A NOTE ON COMPLETENESS OF THE HAUSDORFF FUZZY METRIC SPACES

Changqing Li[1]
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, Fujian 363000
China
e-mail: helen_smile0320@163.com

Yanlan Zhang
College of Computer
Minnan Normal University
Zhangzhou, Fujian 363000
China
e-mail: zyl_1983_2004@163.com

Kexiu Zhang
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, Fujian 363000
China
kxzhang2006@163.com

Abstract. In this paper, completeness and completableness of the Hausdorff fuzzy metric spaces on the family of nonempty finite sets are explored. Also, necessary and sufficient conditions for the Hausdorff fuzzy metric spaces on the family of nonempty compact sets to be complete are found.

Keywords: fuzzy metric; Hausdorff fuzzy metric; complete; completable.
Mathematical Subject Classification: 54A40, 54B20, 54D35, 54E35.

1. Introduction

The concept of fuzzy metric, which has been introduced by many authors from different points of view [2], [3], [10], [12], is important in Fuzzy Topology. In particular, Kramosil and Michalek [12] generalized the concept of probabilistic metric and obtained the concept of fuzzy metric with the help of continuous t-norms in 1975. To make the topology induced by a fuzzy metric to be Hausdorff, George and Veeramani [3] modified the notion given by Kramosil and Michalek and gave

[1]Corresponding author.

a new notion with the help of continuous t-norms. Later, Gregori and Romaguera [7] proved that the topological space induced by the fuzzy metric is metrizable. The new version of fuzzy metric is more restrictive, but it determines the class of spaces that are tightly connected with the class of metrizable topological spaces. So it is interesting to study the new version of fuzzy metric. Many contributions to the study of fuzzy metric spaces can be found in [4]-[6], [9], [11], [13]-[17], [19], [20].

In order to explore the hyperspaces of a fuzzy metric space, Rodríguez-López and Romaguera [18] gave a construction of the Hausdorff fuzzy metric on the set of nonempty compact sets. In this paper, we study completeness and completableness of the Hausdorff fuzzy metric spaces on the family of nonempty finite sets. Moreover, we obtain necessary and sufficient conditions for the Hausdorff fuzzy metric spaces on the family of nonempty compact sets to be complete.

2. Preliminaries

In the section, we recall some concepts. Throughout the paper, the set of all natural numbers will be denoted by $\mathbb{N}$. Our basic reference for general topology is [1].

Definition 2.1.[3] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Clearly, $a * b = a \cdot b$ and $a * b = \min\{a, b\}$ are two common examples of continuous t-norms.

Definition 2.2.[3] A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0, \infty)$:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(v) the function $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will call $(M, *)$ a *fuzzy metric on X*.

Definition 2.3.[3] Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0, 1), t > 0$ and $x \in X$. The set

$$B_M(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}$$

is called the *open ball with center x and radius r with respect to t*.

Clearly, $\{B_M(x,r,t)|x \in X, t > 0, r \in (0,1)\}$ forms a base of a topology in X and the topology is denoted by τ_M. In [3], it was proven that $\{B_M(x, \frac{1}{n}, \frac{1}{n})|n \in \mathbb{N}\}$ is a neighborhood base at x for the topology τ_M for every $x \in X$.

Definition 2.4.[3] Let (X,d) be a metric space. Define $a * b = a \cdot b$ for all $a, b \in [0,1]$, and let M_d be the function on $X \times X \times (0,\infty)$ defined by

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}.$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space and $(M_d, \cdot)$ is called *the standard fuzzy metric induced by d.*

Definition 2.5.[3] Let $(X,M,*)$ be a fuzzy metric space.

(a) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called *Cauchy* if for each $r \in (0,1)$ and $t > 0$, there exists an $N \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq N$.

(b) $(X,M,*)$ is called *complete* if every Cauchy sequence in X is convergent with respect to τ_M.

Definition 2.6.[8] Let $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ be two fuzzy metric spaces.

