DISJOINT J-CLASS OPERATORS

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Abstract. In this paper, we first introduce the notion of disjoint extended limit set for a tuple of bounded linear operators on a separable Banach space X, and we extend some results from a single operator to a tuple of sequences of operators.

Keywords: tuple of sequence, hypercyclic sequences, d-topologically transitive, d-J-class operators, d-J^{mix}-class operators.

1. Introduction

For a infinite-dimensional separable complex Banach space X, $\mathcal{B}(X)$ will denote the algebra of all bounded linear operators on X.

For $x \in X$, the orbit of x under $(T_n)_n$ is the set

$$Orb(T_n, x) = \{T_n x : n \in \mathbb{Z}_+\}.$$

A sequence $(T_n)_n$ of operators is called hypercyclic if there is some x whose orbit under $(T_n)_n$ is dense in X. In such a case, x is called a hypercyclic or universal vector for $(T_n)_n$. A sequence $(T_n)_n$ of operators is called topologically transitive if for every nonempty open subsets U and V of X there is some $n \geq 0$ such that

$$T_n(U) \cap V \neq \emptyset.$$

Let $T_{1,n}, T_{2,n}$ be continuous linear sequences of operators acting on an infinite dimensional separable Banach space X. For $x \in X$, the orbit of x under the pair $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is the set

$$Orb(\mathcal{T}_n, x) = \{T_{1,n}T_{2,n}x : n \in \mathbb{Z}_+\}.$$

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Definition 1.1. Let $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ be a pair sequences of operators acting on an infinite dimensional Banach space X. A vector x is called a hypercyclic vector for $\mathcal{T}_n$ if $Orb(\mathcal{T}_n, x)$ is dense in X and in this case the pair $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is called hypercyclic.

Definition 1.2. We say that the pair of sequence of operators $\mathcal{T}_n$=$\{(T_{1,n},T_{2,n})\}_{n\in\mathbb{Z}_+}$ is topologically transitive if for every nonempty open subsets U and V of X there exists $n \in \mathbb{Z}_+$ such that $T_{1,n}T_{2,n}(U) \cap V \neq \emptyset$.

Definition 1.3. We say that the pair of sequence of operators $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is topologically mixing if for every nonempty open subsets U and V of X there exists $n_0 \in \mathbb{Z}_+$ such that $T_{1,n}T_{2,n}(U) \cap V \neq \emptyset$, $\forall n \geq n_0$.

Definition 1.4. Let $\{n_j\}$ be a strictly increasing sequence of positive integers. We say that $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ the pair of continuous linear sequence of operators on X satisfy the Hypercyclicity Criterion with respect to $\{n_j\}$ provided there exist two dense subsets X_0 and Y_0 in X, and mappings $S_j : Y_0 \longrightarrow X$ satisfying

1. For each $x \in X_0$, $T_{1,n_j}T_{2,n_j}x \to 0$ as $j \to \infty$.
2. For each $y \in Y_0$, $S_j y \to 0$ as $j \to \infty$.
3. For each $y \in Y_0$, $T_{1,n_j}T_{2,n_j}S_j y \to y$ as $j \to \infty$.

For some sources on these topics, see [3], [4], [14]-[18], [22].

The notion of disjoint hypercyclicity, a strengthening of hypercyclicity, concerning a tuple of linear operators, was introduced independently by Bernal [5] and by Bès and Peris [10] in 2007.

For any integer $N \geq 2$, the tuple $(T_1, T_2, ..., T_N)$ of operators, acting on the same topological vector space X is said disjoint hypercyclic, or $d-$hypercyclic for short, provided there exists $(z, ..., z)$ in X^N such that

$$\{(T_1^n z, T_2^n z, ..., T_N^n z) : n \in \mathbb{Z}_+\}$$

is dense in X^N. Such a vector z is called a $d-$hypercyclic vector for the tuple $(T_1, T_2, ..., T_N)$.

Recall that S. Shkarin in [21] gave a short proof of existence of disjoint hypercyclic tuples of operators of any given length on any separable infinite dimensional Fréchet space. Similar argument provides disjoint dual hypercyclic tuples of operators of any length on any infinite dimensional Banach space with separable dual.

We say that the operators $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$ are d-topologically transitive if for any non-empty open subsets $V_0, V_1, ..., V_N$ in $X,$ there exists a positive integer n so that

$$\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap ... \cap T_N^{-n}(V_N)$$

For recent results on disjoint hypercyclicity, see [6]-[9], [19]-[21].

