ON NEARLY $CAP$-EMBEDDED SUBGROUPS OF FINITE GROUPS

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Abstract. We introduce a new subgroup embedding property of a finite group called nearly $CAP$-embedded subgroup. Using this subgroup property, we determine the structure of finite groups with some nearly $CAP$-embedded subgroups of Sylow subgroups. Our results unify and generalize some recent theorems on $p$-nilpotency and supersolvability of finite groups.

Keywords: nearly $CAP$-embedded subgroup, $p$-nilpotency, finite group.

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1. Introduction

In this paper, all groups considered are finite. Let $\pi(G)$ stand for the set of all prime divisors of the order of a group $G$. Let $F$ denote a formation, $U$ the class of supersolvable groups. $H \text{ } \text{Char} \text{ } G$ means that $H$ is a characteristic subgroup of $G$. The other notations and terminology are standard (see[9]).

Let $H$ be a subgroup of $G$, and $A/B$ be a $G$-chief factor. We say that $H$ covers $A/B$ if $HA = HB$; and $H$ avoids $A/B$ if $H \cap A = H \cap B$. $H$ is said to have cover-avoiding property in $G$, in brevity, $H$ is a $CAP$-subgroup of $G$, if $H$ either covers or avoids any $G$-chief factor. In 1962, Gaschütz[5] introduced a certain conjugacy class of subgroups of a solvable group called the pre-Frattini subgroups. These subgroups have cover-avoidance property. Thereafter, many

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authors devoted to find some kind of subgroups of a solvable group having this property, for example, Gillam[6] and Tomkinson[14]. In 1993, Ezquerro[4] considered the converse questions, he gave some characterizations for a group $G$ to be $p$-supersolvable and supersolvable based on the assumption that all maximal subgroups of some subgroups of $G$ are CAP-subgroups. Asaad in [1] obtained further results within the framework of formation theory. As a generalization of CAP-subgroups, Guo and Guo in[7] introduced CAP-embedded subgroups. A subgroup $H$ of $G$ is said to have the CAP-embedded property in $G$ or is called a CAP-embedded subgroup of $G$ if, for each prime $p$ dividing the order of $H$, there exists a CAP subgroup $K$ of $G$ such that a Sylow $p$-subgroup of $H$ is also a Sylow $p$-subgroup of $K$. Moreover, they presented some conditions for a finite group to be $p$-nilpotent and supersolvable under the condition that some subgroups of Sylow subgroup are CAP-embedded.

In recent years, it has been of interest to use some supplemented properties of subgroups to determine the structure of a group. For example, Wang in [15] introduced the concept of c-normal subgroups. A subgroup $H$ of $G$ is c-normal in $G$ if there is a normal subgroup $K_1$ of $G$ such that $G = HK_1$ and $H \cap K_1 \leq H_G = Core_G(H)$. As applications, he gave some criteria for the solvability and supersolvability of groups.

We provide examples in Section 2 to show that CAP-embedded property and c-normality cannot imply from one to the other one. In this paper, we will try an attempt to unify the two concepts and introduce a new subgroup embedding property of a finite group called nearly CAP-embedded subgroup. As applications, we study the influence of nearly CAP-embedded subgroups on the structure of finite groups. We present some sufficient conditions for a group to be $p$-nilpotent, $p$-supersolvable and supersolvable.

## 2. Basic definitions and preliminary results

When we recall the concepts of a c–normal subgroup and a CAP-embedded subgroup, it is easy to see that a normal subgroup $N$ of $G$ is both c-normal and CAP-embedded. The following examples show that c-normal and CAP-embedded are different properties:

**Example 2.1.** Let $G = A_5$, the alternative group of degree 5. Then all Sylow subgroups of $G$ are CAP-embedded subgroups of $G$, but every Sylow subgroup is not a c-normal subgroup of $G$.

