in (-\infty, 0]. \end{cases}$$

It will be assumed that functions f, ϕ and k are sufficiently smooth, moreover D_*^α denotes the fractional differential operator of order $\alpha \in (0, 1)$ in the sense of Caputo and is defined by

$$(1.2)\qquad D_*^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^\alpha} ds.$$

For more properties of the operators D_*^α, one can see [16].

In this paper, we study the numerical solution of an equivalent form of equation (1.1) which is given by the initial value problem

$$(1.3)\quad D_*^\alpha y(t) = f(t, y(t)) + (\psi y)(t) + \int_0^t k(t, s, y(t), y(s))ds,\ y(0) = \phi(0),\ t \in [0, T],$$

where

$$(\psi y)(t) = \int_{-\infty}^{0} k(t, s, y(t), \phi(s))ds, \quad t \in [0, T].$$

The existence and uniqueness of equation (1.3) have received reasonable attention in the last few years (cf. [3] and [5]).

Recently, it turns out that fractional derivatives and integrals can be a valuable tools in the modelling of many phenomena in applied sciences, therefore they have been investigated by many researchers (cf, e.g., [1], [2], [4], [9] and references therein). In particular, fractional integro-differential equations appear in electromagnetic, acoustics, viscoelasticity, heat conduction in materials with memory, fluid dynamics, biological models, chemical kineticsand and many other phenomena (cf, e.g., [12],[13], [14], [17] and references therein).

Numerical solution of integro-differential equations with infinite delay has been attracted by few mathematicians in the literature. Brunner [7] and [8] applied the collocation method to solve integro-differential with finite and infinite delays. In his papers he mentioned two important applications; the Volterra's population equation and integro-differential equations of polymer rheology. Jaradat et al. [11] used the homotopy analysis method to solve the population growth model. In this paper we extend the spline collocation method to fractional order integro-differential with infinite delay. To our best of knowledge, equation (1.1) has received little attention in the literature (cf, e.g., [15] and references therein).

This paper is organized as follows: In Section 2, we define the continuous spline space. In Section 3, a system of equations based on the collocation method is derived. Numerical examples are given in Section 4.

2. Polynomial spline space

For $N \in \mathbb{N}$, let

$$Z_N = \{t_0, t_1, \ldots, t_N : \ 0 = t_0 < t_1 < \ldots < t_N = T\} \tag{2.1}$$

be a partition of the interval $[0, T]$, given by the grid points

$$t_n = nh, \text{ with } h = \frac{T}{N} \text{ and } n = 0, 1, \ldots, N. \tag{2.2}$$

Let

$$\sigma_n = [t_n, t_{n+1}], \ (n = 0, ..., N-1).$$

For given integer $m \geq 1$, let $S_m^{(0)}(Z_N)$ be the spline space of continuous polynomials on the grid (2.1) and (2.2)

$$S_m^{(0)}(Z_N) = \{u(t) : u(t)\Big|_{t \in \sigma_n} = u_n(t) \in \pi_m \text{ on } \sigma_n (n = 0, ..., N-1)\}$$

with

$$u_{n-1}(t_n) = u_n(t_n) \quad t_n \in Z_N - \{0, T\},$$

where π_m denotes the set of all real polynomials of degree not exceeding m. The dimension of $S_m^{(0)}(Z_N)$ is given by

$$\dim\, S_m^{(0)}(Z_N) = mN + 1.$$

3. Derivation of the collocation method

In every subinterval $\sigma_n = [t_n, t_{n+1}]$, $(n = 0, ..., N-1)$, we introduce m interpolation points (called collocation points) $t_{n,1} < \ldots < t_{n,m,}$ with

$$t_{n,j} = t_n + c_j h : j = 1, ..., m; n = 0, ..., N-1,$$

where $c_{1,} \ldots, c_m$ do not depend on n and N and satisfy

$$0 \leq c_1 < ... < c_m \leq 1.$$

Let

$$X(N) = \bigcup_{n=0}^{N-1} X_n \text{ with } X_n = \{t_{n,j} = t_n + c_j h : j = 1, ..., m\} \subset \sigma_n.$$

The exact solution y of (1.3) will be approximated on I by an element $u \in S_m^{(0)}(Z_N)$ (called the collocation solution) satisfying on the set $X(N)$

$$D_*^\alpha u(t) = f(t, u(t)) + (\psi u)(t) + \int_0^t k(t, s, u(t), u(s))ds,\ u(0) = \phi(0),\ t \in X(N), \tag{3.1}$$

where

$$(\psi u)(t) = \int_{-\infty}^0 k(t, s, u(t), \phi(s))ds, \quad t \in X(N).$$

Now, to evaluate the fractional derivative of the spline function $D_*^\alpha u$, we need the following lemma.