(a) A mapping $f : X_1 \to X_2$ is called an *isometry* if for every $x, y \in X_1$ and $t > 0, M_1(x,y,t) = M_2(f(x), f(y), t)$.

(b) $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ are called *isometric* if there exists an isometry from X_1 onto X_2.

(c) A *fuzzy metric completion* of $(X_1, M_1, *_1)$ is a complete fuzzy metric space $(X_2, M_2, *_2)$ such that $(X_1, M_1, *_1)$ is isometric to a dense subspace of X_2.

(d) $(X_1, M_1, *_1)$ is said to be *completable* if it admits a fuzzy metric completion.

3. Completeness of the Hausdorff fuzzy metric on Fin(X)

Given a fuzzy metric space $(X,M,*)$, we shall denote by $\mathcal{P}(X)$, Comp(X) and Fin(X), the set of nonempty subsets, the set of nonempty compact subsets and the set of nonempty finite subsets of (X, τ_M), respectively. Let $M(a,B,t) := \sup_{b\in B} M(a,b,t)$, $M(B,a,t) := \sup_{b\in B} M(b,a,t)$ for all $a \in X$, $B \in \mathcal{P}(X)$ and $t > 0$ (see Definition 2.4 of [20]). Note that $M(a,B,t) = M(B,a,t)$. In the following, $|A|$ denotes the cardinality of A, where $A \subset X$.

Definition 3.1.[18] Let $(X,M,*)$ be a fuzzy metric space. For every $A, B \in$ Comp(X) and $t > 0$, define H_M: Comp$(X)\times$ Comp$(X) \times (0,\infty) \to [0,1]$ by

$$H_M(A,B,t) = \min\{\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\}.$$

Then (Comp(X),$H_M, *$) is a fuzzy metric space. $(H_M, *)$ is called *the Hausdorff fuzzy metric on Comp(X).*

Observe that $H_M(\{x\},\{y\},t) = M(x,y,t)$ for all $x,y \in X$ and $t > 0$, we can regard $(X,M,*)$ as a subspace of $(\mathrm{Comp}(X),H_M,*)$.

Lemma 3.2. [18] *Let $(X,M,*)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0,\infty)$.*

Lemma 3.3. [18] *Let $(X,M,*)$ be a fuzzy metric space. Then, for each $a \in X$, $B \in Comp(X)$ and $t > 0$, there exists a $b_a \in B$ such that $M(a,B,t) = M(a,b_a,t)$.*

Lemma 3.4. *In a fuzzy metric space $(X,M,*)$, the following are obtained.*

(1) *If X is a finite set or a set of isolated points, then Fin(X)=Comp(X).*
(2) *If X is a set of infinite non-isolated points, then Fin(X) is neither a closed subset nor an open subset of Comp(X).*

Proof. (1) is obviously satisfied.

(2) Suppose that X is a set of infinite non-isolated points. Then we can find a point $a \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_i \neq x_j$ whenever $i \neq j$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to a. Put $A = \{x_n | n \in \mathbb{N}\} \cup \{a\}$. Then $A \in \mathrm{Comp}(X) \backslash \mathrm{Fin}(X)$. Let $r_1 \in (0,1)$ and $t_1 > 0$. Then there exists an $m \in \mathbb{N}$ such that if $n \geq m$, we get that $x_n \in B_M(a, \frac{r_1}{2}, t_1)$. Since $B = \{x_1, x_2, \cdots, x_{m-1}\} \cup \{a\} \in \mathrm{Fin}(X)$, we have that $H_M(A,B,t_1) \geq 1 - \frac{r_1}{2} > 1 - r_1$. Hence $B \in B_{H_M}(A, r_1, t_1)$. So Fin($X$) is not a closed subset of $(\mathrm{Comp}(X), \tau_{H_M})$. On the other hand, let $r_2 \in (0,1)$ and $t_2 > 0$. Then there exists an $l \in \mathbb{N}$ such that if $n \geq l$, we have that $x_n \in B_M(a, \frac{r_2}{2}, t_2)$. Put $C = \{x_n | n \geq l\}$. Then $H_M(\{a\}, C, t_2) \geq 1 - \frac{r_2}{2} > 1 - r_2$, which implies that $C \in B_{H_M}(\{a\}, r_2, t_2)$. Since $\{a\} \in \mathrm{Fin}(X)$, we deduce that Fin($X$) is not an open subset of $(\mathrm{Comp}(X), \tau_{H_M})$. This completes the proof.