**Example 2.2.** Let $A_4$ be the alternative group of degree 4 and $C = \langle c \rangle$ be a cyclic group of order 2. Let $G = C \times A_4$. Then $A_4 = [K_4]C_3$, where $K_4 = \langle a, b \rangle$ is the Klein Four Group with generators $a$ and $b$ of order 2 and $C_3$ is the cyclic group of order 3. Take $H = \langle ac \rangle$ be the cyclic subgroup of order 2 of $G$. Then $G = HA_4$ and $H \cap A_4 = 1$. By definition, $H$ is c-normal in $G$. However, $H$ is not a CAP-embedded subgroup of $G$, if not, then there exists a CAP-subgroup $B$ of $G$ such that $H \in Syl_2(B)$, so $B$ covers or avoids $(C \times K_4)/C$, it is impossible.
In the c-normal case, $G = HK_1$, if we let $K_2 = H GK_1$, then $G = HK_2$ and $H \cap K_2 = HK$; $H \cap K_2$ is, of course, a CAP-embedded subgroup of $G$. Based on the observation, we introduce the following:

**Definition 2.3.** A subgroup $H$ of a group $G$ is said to be nearly $CAP$-embedded in $G$ if there are a subnormal subgroup $T$ of $G$ and a $CAP$-embedded subgroup $H_{ce}$ of $G$ contained in $H$ such that $G = HT$ and $H \cap T \leq H_{ce}$.

If $H$ is a $CAP$-embedded subgroup of $G$, taking $T = G$, we get $H$ is a nearly $CAP$-embedded subgroup of $G$. Hence nearly $CAP$-embedded subgroup is a real uniform generalization of a $c$-normal subgroup and a $CAP$-embedded subgroup.

For the sake of convenience, we list here some known results which will be useful in the sequel.

**Lemma 2.4** ([[7, Lemma 1]]). Suppose that $U$ is $CAP$-embedded in a group $G$ and $N \leq G$. Then $UN/N$ is $CAP$-embedded in $G/N$.

**Lemma 2.5.** ([19, Lemma 2.4]) Let $H$ be a normal subgroup of a group $G$ such that $G/H$ is $p$-nilpotent and let $P$ be a Sylow $p$-subgroup of $H$, where $p$ is a prime divisor of $|G|$. If $|P| \leq p^2$ and one of the following conditions holds, then $G$ is $p$-nilpotent:

1. $(|G|, p - 1) = 1$ and $|P| \leq p$;
2. $G$ is $A_4$-free if $p = \min \pi(G)$;
3. $(|G|, p^2 - 1) = 1$.

**Lemma 2.6.** ([20, Theorem 3.1]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and $G$ a group with a normal subgroup $N$ such that $G/N \in \mathcal{F}$. If all Sylow subgroups of $F^*(N)$ are cyclic, then $G \in \mathcal{F}$.

**Lemma 2.7.** ([17, Theorem 3.1]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, $G$ a group with a solvable normal subgroup $H$ such that $G/H \in \mathcal{F}$. If for any maximal subgroup $M$ of $G$, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.

**Lemma 2.8.** Let $U$ be a nearly $CAP$-embedded subgroup and $N$ a normal subgroup of a group $G$. Then

1. If $N \leq U$, then $UN/N$ is nearly $CAP$-embedded in $G/N$.
2. If $(|U|, |N|) = 1$, then $UN/N$ is nearly $CAP$-embedded in $G/N$.

**Proof.** By the hypotheses, there are a subnormal subgroup $T$ of $G$ and a $CAP$-embedded subgroup $U_{ce}$ of $G$ contained in $U$ such that $G = UT$ and $U \cap T \leq U_{ce}$.

1. $G/N = (U/N)(TN/N)$, $TN/N \triangleleft G/N$ by [3, Chap A, Lemma 14.1(b)], and $(U/N) \cap (TN/N) = (U \cap TN)/N = (U \cap T)N/N \leq (U_{ce}N)/N$. By Lemma 2.4, $(U_{ce}N)/N$ is $CAP$-embedded in $G/N$. Hence $U/N$ is nearly $CAP$-embedded in $G/N$.