Lemma 3.1. *Let $\alpha \in (0, 1)$, then*

$$D_*^\alpha u(t_{n,j}) = \sum_{i=0}^{n-1} \sum_{k=1}^{m} D_{i,k} Y_{i,k} + \sum_{k=1}^{m} D_{n,k}(c_j) Y_{n,k}$$

where

$$\begin{aligned} Y_{n,k} &= u_n'(t_{n,k}), \\ D_{i,k} &= \frac{h}{\Gamma(1-\alpha)} \int_0^1 \frac{L_k(v)}{(t_{n,j} - t_i - vh)^\alpha} dv, \\ D_{n,k}(c_j) &= \frac{h}{\Gamma(1-\alpha)} \int_0^{c_j} \frac{L_k(v)}{(t_{n,j} - t_n - vh)^\alpha} dv, \\ L_k(v) &= \prod_{l=1, l\neq k}^{m} (v - c_l)/(c_k - c_l). \end{aligned}$$

Proof. On the interval σ_n, u_n' is a polynomial of degree $m-1$, thus it can be written in the form

$$u_n'(t_n+vh)=\sum_{k=1}^{m}L_k(v)Y_{n,k} \tag{3.2}$$

where $Y_{n,k}=u_n'(t_{n,k})$ and $L_k(v)=\prod_{l=1,l\neq k}^{m}(v-c_l)/(c_k-c_l)$.

If we substitute $s=t_i+vh$ in formula (1.2), we obtain

$$\begin{aligned}
D_*^\alpha u(t_{n,j}) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{n,i}}\frac{u'(s)}{(t_{n,j}-s)^\alpha}ds\\
&= \frac{1}{\Gamma(1-\alpha)}\Big(\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{u_i'(s)}{(t_{n,j}-s)^\alpha}ds+\int_{t_n}^{t_{n,j}}\frac{u_n'(s)}{(t_{n,j}-s)^\alpha}ds\Big)\\
&= \frac{h}{\Gamma(1-\alpha)}\Big(\sum_{i=0}^{n-1}\int_0^1\frac{\sum_{k=1}^m L_k(v)Y_{i,k}}{(t_{n,j}-t_i-vh)^\alpha}dv+\int_0^{c_j}\frac{\sum_{k=1}^m L_k(v)Y_{n,k}}{(t_{n,j}-t_n-vh)^\alpha}dv\Big)\\
&= \frac{h}{\Gamma(1-\alpha)}\Big(\sum_{i=0}^{n-1}\sum_{k=1}^{m}\big(\int_0^1\frac{L_k(v)}{(t_{n,j}-t_i-vh)^\alpha}dv\big)Y_{i,k}\\
&+ \sum_{k=1}^{m}\big(\int_0^{c_j}\frac{L_k(v)}{(t_{n,j}-t_n-vh)^\alpha}dv\big)Y_{n,k}\Big)\\
&= \sum_{i=0}^{n-1}\sum_{k=1}^{m}D_{i,k}Y_{i,k}+\sum_{k=1}^{m}D_{n,k}(c_j)Y_{n,k}.
\end{aligned}$$

■

Note that Lemma 3.1 provides a new technique for finding $D_*^\alpha u$, for other techniques one can see [6] and [10].

Now, from equation (3.2), we get

$$u_n(t_n+vh)=y_n+h\sum_{k=1}^{m}a_k(v)Y_{n,k} \tag{3.3}$$

where $y_n=u_n(t_n)=u_{n-1}(t_n)$, $y_0=y(0)=\phi(0)$, and $a_k(v)=\int_0^v L_k(z)dz$.