Corollary 3.5. *Let $(X,M,*)$ be a fuzzy metric space. If Fin(X) is either a closed subset or an open subset of Comp(X), then Fin(X)=Comp(X).*

Lemma 3.6. [18] *Let $(X,M,*)$ be a fuzzy metric space. Then $(Comp(X), H_M, *)$ is complete if and only if $(X,M,*)$ is complete.*

Lemma 3.7. *Let $(X,M,*)$ be a complete fuzzy metric space and $A \subseteq X$. Then A is a closed subset of (X,τ_M) if and only if $(A,M,*)$ is complete.*

Proof. Suppose that A is a closed subset of (X,τ_M). Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(A,M,*)$. Then $\{x_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in $(X,M,*)$. Since $(X,M,*)$ is complete, we can find an $x_0 \in X$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0. Hence $x_0 \in A$. Thus, $(A,M,*)$ is complete.

Conversely, suppose that $(A,M,*)$ is complete. If A fails to be a closed subset of (X,τ_M), then there exists an $x \in X \setminus A$ such that $B_n(x, \frac{1}{n}, \frac{1}{n}) \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. Take $x_n \in B_M(x, \frac{1}{n}, \frac{1}{n}) \cap A$ for every $n \in \mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $(A,M,*)$ and $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X. Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(A,M,*)$. So $x \in A$, which is a contradiction. Consequently, A is a closed subset of (X,τ_M).

Theorem 3.8. *Let $(X, M, *)$ be a fuzzy metric space. If (Fin$(X), H_M, *$) is complete, then $(X, M, *)$ is complete.*

Proof. Let $t > 0$ and $A \in \text{Fin}(X)$ with $|A| \geq 2$. Take $a, b \in A$ with $a \neq b$. Put $M(a, b, 2t) = \varepsilon_0$. Then there exists a $\varepsilon_1 \in (\varepsilon_0, 1)$ such that $\varepsilon_1 * \varepsilon_1 > \varepsilon_0$. We claim that

$$B_M(a, 1 - \varepsilon_1, t) \cap B_M(b, 1 - \varepsilon_1, t) = \emptyset.$$

Indeed, if not, we can choose a $c \in B_M(a, 1 - \varepsilon_1, t) \cap B_M(b, 1 - \varepsilon_1, t)$. Hence

$$M(a, b, 2t) \geq M(a, c, t) * M(c, b, t) \geq \varepsilon_1 * \varepsilon_1 > \varepsilon_0 = M(a, b, 2t),$$

which is a contradiction. Let $x \in X$. If $x \in \overline{B_M(a, 1 - \varepsilon_1, t)}$, where $\overline{B_M(a, 1 - \varepsilon_1, t)}$ is the closure of $B_M(a, 1 - \varepsilon_1, t)$, then $x \notin B_M(b, 1 - \varepsilon_1, t)$. Hence $M(b, x, t) \leq \varepsilon_1$. Thus

$$H_M(A, \{x\}, t) \leq \inf_{y \in A} M(y, \{x\}, t) = \inf_{y \in A} M(y, x, t) \leq M(b, x, t) \leq \varepsilon_1.$$

If $x \notin \overline{B_M(a, 1 - \varepsilon_1, t)}$, then $x \notin B_M(a, 1 - \varepsilon_1, t)$, whence $M(a, x, t) \leq \varepsilon_1$. Hence $H_M(A, \{x\}, t) \leq \varepsilon_1$. So

$$\{\{x\} | x \in X\} \cap H_M(A, 1 - \varepsilon_1, t) = \emptyset,$$

which implies that X is a closed subset of (Fin(X),τ_{H_M}). Due to Lemma 3.7, we deduce that $(X, M, *)$ is complete.