2. Let $\pi$ be the set of all prime divisors of $|U|$, then $N$ is a normal $\pi'$-subgroup and $U$ is a $\pi$-subgroup of $G$. Since $|G|_{\pi'} = |T|_{\pi'} = |TN|_{\pi'}$, we have that $|T \cap N| =
Let $P_G = N$ be a $P$-embedded subgroup of $G$. Therefore, $G/N = (UN/N)(T/N)$, $T/N < G/N$ by [3, Ch. A, Lemma 14.1(b)], and $(UN/N) \cap T/N = (U \cap T)N/N \leq (U_GN)/N$. By Lemma 2.4, we have $(U_GN)/N$ is CAP-embedded in $G/N$. Hence, $(UN)/N$ is nearly CAP-embedded in $G/N$.

3. Main results and their proofs

**Theorem 3.1.** Let $G$ be a group, $N$ a normal subgroup of $G$ such that $G/N$ is $p$-nilpotent and $P$ a Sylow $p$-subgroup of $N$, where $p \in \pi(G)$ with $|G|, p - 1 = 1$. If all maximal subgroups of $P$ are nearly CAP-embedded in $G$, then $G$ is $p$-nilpotent.

**Proof.** Assume that the result is false. Let $G$ be a minimal counterexample with least $|N| + |G|$.

(1) $G$ has a unique minimal normal subgroup $L$ contained in $N$, $G/L$ is $p$-nilpotent and $L \not\leq \Phi(G)$.

Let $L$ be a minimal normal subgroup of $G$ contained in $N$. Consider the factor group $\overline{G} = G/L$. Clearly $\overline{G}/\overline{N} \cong G/N$ is $p$-nilpotent and $\overline{P} = PL/L$ is a Sylow $p$-subgroup of $\overline{N}$, where $\overline{N} = N/L$. Now let $\overline{P}_1 = P_1 L/L$ be a maximal subgroup of $\overline{P}$. We may assume that $\overline{P}_1$ is a maximal subgroup of $P$. Then $P_1 \cap L = P \cap L$ is a Sylow $p$-subgroup of $L$. By the hypothesis, there are a subnormal subgroup $T$ of $G$ and a CAP-embedded subgroup $(P_1)_{ce}$ contained in $P_1$ of $G$ such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)_{ce}$. Clearly $TL/L < G/L$.

Now we let $\pi(G) = \{p_1, p_2, \cdots , p_n\}$ where $p_1 = p$, and $T_{p_i}$ be a Sylow $p_i$-subgroup of $G$. Then $T_{p_i}$ is also a Sylow $p_i$-subgroup of $G$, hence $T_{p_i} \cap L$ is a Sylow $p_i$-subgroup of $L (i = 2, \cdots , n)$. Write $V = L \cap \langle L \cap T_{p_1}, \cdots , L \cap T_{p_n} \rangle$, then $V \leq T \cap L$. Note that $(|L : P_1 \cap L|, |L : V|) = 1$, $L = (P_1 \cap L)V$, thus $P_1 L \cap TL = (P_1 L \cap T)L = (P_1 V \cap T)L = (P_1 \cap T)VL = (P_1 \cap T)L$. It follows from Lemma 2.4 that $(P_1 L/L) \cap (TL/L) = (P_1 \cap T)L/L \leq (P_1)_{ce}L/L$ and $(P_1)_{ce}L/L$ is CAP-embedded in $G/L$. Therefore $\overline{P}_1$ is nearly CAP-embedded in $\overline{G}$. The choice of $G$ implies that $\overline{G}$ is $p$-nilpotent. Since the class of $p$-nilpotent groups is a saturated formation, $L$ is a unique minimal normal subgroup of $G$ contained in $N$ and $L \not\leq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