Definition 1.5. Let $T : X \to X$ be a bounded linear operator on a Banach space X. For every $x \in X$, the sets

$J(x) = \{y \in X$: there exist a strictly increasing sequence of positive integers k_n and a sequence $x_n \subset X$ such that $x_n \to x$ and $T^{k_n} x_n \to y\}$

$J^{mix}(x) = \{y \in X$: there exist a sequence $x_n \subset X$ such that $x_n \to x$ and $T^n x_n \to y\}$

will be called the extended limit set of x under T and the extended mixing limit set of x under T respectively.

The notions of the limit and extended limit sets are well known in the theory of topological dynamics, see [11].

For $x \in X$, the orbit of x under T is the set $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$. A vector x is called a hypercyclic vector for T if $Orb(T, x)$ is dense in X and the operator T is said to be hypercyclic if there is some vector $x \in X$ such that $Orb(T, x)$ is dense in X. We say $T \in \mathcal{B}(X)$ is topologically transitive if for every open sets U, V there exists a non negative integer n such that $T^{-n}(U) \cap V \neq \emptyset$. It is well known that $T \in \mathcal{B}(X)$ is hypercyclic if and only if T is topologically transitive. For more information on the topics of the hypercyclicity, see [1], [3], [15]-[17]. It is not difficult to show that T is topologically transitive if and only if $J(x) = X$ for every $x \in X$ and that T is topologically mixing if and only if $J^{mix}(x) = X$ for every $x \in X$ see [13]. For more information on the J-class set see [2], [12], [13].

In this paper, we introduced the notion of disjoint extended limit set for a tuple of bounded linear operators on a separable Banach space X, and we extend some results known for a single operator to a tuple of sequences of operators.

2. Main results

Definition 2.6. Let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. For every $x_0 \in X$, the sets

d-$J_{(T_1,T_2,...,T_N)}(x_0)$=$\{(x_1, ..., x_N) \in X^N$: for every neighborhoods $V_0, V_1, ..., V_N$ of $x_0, x_1, ..., x_N$ respectively, there exists a positive integer n, so that $\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap ... \cap T_N^{-n}(V_N)\}$

d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0)$=$\{(x_1, ..., x_N) \in X^N$: for every neighborhoods $V_0, V_1, ..., V_N$ of $x_0, x_1, ..., x_N$ respectively, there exists a positive integer m, so that $\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap ... \cap T_N^{-n}(V_N)$ for every $n \geq m\}$

will be called the extended limit set of x_0 under $T_1, T_2, ..., T_N$ and the extended mixing limit set of x_0 under $T_1, T_2, ..., T_N$ respectively.

Proposition 2.1. *An equivalent definition for the sets d-$J_{(T_1,T_2,...,T_N)}(x_0)$ and d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0)$ is the following* *d-$J_{(T_1,T_2,...,T_N)}(x_0) = \{(x_1,...,x_N) \in X^N$: there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_i^{k_n} x_n \to x_i$ for all $1 \le i \le N\}$, and*

d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0) = \{(x_1,...,x_N) \in X^N$: there exists a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_i^n x_n \to x_i$ for all $1 \le i \le N\}$.

Proof. We give the proof for the d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0)$, because the proof for the d-$J_{(T_1,T_2,...,T_N)}(x_0)$ is similar. Let us prove that

d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0) \subset \{(x_1,...,x_N) \in X^N$: there exists a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_i^n x_n \to x_i$ for all $1 \le i \le N\}$.

Let $(x_1,...,x_N) \in$ d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0)$ and consider the open balls

$V_{0,n} = B\left(x_0, \frac{1}{n}\right), V_{i,n} = B\left(x_i, \frac{1}{n}\right)$ centered at $x_0, x_i \in X$ and radius $1/n$ for $n = 1, 2, ...,$ and $1 \le i \le N$.

Then there exists a positive integer m so that $\emptyset \neq V_{0,n} \cap T_1^{-n}(V_{1,n}) \cap T_2^{-n}(V_{2,n}) \cap ... \cap T_N^{-n}(V_{N,n})$ for every $n \ge m$. Hence there exists $x_n \in V_{0,n} = B(x_0, \frac{1}{n})$ such that $x_n \in \bigcap_{i=1}^{N} T_i^{-n}(V_{i,n})$, this implies that $T_i^n(x_n) \in V_{i,n}$ for all $i = 1, ..., N$. Therefore, there exists a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_i^n x_n \to x_i$ for all $1 \le i \le N$. The converse is obvious. ∎

Theorem 2.1. *Let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \ge 2$. Then the following conditions are equivalent:*

1. *$T_1, T_2, ..., T_N$ is d-transitive.*
2. *For every $x_0 \in X$, d-$J_{(T_1,T_2,...,T_N)}(x_0) = X^N$.*

Proof. We first prove that (1) implies (2). Let $x_i \in V_i$ $(0 \le i \le N)$ and $V_0, V_1, ..., V_N$ be relatively open subsets of X. There exists $n \in \mathbb{N}$ such that $\emptyset \neq V_0 \cap \bigcap_{i=1}^{N} T_i^{-n}(V_i)$. Thus $(x_1, ..., x_N) \in$ d-$J_{(T_1,T_2,...,T_N)}(x_0)$, and consequently d-$J_{(T_1,T_2,...,T_N)}(x_0) = X^N$.