The converse of the above theorem is false. We illustrate this with the following example.

Example 3.9. Let $X = \{0, \frac{1}{2}, \cdots, 1 - \frac{1}{n}, \cdots\} \cup \{1\}$ and d be the Euclidian metric of X. Denote $a * b = a \cdot b$ for all $a, b \in [0, 1]$. Define the function M by

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a complete fuzzy metric space. Thanks to Lemma 3.6, we have that (Comp(X),$H_M, *$) is complete. Due to Lemma 3.4 (2), we get that Fin(X) is not a closed subset of (Comp$(X), \tau_{H_M}$). By Lemma 3.7, we conclude that (Fin$(X), H_M, *$) fails to be complete.

Now, according to Lemma 3.4, Lemma 3.6 and Lemma 3.7, it is easy to obtain the following theorem.

Theorem 3.10. *In a complete fuzzy metric space $(X, M, *)$, the following hold:*

(1) *If X is a finite set or an isolated set, then (Fin$(X), H_M, *$) is complete.*

(2) *If X is an infinite non-isolated set, then (Fin$(X), H_M, *$) is not complete.*

Lemma 3.11. [18] *Let $(X, M, *)$ be a fuzzy metric space. Then $(Comp(X), H_M, *)$ is completable if and only if $(X, M, *)$ is completable.*

Theorem 3.12. *Let $(X, M, *)$ be a fuzzy metric space. Then $(Fin(X), H_M, *)$ is completable if and only if $(X, M, *)$ is completable.*

Proof. Assume that $(\mathrm{Fin}(X), H_M, *)$ is completable. Then there exists an isometry $i : (\mathrm{Fin}(X), H_M, *) \to (\widetilde{\mathrm{Fin}(X)}, \widetilde{H_M}, \star)$ such that $i(\mathrm{Fin}(X))$ is dense in $\widetilde{\mathrm{Fin}(X)}$. Observe that for each $x \in X, \{x\} \in \mathrm{Fin}(X)$, the restriction i_X of i is an isometry between $(X, M, *)$ and $(i_X(X), \widetilde{H_M}, \star)$. Let $\overline{i_X(X)}$ be the closure of $i_X(X)$. By Lemma 3.7, we have that $(\overline{i_X(X)}, \widetilde{H_M}, \star)$ is a complete fuzzy metric space that has $i_X(X)$ as a dense subspace. Thus $(X, M, *)$ is completable.

Conversely, assume that $(X, M, *)$ is completable. Then, by Lemma 3.11, $(\mathrm{Comp}(X), H_M, *)$ is completable. Hence there exists an isometry $i : (\mathrm{Comp}(X), H_M, *) \to (\widetilde{\mathrm{Comp}(X)}, \widetilde{H_M}, \star)$ such that $i(\mathrm{Comp}(X))$ is dense in $\widetilde{\mathrm{Comp}(X)}$.

Note that the restriction $i_{\mathrm{Fin}(X)}$ of i is an isometry between $(\mathrm{Fin}(X), H_M, *)$ and $(i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X)), \widetilde{H_M}, \star)$. Let $\overline{i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X))}$ be the closure of $i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X))$. It follows from Lemma 3.7 that $(\overline{i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X))}, \widetilde{H_M}, \star)$ is a complete fuzzy metric space that has $i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X))$ as a dense subspace. So $(\mathrm{Fin}(X), H_M, *)$ is completable.

4. Completeness of the Hausdorff fuzzy metric on Comp(X)

In the section, we will give necessary and sufficient conditions for the Hausdorff fuzzy metric spaces on $\mathrm{Comp}(X)$ to be complete.