If $E = O_{p'}(G) \neq 1$, we consider $\overline{G} = G/E$. Clearly, $\overline{G}/\overline{N} \cong G/NE$ is $p$-nilpotent because $G/N$ is, where $\overline{N} = NE/E$. Let $\overline{P}_1 = P_1 E/E$ be a maximal subgroup of $\overline{P}$. We may assume that $P_1$ is a maximal subgroup of $P$. Since $P_1$ is nearly CAP-embedded in $G$, $P_1 E/E$ is nearly CAP-embedded in $G/E$ by Lemma 2.8 (2). The minimality of $G$ yields that $G$ is $p$-nilpotent, therefore $G$ is $p$-nilpotent, a contradiction.

(3) $O_p(N) = 1$ and so $L$ is not $p$-nilpotent.

If not, then by (1), $L \leq O_p(N)$ and, there is a maximal subgroup $M$ of $G$ such that $G = LM$ and $L \cap M = 1$. Since $M_p < P$, where $M_p \in Syl_p(M)$, we may let $P_1$ be a maximal subgroup of $P$ containing $M_p$. Because $P_1$ is a nearly CAP-embedded subgroup of $G$, there are a subnormal subgroup $T$ of $G$ and a CAP-embedded subgroup $(P_1)_{ce}$ contained in $P_1$ of $G$ such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where $K$ is a CAP subgroup of $G$. If $K$ covers
Let $T \in \text{P}\text{-embedded in } \text{CAP}$, then $L$ that be the normal $p$-complement of $T$. Then $T'_{p'}$ is a Hall $p'$-subgroup of $G$ and $T'_{p'} \cap \text{Char } T \leq L$, so $T'_{p'} \leq G$, contrary to $O_{p'}(G) = 1$.

If $L$ is $p$-nilpotent, then $L'_{p'} \cap \text{Char } L \leq L$, so $L'_{p'} \leq O_{p'}(N) \leq O_{p'}(G) = 1$ by (2). Thus $L$ is a $p$-group, $L \leq O_{p}(N) = 1$, a contradiction. Hence (3) holds.

(4) The final contradiction.

If $P \cap L \leq \Phi(P)$, then $L$ is $p$-nilpotent by Tate's theorem [9, IV, Th 4.7], contrary to (3). Consequently, there exists a maximal subgroup $P_1$ of $P$ such that $P = (L \cap P)P_1$. Let $T$ be a subnormal supplement of $P_1$ in $G$, we have $P_1 \cap T \leq (P_1)_{ce} \leq \text{Syl}_p(K)$, where $K$ is a $\text{CAP}$ subgroup of $G$. If $K$ covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \leq \text{Syl}_p(K)$ that $P_1 \cap K = (P_1)_{ce} \leq \text{Syl}_p(K)$, then $P_1 \cap L \leq \text{Syl}_p(L)$. Thus $L \cap P = L \cap P_1$. We obtain $P = (L \cap P)P_1 = P_1$, a contradiction. So $K$ must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|P \cap T \cap L| \leq p$. Since $T \cap L \cap T \geq TL/CapL \leq G/L$, $T \cap L \cap T$ is $p$-nilpotent. It follows that $T$ is $p$-nilpotent by Lemma 2.5. Let $T'_{p'}$ be the normal $p$-complement of $T$. Then $T'_{p'}$ is a Hall $p'$-subgroup of $G$ and $T'_{p'} \cap \text{Char } T \leq L$, so $T'_{p'} \leq G$, contrary to $O_{p'}(G) = 1$. This contradiction completes the proof.

**Theorem 3.2.** Let $p$ be a prime dividing the order of the group $G$ and let $N$ be a $p$-solvable normal subgroup of $G$ such that $G/N$ is $p$-supersolvable. If there exists a Sylow $p$-subgroup $P$ of $N$ such that every maximal subgroup of $P$ is nearly $\text{CAP}$-embedded in $G$, then $G$ is $p$-supersolvable.