We will show that (2) implies (1). Let $V_0, V_1, ..., V_N$ the nonempty open. Consider $x_i \in V_i$ $(0 \le i \le N)$. Since d-$J_{(T_1,T_2,...,T_N)}(x_0) = X^N$. There exists $n \in \mathbb{N}$ such that $\emptyset \neq V_0 \cap \bigcap_{i=1}^{N} T_i^{-n}(V_i)$. By definition, $T_1, T_2, ..., T_N$ is d-transitive. ∎

Proposition 2.2. *Let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \ge 2$. Then*

$$d - J_{(\lambda_1 T_1, \lambda_2 T_2, ..., \lambda_N T_N)}(0) = d - J_{(T_1,T_2,...,T_N)}(0)$$

for every $|\lambda_i| = 1 \quad (1 \le i \le N)$.

Proof. Let $(x_1,...,x_N) \in$ d-$J_{(\lambda_1T_1,\lambda_2T_2,...,\lambda_NT_N)}(0)$. Then there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to 0$ and $\lambda_i^{k_n}T_i^{k_n}x_n \to x_i$ for all $1 \le i \le N$. Since $|\lambda_i| = 1$ then $\lambda_i^{k_n}x_n \to 0$ and since $T_i^{k_n}(\lambda_i^{k_n}x_n) \to x_i$ it follows that $(x_1,...,x_N) \in$ d-$J_{(T_1,T_2,...,T_N)}(0)$.

Let $(x_1,...,x_N) \in$ d-$J_{(T_1,T_2,...,T_N)}(0)$. Then there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to 0$ and $T_i^{k_n}x_n \to x_i$ for all $1 \le i \le N$. Since $|\lambda_i| = 1 \quad (1 \le i \le N)$, without loss of generality we can assume that $\lambda_i^{k_n} \to \mu_i$ for some $|\mu_i| = 1$. Hence $\lambda_i^{k_n}T_i^{k_n}(\frac{x_n}{\mu_i}) \to x_i$ for all $1 \le i \le N$ and since $\frac{x_n}{\mu_i} \to 0$ then $(x_1,...,x_N) \in$ d-$J_{(\lambda_1T_1,\lambda_2T_2,...,\lambda_NT_N)}(0)$. ∎

Proposition 2.3. *Let $T_1,T_2,...,T_N$ in $\mathcal{B}(X)$ with $N \ge 2$ and $x_{0,n},x_{1,n},...,x_{N,n}$ be the sequences in X such that $x_{0,n} \to x_0$ and $x_{i,n} \to x_i$ for some $x_0,x_i \in X$ $(1 \le i \le N)$.*

(1) *If $(x_{1,n},...,x_{N,n}) \in$ d-$J_{(T_1,T_2,...,T_N)}(x_{0,n})$ for every $n = 1,2,...$, then $(x_1,...,x_N) \in$ d-$J_{(T_1,T_2,...,T_N)}(x_0)$.*

(2) *If $(x_{1,n},...,x_{N,n}) \in$ d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_{0,n})$ for every $n = 1,2,...$, then $(x_1,...,x_N) \in$ d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0)$.*

Proof. (1) For $n = 1$ there exists a positive integer k_1 such that

$$||x_{0,k_1} - x_0|| < \tfrac{1}{2} \quad \text{and} \quad ||x_{i,k_1} - x_i|| < \tfrac{1}{2} \ (1 \le i \le N).$$

Since $(x_{1,k_1},...,x_{N,k_1}) \in$ d-$J_{(T_1,T_2,...,T_N)}(x_{0,k_1})$ we may find a positive integer l_1 and $z_1 \in X$ such that

$$||z_1 - x_{0,k_1}|| < \tfrac{1}{2} \quad \text{and} \quad ||T_i^{l_1}z_1 - x_{i,k_1}|| < \tfrac{1}{2} \ (1 \le i \le N).$$

Therefore,

$$||z_1 - x_0|| < 1 \quad \text{and} \quad ||T_i^{l_1}z_1 - x_i|| < 1 \ (1 \le i \le N).$$

Proceeding inductively we find a strictly increasing sequence of positive integers l_n and a sequence z_n in X such that

$$||z_n - x_0|| < \tfrac{1}{n} \quad \text{and} \quad ||T_i^{l_n}z_n - x_i|| < \tfrac{1}{n} \ (1 \le i \le N).$$

This completes the proof of assertion (1).