In a fuzzy metric space $(X, M, *)$, put

$$\mathrm{Comp}_K(X) = \{K' \in \mathrm{Comp}(X) | K \subseteq K'\}$$

for every $K \in \mathrm{Comp}(X)$.

Lemma 4.1. *Let $(X, M, *)$ be a fuzzy metric space and $K \in Comp(X)$. Then $Comp_K(X)$ is a closed subset of $(Comp(X), \tau_{H_M})$.*

Proof. Let $A \in \mathrm{Comp}(X) \setminus \mathrm{Comp}_K(X)$. Then there exists an $b \in K$ such that $b \notin A$. Let $t > 0$. Put $M(b, A, t) = \varepsilon_0$. Since $A \in \mathrm{Comp}(X)$, then, by Lemma 3.3, there exists an $a_b \in A$ such that $M(b, a_b, t) = M(b, A, t)$. Hence $0 < \varepsilon_0 < 1$. Let $B \in \mathrm{Comp}_K(X)$. Then $b \in B$. Since

$$\begin{aligned} H_M(A, B, t) &= \min\left\{\inf_{x\in A} M(x, B, t), \inf_{y\in B} M(A, y, t)\right\} \\ &\leq \inf_{y\in B} M(A, y, t) \leq M(A, b, t) \\ &= \varepsilon_0, \end{aligned}$$

we have that $B \notin B_{H_M}(A, 1-\varepsilon_0, t)$. It follows that

$$B_{H_M}(A, 1-\varepsilon_0, t) \cap \mathrm{Comp}_K(X) = \emptyset.$$

So $\mathrm{Comp}_K(X)$ is a closed subset of $(\mathrm{Comp}(X), \tau_{H_M})$.

Theorem 4.2. *Let $(X, M, *)$ be a fuzzy metric space. Then $(X, M, *)$ is complete if and only if $(Comp_K(X), H_M, *)$ is complete for all $K \in Comp(X)$.*

Proof. Assume that $(X, M, *)$ is complete. Then, by Lemma 3.6, we have that $(\mathrm{Comp}(X), H_M, *)$ is complete. Let $K \in \mathrm{Comp}(X)$. Due to Lemma 4.1, we obtain that $\mathrm{Comp}_K(X)$ is a closed subset of $\mathrm{Comp}(X)$. Consequently, according to Lemma 3.7, $(\mathrm{Comp}_K(X), H_M, *)$ is complete.

Conversely, assume that $(\mathrm{Comp}_K(X), H_M, *)$ is complete for all $K \in \mathrm{Comp}(X)$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, M, *)$. Take $a \in X$. For each $n \in \mathbb{N}$, put $A_n = \{a, a_n\}$. Then $A_n \in \mathrm{Comp}_{\{a\}}(X)$. Let $r \in (0,1)$ and $t > 0$. Then there exists an $N \in \mathbb{N}$ such that $M(a_n, a_m t) > 1 - r$ for all $n, m \geq N$.

Note that

$$\begin{aligned} H_M(A_n, A_m, t) &= \min\left\{\inf_{x\in A_n} M(x, A_m, t), \inf_{y\in A_m} M(A_n, y, t)\right\} \\ &= \min\{M(a_n, A_m, t), M(A_n, a_m, t)\}. \end{aligned}$$