**Proof.** Assume that the result is false and let $G$ be a counterexample of minimal order. Now, arguing as in the proof of Theorem 3.1, the following statements (1) and (2) about $G$ are true.

(1) $G$ has a unique minimal normal subgroup $L$ contained in $N$, $G/L$ is $p$-supersolvable and $L \nleq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

Since $G$ is $p$-solvable and $O_{p'}(G) = 1$, $L$ is a $p$-group and $L \leq P$. If $L \leq \Phi(P)$, by [12, Theorem 5.2.13], $L \leq \Phi(G)$, a contradiction. Consequently, there exists a maximal subgroup $P_1$ of $P$ such that $P_1L = P$. Since $P_1$ is a nearly $\text{CAP}$-embedded subgroup of $G$, there are a subnormal $T$ of $G$ and a $\text{CAP}$-embedded subgroup $(P_1)_{ce}$ of $G$ such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \leq \text{Syl}_p(K)$, where $K$ is a $\text{CAP}$ subgroup of $G$. If $K$ covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \leq \text{Syl}_p(K)$ that $L \leq P_1$, thus $P = LP_1 = P_1$, a contradiction. So $K$ must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq p$. Noting that $G/T_G$ is $p$-group, so $N \cap T_G \neq 1$. If not, then $G = G/N \cap T_G \leq G/N \times G/T_G$ is $p$-supersolvable, a contradiction. So $L \leq N \cap T_G$ by (1). Hence $|L| = |T \cap L| = p$. The $p$-supersolvability of $G/L$ implies that $G$ is $p$-supersolvable, final contradiction.
Remark 3.3. The hypothesis that $N$ is $p$-solvable in Theorem 3.2 is essential. For example, if we let $G$ be the alternating group $A_5$ of degree 5, $N = G$ and $p = 3$, then it is clear that the statement of Theorem 3.2 does not hold.

Theorem 3.4. Let $G$ be a group. Then $G$ is supersolvable if and only if there exists a normal subgroup $N$ such that $G/N$ is supersolvable and all maximal subgroups of any Sylow subgroup of $N$ are nearly CAP-embedded in $G$.

Proof. The necessity part can be obtained if we let $N = G$ and apply a result due to Ezquerro[4]. So we need to prove the sufficiency part.

Let $p$ be the smallest prime divisor of $|G|$. The supersolvability of $G/N$ implies that $G/N$ is $p$-nilpotent. By Theorem 3.1, $G$ is $p$-nilpotent. Furthermore $G$ is solvable. Applying Theorem 3.2, it is easy to see that $G$ is supersolvable. ■

Theorem 3.5. Let $\mathcal{F}$ be a saturated formation containing $U$. Suppose that $G$ is a group with a normal subgroup $N$ such that $G/N \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $N$ are nearly CAP-embedded in $G$, then $G \in \mathcal{F}$.

Proof. Let $G$ be a minimal counterexample. With similar arguments as in the proof of Theorem 3.1, we have the following claim (1).

(1) $G$ has a unique minimal normal subgroup $L$ contained in $N$ such that $G/L \in \mathcal{F}$ and $L \not\in \Phi(G)$.

(2) $L$ is an elementary abelian $p$-group for some prime $p$.