(2) For $n = 1$ there exists a positive integer k_1 such that

$$||x_{0,k_1} - x_0|| < \tfrac{1}{2} \quad \text{and} \quad ||x_{i,k_1} - x_i|| < \tfrac{1}{2} \ (1 \le i \le N).$$

Since $(x_{1,k_1},...,x_{N,k_1}) \in$ d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_{0,k_1})$ we may find a positive integer l_1 and $z_n \in X$ such that

$$||z_n - x_{0,k_1}|| < \tfrac{1}{2} \quad \text{and} \quad ||T_i^{n}z_n - x_{i,k_1}|| < \tfrac{1}{2} \ (1 \le i \le N).$$

For every $n \ge l_1$. Therefore,

$$||z_n - x_0|| < 1 \quad \text{and} \quad ||T_i^n z_n - x_i|| < 1 \ (1 \leq i \leq N).$$

For every $n \geq l_1$. Proceeding inductively we find a strictly increasing sequence of positive integers $l_2 > l_1$ and a sequence $w_n \subset X$ such that

$$||w_n - x_0|| < \tfrac{1}{2} \quad \text{and} \quad ||T_i^n w_n - x_i|| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

For every $n \geq l_2$. Set $v_n = z_n$ for every $l_1 \leq n \leq l_2$, hence

$$||v_n - x_0|| < 1 \quad \text{and} \quad ||T_i^n v_n - x_i|| < 1 \ (1 \leq i \leq N).$$

Proceeding inductively we find a strictly increasing sequence of positive integers n_k and a sequence $v_n \subset X$ such that if $n \geq k$ then

$$||v_n - x_0|| < \tfrac{1}{k} \quad \text{and} \quad ||T_i^n v_n - x_i|| < \tfrac{1}{k} \ (1 \leq i \leq N).$$

Take any $\varepsilon > 0$. There exists a positive integer k_0 such that $\frac{1}{k_0} < \varepsilon$. Hence for every $n \geq n_{k_0}$ we get

$$||v_n - x_0|| < \tfrac{1}{k_0} < \varepsilon \quad \text{and} \quad ||T_i^n v_n - x_i|| < \tfrac{1}{k_0} < \varepsilon \ (1 \leq i \leq N).$$

This completes the proof of assertion (2). ■

If $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$, we denote

d-$L(x_0) := \{(x_1, ..., x_N) \in X^N$: there exists a strictly increasing sequence of positive integers k_n such that $T_i^{k_n} x_0 \to x_i$ for all $1 \leq i \leq N\}$.

Proposition 2.4. *Let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. If T_i is power bounded for all $1 \leq i \leq N$ then d-$J_{(T_1,T_2,...,T_N)}(x_0) =$ d-$L(x_0)$ for every $x_0 \in X$.*

Proof. Since T_i is power bounded for all $1 \leq i \leq N$ there exists a positive number M such that $||T_i^n|| \leq M$ for every positive integer n and $1 \leq i \leq N$. Let $x_0 \in X$. If d-$J_{(T_1,T_2,...,T_N)}(x_0) = \emptyset$ there is nothing to prove. Therefore assume that d-$J_{(T_1,T_2,...,T_N)}(x_0) \neq \emptyset$. Since the inclusion d-$L(x_0) \subset$ d-$J_{(T_1,T_2,...,T_N)}(x_0)$ is always true, it suffices to show that d-$J_{(T_1,T_2,...,T_N)}(x_0) \subset$ d-$L(x_0)$. Take $(x_1, ..., x_N) \in$ d-$J_{(T_1,T_2,...,T_N)}(x_0)$. There exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_i^{k_n} x_n \to x_i$ for all $1 \leq i \leq N$. Then we have

$$\begin{aligned} ||T_i^{k_n} x_0 - x_i|| &\leq ||T_i^{k_n} x_0 - T_i^{k_n} x_n|| + ||T_i^{k_n} x_n - x_i|| \\ &\leq M||x_0 - x_n|| + ||T_i^{k_n} x_n - x_i|| \end{aligned}$$

and, letting n goes to infinity to the above inequality, we get that $(x_1, ..., x_N) \in$ d-$L(x_0)$. ■