Since $M(a_n, A_m, t) \geq M(a_n, a_m, t)$ and $M(A_n, a_m, t) \geq M(a_n, a_m, t)$, we get that

$$H_M(A_n, A_m, t) \geq M(a_n, a_m, t) > 1 - r,$$

which means that $\{A_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Comp}_{\{a\}}(X)$. So $\{A_n\}_{n\in\mathbb{N}}$ converges to $A \in \mathrm{Comp}_{\{a\}}(X)$. If $|A| \geq 3$, then we can choose an $A' \subseteq A$ with $|A'| = 3$. Put $\varepsilon' = \max\{M(x, y, 2t) | x, y \in A'\}$. Then there exists a $\varepsilon_0 \in (\varepsilon', 1)$ such that $\varepsilon_0 * \varepsilon_0 > \varepsilon'$. Put $\mathcal{A} = \{A_n | n \in \mathbb{N}\}$. Then $B_{H_M}(A, 1-\varepsilon_0, t) \cap \mathcal{A} = \emptyset$, which contradicts that $\{A_n\}_{n\in\mathbb{N}}$ converges to A. In fact, if there exists a $B \in B_{H_M}(A, 1-\varepsilon_0, t) \cap \mathcal{A}$, then

$$H_M(A, B, t) = \min\{\inf_{x\in A} M(x, B, t), \inf_{y\in B} M(A, y, t)\} > \varepsilon_0.$$

Hence $\inf_{x\in A} M(x, B, t) > \varepsilon_0$. Since $A' \subseteq A$, we have that $\inf_{x\in A'} M(x, B, t) > \varepsilon_0$. Therefore $M(x, B, t) > \varepsilon_0$ for all $x \in A'$. Thus, due to Lemma 3.3, for each $x \in A'$, there exists a $y_x \in B$ such that $M(x, y_x, t) = M(x, B, t) > \varepsilon_0$. Hence $x \in B_M(y_x, 1-\varepsilon_0, t)$ for all $x \in A'$. So

$$A' \subseteq \bigcup_{y_x\in B} B_M(y_x, 1-\varepsilon_0, t) \subseteq \bigcup_{y\in B} B_M(y, 1-\varepsilon_0, t).$$

Since $|B| = 2$ and $|A'| = 3$, there exist $x_1, x_2 \in A'$ and $y_1 \in B$ such that $x_1, x_2 \in B_M(y_1, 1-\varepsilon_0, t)$. We get that

$$M(x_1, x_2, 2t) \geq M(x_1, y_1, t) * M(y_1, x_2, t) \geq \varepsilon_0 * \varepsilon_0 > \varepsilon' \geq M(x_1, x_2, 2t),$$

which is a contradiction. So $|A| \leq 2$. Let $r \in (0,1)$ and $t > 0$. Then there exists an $N_1 \in \mathbb{N}$ such that

$$H_M(A, A_n, t) = \min\{\inf_{x\in A} M(x, A_n, t), \inf_{y\in A_n} M(A, y, t)\} > 1 - r$$

for all $n \geq N_1$. In case $A = \{a\}$ we have that

$$\begin{aligned} H_M(A, A_n, t) &= \min\{M(a, A_n, t), \inf_{y\in A_n} M(a, y, t)\} \\ &= \min\{1, M(a, a_n, t)\} = M(a, a_n, t). \end{aligned}$$

Hence $M(a, a_n, t) > 1 - r$. Consequently, $\{a_n\}_{n\in\mathbb{N}}$ converges to a. Let $A = \{a, b\}$. Set $M(b, a, t) = \varepsilon_1$. Then there exists $r_1 \in (0, r)$ such that $1 - r_1 > \varepsilon_1$. Then there exists an $N_2 \in \mathbb{N}$ such that

$$\begin{aligned} H_M(A, A_n, t) &= \min\{\inf_{x\in A} M(x, A_n, t), \inf_{y\in A_n} M(A, y, t)\} \\ &= \min\{M(b, A_n, t), M(A, a_n, t)\} \\ &> 1 - r_1 \end{aligned}$$

for all $n \geq N_2$. Hence

$$M(b, A_n, t) = \max\{M(b, a, t), M(b, a_n, t)\} > 1 - r_1.$$

Since $M(b, a, t) = \varepsilon_1 < 1 - r_1$, we get that $M(b, a_n, t) > 1 - r_1 > 1 - r$. Therefore, $\{a_n\}_{n\in\mathbb{N}}$ converges to b. So $(X, M, *)$ is complete.