Let $q$ be the smallest prime divisor of $|N|$, $Q$ a Sylow $q$-subgroup of $N$. If $Q \cap L \not\in \Phi(Q)$, then there exists a maximal subgroup $Q_1$ of $Q$ such that $Q = (Q \cap L)Q_1$. By the hypotheses, there are a subnormal $T$ of $G$ and a CAP-embedded subgroup $(Q_1)_{ce}$ of $G$ such that $G = Q_1T$ and $Q_1 \cap T \leq (Q_1)_{ce} \in Syl_q(K)$, where $K$ is a CAP subgroup of $G$. If $K$ covers $L/1$, then $L \leq K$. It follows from $(Q_1)_{ce} \in Syl_q(K)$ that $L \cap Q_1 = L \cap Q$, thus $Q = (Q \cap L)Q_1 = (Q_1 \cap L)Q_1 = Q_1$, a contradiction. So $K$ must avoid $L/1$, i.e., $K \cap L = 1$, hence $Q_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq q$. Noting that $G/T_G$ is $q$-group, so $N \cap T_G \neq 1$. If not, then $G = G/N \cap T_G \leq G/N \times G/T_G$ belongs to $\mathcal{F}$, a contradiction. So $L \leq N \cap T_G$ by (1). Hence $|L| = |T \cap L| = q$. By applying Lemma 2.6, we obtain $G \in \mathcal{F}$, a contradiction. Therefore, $Q \cap L \not\in \Phi(Q)$, then $L$ is $q$-nilpotent by Tate’s theorem [9, IV, Th 4.7] and, by the Odd Order Theorem, $L$ is solvable, statement (2) is true.

(3) A final contradiction.

From (1) and (2), there exists a maximal subgroup $M$ of $G$ such that $G = LM$ and $L \cap M = 1$. Let $P$ be a Sylow $p$-subgroup of $N$. Then $P = LM_p$ where $M_p \in Syl_p(G)$. Since $M_p < P$, we may let $P_1$ be a maximal subgroup of $P$ such that $M_p \leq P_1$. By the hypotheses, there are a subnormal $T$ of $G$ and a CAP-embedded subgroup $(P_1)_{ce}$ of $G$ such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where $K$ is a CAP subgroup of $G$. If $K$ covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $L \leq P_1$, thus $P = LM_p \leq P_1$, a contradiction. So $K$ must avoid $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq p$. Noting that $G/T_G$ is $p$-group, so $N \cap T_G \neq 1$. If not, then $G = G/N \cap T_G \leq G/N \times G/T_G$
belongs to $\mathcal{F}$, a contradiction. So $L \leq N \cap T_G$ by (1). Hence $|L| = |T \cap L| = q$.
By applying Lemma 2.6, we obtain $G \in \mathcal{F}$, final contradiction. We are done.

**Theorem 3.6.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $N$ be a solvable normal subgroup of $G$ such that $G/N \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F(N)$ are nearly CAP-embedded subgroups of $G$, then $G \in \mathcal{F}$.