Applying these representation in the collocation equation (3.1) and using Lemma 3.1, we obtain the following nonlinear system

$$\begin{aligned}
\sum_{i=0}^{n-1}\sum_{k=1}^{m}D_{i,k}Y_{i,k}+\sum_{k=1}^{m}D_{n,k}(c_j)Y_{n,k} &= f(t_{n,j},W_{n,j})+(\psi u)(t_{n,j})\\
&+h\sum_{i=0}^{n-1}\int_0^1 k\Big(t_{n,j},t_i+vh,W_{n,j},y_i+h\sum_{k=1}^{m}a_k(v)Y_{i,k}\Big)dv\\
&+h\int_0^{c_j}k\Big(t_{n,j},t_n+vh,W_{n,j},y_n+h\sum_{k=1}^{m}a_k(v)Y_{n,k}\Big)dv,
\end{aligned} \tag{3.4}$$

where

$$W_{n,j} = u_n(t_{n,j}) = y_n(t_n) + h\sum_{k=1}^{m} a_k(c_j)Y_{n,k}$$

and

$$(\psi u)(t_{n,j}) = \int_{-\infty}^{0} k(t_{n,j}, s, W_{n,j}, \phi(s))ds.$$

For each $n = 0, 1, \ldots, N-1$, (3.4) represents a nonlinear system with the variables

$$Y_n = \begin{pmatrix} Y_{n,1} \\ \vdots \\ Y_{n,m} \end{pmatrix},$$

where $k, j = 1, 2, \ldots, m$. Once the vectors Y_n are known, the collocation solution $u \in S_m^{(0)}(Z_n)$ that is given by (3.3) is completely determined.

4. Numerical illustration

We solve the nonlinear system (3.4) in the space $S_2^{(0)}(Z_N)$ for a couple of examples:

1) Consider the following fractional integro-differential equation with infinite delay

$$\begin{cases} D_*^{0.25}y(t) = f(t, y(t)) + \int_{-\infty}^{t} e^{4t}sy(s)ds, & t \in [0, 1], \\ y(t) = \frac{t}{t^4+2}, & t \in (-\infty, 0], \end{cases} \tag{4.1}$$

where f have been chosen in such a way that the exact solution of (4.1) is $y(t) = t^3$.

2) Consider the following fractional integro-differential equation with infinite delay

$$\begin{cases} D_*^{0.5}y(t) = f(t, y(t)) + \int_{-\infty}^{t}(t+s)y(s)ds, & t \in [0, 1], \\ y(t) = e^t, & t \in (-\infty, 0], \end{cases} \tag{4.2}$$

where

$$f(t, y(t)) = \sqrt{2}e^{2t}erf(\sqrt{2t}) - te^{2t} - \frac{4te^{2t} - e^{2t} - 2t - 3}{4} - te^{-2t}y(t)$$

and the exact solution of (4.2) is $y(t) = e^{2t}$. Here erf is the error function, and is defined by

$$erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\tau^2}d\tau.$$

In both examples we choose the collocation parameters to be the Gauss points ($c_1 = \frac{\sqrt{3}+1}{2\sqrt{3}}$ $c_2 = \frac{\sqrt{3}-1}{2\sqrt{3}}$) and the Radau II points ($c_1 = \frac{1}{3}$ and $c_2 = 1$).

The absolute error $|y(t_n) - u(t_n)|$ where $u \in S_2^{(0)}(Z_N)$ at certain values of $t \in [0, 1]$ is listed in Table 4.1 and Table 4.2.

Table 4.1 for Example 4.1

t_n	N	Gauss points $\lvert y(t_n) - u(t_n)\rvert$	Radau II points $\lvert y(t_n) - u(t_n)\rvert$
0.1	10	0.881×10^{-4}	0.577×10^{-4}
	20	0.154×10^{-4}	0.939×10^{-4}
	40	0.224×10^{-5}	0.149×10^{-5}
0.5	10	0.117×10^{-3}	0.326×10^{-3}
	20	0.191×10^{-4}	0.541×10^{-4}
	40	0.290×10^{-5}	0.873×10^{-5}
1	10	0.71×10^{-2}	6.02
	20	0.399×10^{-2}	0.876
	40	0.103×10^{-2}	0.131