From Lemma 3.6 and Theorem 4.2 we immediately deduce the next corollary.

Corollary 4.3. *Let $(X, M, *)$ be a fuzzy metric space. Then the following are equivalent.*

(i) *$(X, M, *)$ is complete.*

(ii) *$(Comp(X), H_M, *)$ is complete.*

(iii) *$(Comp_K(X), H_M, *)$ is complete for all $K \in Comp(X)$.*

Acknowledgments. The authors are grateful to the referees for their valuable suggestions.

This work was supported by Grants from the National Natural Science Foundation of China (Nos. 11526109, 11471153, 11571158, 61379021), Natural Science Foundation of Fujian (Nos. 2016J01671, 2015J05011, JK2014028), the Foundation of Fujian Province (No. JA14200), and the outstanding youth foundation of the Education Department of Fujiang Province.

References

[1] Engelking, R., *General Topology*, PWN-Polish Science Publishers, Warsaw, 1977.

[2] Erceg, M.A., *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl., 69 (1979), 205–230.

[3] George, A., Veeramani, P., *On some resules in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), 395–399.

[4] George, A., Veeramani, P., *Some theorems in fuzzy metric spaces*, J. Fuzzy Math., 3 (1995), 933–940.

[5] George, A., Veeramani, P., *On some resules of analysis for fuzzy metric spaces*, Fuzzy Sets and Systems, 90 (1997), 365–368.

[6] Gregori, V., Mascarell, J.A., Sapena, A., *On completion of fuzzy quasi-metric spaces*, Topology and its Applications, 153 (2005), 886–899.

[7] Gregori, V., Romaguera, S., *Some properties of fuzzy metric spaces*, Fuzzy Sets and Systems, 115 (2000), 485–489.

[8] Gregori, V., Romaguera, S., *On completion of fuzzy metric spaces*, Fuzzy Sets and Systems, 130 (2002), 399–404.

[9] Gregori, V., Romaguera, S., *Characterizing completable fuzzy metric spaces*, Fuzzy Sets and Systems 144 (2004) 411–420.

[10] Kaleva, O., Seikkala, S., *On fuzzy metric spaces*, Fuzzy Sets and Systems, 12 (1984), 215–229.

[11] Kočinac, Lj.D.R., *Selection properties in fuzzy metric spaces*, Filomat, 26 (2) (2012), 305–312.

[12] Kramosil, I., Michalek, J., *Fuzzy metric and statistical metric spaces*, Kybernetika, 11 (1975), 326–334.

[13] Li, C.Q., *On some results of metrics induced by a fuzzy ultrametric*, Filomat, 27 (6) (2013), 1133–1140.

[14] Li, C.Q., Yang, Z.Q., *Fuzzy ultrametrics based on idempotent probability measures*, J. Fuzzy Math., 22 (2) (2014), 463–476.

[15] Li, C.Q., *Some properties of intuitionstic fuzzy metric spaces*, Journal of Computational Analysis and Applications, 16 (4) (2014), 670–677.

[16] Park, J.H., Park, Y.B., Saadati, R., *Some results in intuitionistic fuzzy metric spaces*, Journal of Computational Analysis and Applications, 10 (4) (2008), 441-451.

[17] Repovš, D., Savchenko, A., Zarichnyi, M., *Fuzzy Prokhorov metric on the of probability measures*, Fuzzy Sets and Systems, 175 (2011), 96–104.

[18] Rodríguez-López, J., Romaguera, S., *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems, 147 (2004), 273–283.

[19] Savchenko, A., Zarichnyi, M., *Fuzzy ultrametrics on the set of probability measures*, Topology, 48 (2009), 130–136.

[20] Veeramani, P., *Best approximation in fuzzy metric spaces*, J. Fuzzy Math., 9 (2001), 75–80.

Accepted: 09.12.2015