**Proof.** Assume that the result is false and let $G$ be a counterexample of minimal order. First we have $\Phi(G) = 1$. Suppose that $\Phi(G) \neq 1$ and take a prime $p$ dividing $|\Phi(G)|$. Denote $D = O_p(\Phi(G)) \neq 1$. Clearly $D \trianglelefteq G$. Let $F(ND/D) = L/D$. By $L/D$ Char $ND/D \trianglelefteq G/D$, $L/D \trianglelefteq G/D$. Hence $L \trianglelefteq G$. Since $L/D$ is a normal nilpotent subgroup of $G/D$ and $D \leq \Phi(G)$, applying a result due to Gaschütz[9, III, Theorem 3.5], we have that $L$ is a normal nilpotent subgroup of $ND$. Thus $L \leq F(ND)$. Consequently $F(ND/D) = F(ND)/D = L/D$. By [2, Lemma 3.1], $F(ND/D) = F(N)D/D$. It is clear that $(G/D)/(ND/D) \cong G/ND \cong (G/N)/(ND/N)$ belongs to $\mathcal{F}$. Now, by Lemma 2.8(1), the hypotheses of the theorem hold in $G/D$. By the minimality of $G$, $G/D \notin \mathcal{F}$. Since $\mathcal{F}$ is saturated, $G \notin \mathcal{F}$, a contradiction. We obtain $\Phi(N) \leq \Phi(G) = 1$. Let $M$ be a maximal subgroup of $G$ such that $F(N) \nsubseteq M$. Then there exists a prime $p$ such that $O_p(N) \nsubseteq M$. It follows that $G = O_p(N)M$. Clearly, $O_p(N) \cap M < O_p(N)$, so we may take a maximal subgroup $P_1$ of $O_p(N)$ containing $O_p(N) \cap M$. Then $P_1 \cap M = O_p(N) \cap M \leq G$, therefore $P_1 \cap M \leq (P_1)_G$. If $(P_1)_GM = G$, then $O_p(N) = O_p(N) \cap (P_1)_GM = (P_1)_G(O_p(N) \cap M) = (P_1)_G$, a contradiction. Thus $(P_1)_GM < G$, so $(P_1)_G \leq O_p(N) \cap M$ and $P_1 \cap M = O_p(N) \cap M = (P_1)_G$. Let $O_p(N)/K$ be a chief factor of $G$ with $O_p(N)\cap M \leq K$. Then $O_p(N) \cap M = K \cap M$. If $KM = G$, then $O_p(N) = O_p(N) \cap KM = K(O_p(N) \cap M) = K$, a contradiction. Thus $KM < G$, so $K \leq M$ and $K = O_p(N) \cap M = (P_1)_G$. Since $P_1$ is a nearly CAP-embedded subgroup of $G$, there are a normal $T$ of $G$ and a CAP-embedded subgroup $(P_1)_{ce}$ of $G$ such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(B)$, where $B$ is a CAP subgroup of $G$. Clearly $(P_1)_G(O_p(N) \cap T)$ is normal in $G$. From the fact that $O_p(N)/(P_1)_G$ is a $G$-chief factor, we know that either $(P_1)_G = (P_1)_G(O_p(N) \cap T)$ or $(P_1)_G(O_p(N) \cap T) = O_p(N)$. If the former holds, then $O_p(N) \cap T \leq (P_1)_G$. Furthermore, $O_p(N) \cap T = P_1 \cap T$ and $O_p(N) = P_T = O_p(N)T = G$, a contradiction. So $(P_1)_G(O_p(N) \cap T) = O_p(N)$, we obtain $O_p(N) \leq (P_1)_GT$. Thus $G = P_1T = (P_1)_GT$. Noting that $B$ is a CAP subgroup of $G$. If $B$ covers $O_p(N)/(P_1)_G$, then $O_p(N) \leq B(P_1)_G$. It follows from $(P_1)_{ce} \in Syl_p(B)$ that $O_p(N) \leq P_1$, a contradiction. So $B$ must avoids $O_p(N)/(P_1)_G$, i.e., $(P_1)_{ce} = B \cap O_p(N) = B \cap (P_1)_G$, hence $(P_1)_{ce} \leq (P_1)_G$. Consequently $(P_1)_G \cap T = P_1 \cap T$, we have $P_1 = (P_1)_G = O_p(N) \cap M$. Therefore $|G : M| = |O_p(N) : O_p(N) \cap M| = p$. By Lemma 2.7, we get $G \in \mathcal{F}$, a final contradiction.

**Remark 3.7.** The hypothesis that $N$ is solvable in Theorem 3.6 cannot be removed. For example, if we let $G = SL(2,5)$ and $N = G$, then $F(N)$ is a group of order 2. Thus all maximal subgroups of any Sylow subgroup of $F(N)$ have the nearly CAP-embedded property in $G$, but $G$ is not supersolvable.
4. Some applications

Since many relevant families of subgroups, such as normal subgroups, $c$-normal subgroups, $CAP$ subgroups, $CAP$-embedded subgroups and $c^\sharp$-normal subgroups, enjoy the nearly $CAP$-embedded property, a lot of nice results can be obtained according to our theorems.

Recall first the concept of $c^\sharp$-normal subgroups mentioned above. Let $H$ be a subgroup of $G$. We call $H$ a $c^\sharp$-normal subgroup of $G$ if there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is a $CAP$ subgroup of $G$ (see[16]).

Now, we here list special cases of our theorems which can be found in the literature.