Table 4.2 for Example 4.2

t_n	N	Gauss points $\lvert y(t_n) - u(t_n)\rvert$	Radau II points $\lvert y(t_n) - u(t_n)\rvert$
0.1	10	0.101×10^{-3}	0.998×10^{-4}
	20	0.154×10^{-4}	0.171×10^{-4}
	40	0.207×10^{-5}	0.293×10^{-5}
0.5	10	0.279×10^{-3}	0.378×10^{-3}
	20	0.305×10^{-4}	0.646×10^{-4}
	40	0.497×10^{-5}	0.110×10^{-5}
1	10	0.757×10^{-3}	0.137×10^{-2}
	20	0.104×10^{-3}	0.238×10^{-3}
	40	0.139×10^{-4}	0.410×10^{-4}

It is clear that in both examples, the spline collocation method is convergent to the exact solution and the method is much better if we choose the collocation parameters to be the Gauss points. This is because the m-point interpolatory quadrature formula has the highest degree of precision $2m-1$ on the interval $[0, 1]$.

Acknowledgement. This paper was supported by the Scientific Research and Graduate Studies at Yarmouk University.

References

[1] Agarwal, R.P., de Andrade, B., Cuevas, C., *On type of periodicity and ergodicity to a class of fractional order differential equations*, Advances in Difference Equations, 2010 (2010) Article ID 179750.

[2] Ahmed, H.M., *Boundary controllability of nonlinear fractional integro-differential systems*, Advances in Difference Equations, 2010 (2010) Article ID 279493.

[3] Aissani, K., Benchohra, M., *Semlinear Fractional order Integro-Differential Equations with Infinite Delay in Banach Spaces*, Archivum Mathematicum, 49 (2013), 105-117.

[4] Alsaedi, A., Ahmad, B., *Existence of solutions for nonlinear fractional integro-differential equations with three-point non local fractional boundary conditions*, Advances in Difference Equations, 2010 (2010) Article ID 691721.

[5] Benchohra, M., Hendersonb, J., Ntouyasc, S.K., Ouahaba, A., *Existence results for fractional order functional differential equations with infinite delay*, Journal of Mathematical Analysis and Applications, 338 (2) (2008), 1340-1350.

[6] Blank, L., *Numerical treatment of differential equation of fractional order*, Numerical Analysis Report 287. Manchester Center for Numerical Computational Mathematics, 1996.

[7] Brunner, H., *The numerical solution of neutral Volterra integro-differential equations with delay arguments*, Ann. Numer. Math., 1 (1994), 309-322.

[8] Brunner, H., *Collocation Methods for Nonlinear Volterra Integro-Differential Equations with Infinite Delay*, Math., 35 (488) (1989), 571-587.

[9] El-Borai, M.M., *Some probability densities and fundamental solutions of fractional evolution equations*, Chaos, Solitons and Fractals, 14 (2) (2002), 433-440.

[10] Hamarsheh, M.H., Rawashdeh, E.A., *A Numerical Method for Solution of Semidifferential Equations,* Matematicki Vesnik, 62 (2010), 117-126.

[11] Jaradat, H.M., Rawashdeh, E.A., Awaedeh, F., *On Volteera's Population Growth Models*, Annals of the University of Craiova, Mathematics and Computer Science Series, 38 (2) (2011), 18-25.

[12] Liang, J., Xiao, T.-J., van Casteren, J., *A note on semilinear abstract functional differential and integro-differential equations with infinite delay*, Applied Mathematics Letters, 17 (4) (2004), 473- 477.

[13] Liang, J., Xiao, T.-J., *Semilinear integro-differential equations with non local initial conditions*, Computers & Mathematics with Applications, 47 (6-7) (2004), 863-875.

[14] Liang, J., Liu, J.H., Xiao, T.-J., *Non local problems for integro-differential equations*, Dynamics of Continuous, Discrete & Impulsive Systems, Series A 15 (6) (2008), 815-824.

[15] Lil, F., Zhang, J., *Existence of Mild Solutions to Fractional Integro-differential Equations of Neutral Type with Infinite Delay*, Advances in Difference Equations, 2011 (2011) Article ID 963463.

[16] Podlubny, I., Fractional differential equations: An introduction to fractional Derivatives, Fractional Differential Equations, Some Methods of their solution and some of their applications, Mathematics in Science and Engineering, vol.198, Academic Press, San Diego, 1999.

[17] Xiao, T.-J., Liang, J., *Blow-up and global existence of solutions to integral equations with infinite delay in Banach spaces*, Nonlinear Analysis: Theory, Methods & Applications, 71 (12) (2009), 1442-1447.

Accepted: 20.01.2016