Theorem 3.1 immediately implies:

Corollary 4.1. ([7, Theorem 3.1]) Let $p$ be a prime dividing the order of the group $G$ with $(|G|, p - 1) = 1$ and let $H$ be a normal subgroup of $G$ such that $G/H$ is $p$-nilpotent. If there exists a Sylow $p$-subgroup $P$ of $H$ such that $P$ is cyclic or every maximal subgroup of $P$ is $CAP$-embedded in $G$, then $G$ is $p$-nilpotent.

Proof. If $P$ is a cyclic group, by [9, p. 420, Theorem 2.8], we have $G$ is $p$-nilpotent. So every maximal subgroup of $P$ has the $CAP$-embedded property in $G$. Hence $G$ is $p$-nilpotent by Theorem 3.1.

Corollary 4.2. ([8, Theorem 3.4]) Let $p$ be the smallest prime number dividing the order of a group $G$ and $P$ a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is $c$-normal in $G$, then $G$ is $p$-nilpotent.

Corollary 4.3. ([16, Theorem 3.1]) Let $G$ be a group, $H$ a normal subgroup of $G$ such that $G/H$ is $p$-nilpotent and $P$ a Sylow $p$-subgroup of $H$, where $p$ is a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. If all maximal subgroups of $P$ are $c^\sharp$-normal in $G$, then $G$ is $p$-nilpotent. In particular, $G$ is $p$-supersolvable.

From Theorem 3.2 we obtain:

Corollary 4.4. ([7, Theorem 4.1]) Let $p$ be a prime dividing the order of the group $G$ and let $H$ be a $p$-solvable normal subgroup of $G$ such that $G/H$ is $p$-supersolvable. If there exists a Sylow $p$-subgroup $P$ of $H$ such that every maximal subgroup of $P$ is $CAP$-embedded in $G$, then $G$ is $p$-supersolvable.

Corollary 4.5. ([16, Theorem 3.4]) Let $G$ be a $p$-solvable group, $H$ a normal subgroup of $G$ such that $G/H$ is $p$-supersolvable and $P$ a Sylow $p$-subgroup of $H$, where $p$ is a prime. If all maximal subgroups of $P$ are $c^\sharp$-normal in $G$, then $G$ is $p$-supersolvable.

By Theorem 3.5 we have:

Corollary 4.6. ([13, Theorem 1]) If the maximal subgroups of the Sylow subgroups of $G$ are normal in $G$, then $G$ is supersolvable.

Corollary 4.7. ([11, Theorem 3.5]) Assume that $G/H$ is supersolvable and all maximal subgroups of the Sylow subgroups of $H$ are normal in $G$. Then $G$ is supersolvable.
Corollary 4.8. ([15, Theorem 4.1]) If the maximal subgroups of the Sylow subgroups of $G$ are $c$-normal in $G$, then $G$ is supersoluble.

Corollary 4.9. ([16, Theorem 4.1]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $H$ are $c^\sharp$-normal in $G$, then $G \in \mathcal{F}$.

As immediate corollaries of Theorem 3.6, we have the following:

Corollary 4.10. ([11, Theorem 3.1]) Assume that $G$ is solvable and every maximal subgroup of the Sylow subgroups of $F(G)$ is normal in $G$. Then $G$ is supersolvable.

Corollary 4.11. [7, Theorem 4.3] Let $G$ be a group. Then $G$ is supersolvable if and only if there exists a solvable normal subgroup $H$ such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F(H)$ have the CAP-embedded property in $G$.

Corollary 4.12. ([10, Theorem 2]) Let $G$ be a group and $E$ a soluble normal subgroup of $G$ such that $G/E$ is supersoluble. If all maximal subgroups of the Sylow subgroups of $F(E)$ are $c$-normal in $G$, then $G$ is supersoluble.

Corollary 4.13. [1, Theorem 4.4] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a solvable group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are CAP-subgroups of $G$, then $G \in \mathcal{F}$.

Corollary 4.14. ([18, Theorem 1]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a soluble normal subgroup $H$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are $c$-normal in $G$, then $G \in \mathcal{F}$.

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References


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