From now on, let $T_{1,n}, T_{2,n}, ..., T_{2,N}$, with $N \geq 2$, be continuous linear sequences of operators acting on an infinite dimensional separable Banach space X. We extend some results from a single operator to a tuple of sequences of operators. For simplicity we state and prove our results for a pair that is a tuple with $N = 2$, and the general case follows by a similar method. Let $x \in X$, the orbit of x under the pair $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is the set

$$Orb(\mathcal{T}_n, x) = \{T_{1,n}T_{2,n}x : n \in \mathbb{Z}_+\}.$$

In the proof of the following lemma, we use a method of the proof of [15, Theorem 1.2] to extend results for tuples. We will use $HC(\mathcal{T}_n)$ for the collection of hypercyclic vectors for the pair of sequence $\mathcal{T}_n$

Lemma 2.1. *Let X be a separable infinite dimensional Banach space and $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ be the pair of the sequence operators $T_{1,n}$ and $T_{2,n}$. Then, $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is topologically transitive if and only if $HC(\mathcal{T}_n)$ is dense in X.*

Proof. Fix an enumeration $\{B_n, n \in \mathbb{Z}_+\}$ of the open balls in X with rational radii, and centers in a countable dense subset of X. By the continuity of the sequences $T_{1,n}$ and $T_{2,n}$ the sets

$$G_k = \cup\{T_{1,n}^{-1}T_{2,n}^{-1}(B_k) : n \in \mathbb{Z}_+\}$$

are open. Clearly, $HC(\mathcal{T}_n)$ is equal to

$$\cap\{G_k : k \in \mathbb{Z}_+\}.$$

Now, let $\mathcal{T}_n$ be topologically transitive and let U by an arbitrary nonempty open set in X. Then for all $k \in \mathbb{Z}_+$, there exist $n(k)$ in $\mathbb{Z}_+$ such that

$$T_{1,n(k)}T_{2,n(k)}(U) \cap B_k \neq \emptyset$$

which implies that $U \cap G_k \neq \emptyset$ for all k. Thus each G_k is dense in X and so by the Bair Category Theorem $HC(\mathcal{T}_n)$ is also dense in X.

Conversely, if $HC(\mathcal{T}_n)$ is dense in X, then each set G_k. This implies that $\mathcal{T}_n$ is topologically transitive. ∎

Theorem 2.2. (The Hypercyclicity Criterion for a Tuple of sequence) *Suppose that X is a separable infinite dimensional Banach space and $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a pair of continuous linear sequence of operators on X. If there exist two dense subsets X_0 and Y_0 in X, and a strictly increasing sequence of positive integers $\{n_j\}$ and mappings $S_j : Y_0 \longrightarrow X$ such that*

1. *For each $x \in X_0$, $T_{1,n_j}T_{2,n_j}x \to 0$ as $j \to \infty$.*
2. *For each $y \in Y_0$, $S_jy \to 0$ as $j \to \infty$.*
3. *For each $y \in Y_0$, $T_{1,n_j}T_{2,n_j}S_jy \to y$ as $j \to \infty$.*

Then, $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a hypercyclic tuple.

Proof. Let U and V are two nonempty open sets in X, then chose $x \in X_0 \cap U$ and $y \in V \cap Y_0$ and let $z_j = x + S_jy$. Then, as $j \to \infty$, $z_j \to x$ and $T_{1,n_j}T_{2,n_j}z_j = T_{1,n_j}T_{2,n_j}x + T_{1,n_j}T_{2,n_j}S_jy \to y$. Thus, for large j, we have $z_j \in U$ and $T_{1,n_j}T_{2,n_j}z_j \in V$. By Lemma 2.1, $HC(\mathcal{T}_n)$ is dense in X and this implies, clearly, that the pair $\{(T_{1,n},T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a hypercyclic pair. ∎

Proposition 2.5. *Let $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ satisfy the Hypercyclicity Criterion with respect to a sequence $\{n_j\}$. Then the pair $(T_{1,n_j}, T_{2,n_j})_{n\in\mathbb{Z}_+}$ are topologically mixing. In particular $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ are hypercyclic.*

Proof. We show that $\{(T_{1,n_j}, T_{2,n_j})\}_{n\in\mathbb{Z}_+}$ are topologically mixing. Let X_0 and Y_0 be dense sets in X, that are given in the hypercyclicity criterion of Definition 1.4. Let U and V are two nonempty open sets in X, then choose $x \in X_0 \cap U$ and $y \in V \cap Y_0$ and $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U$ and $B(y,\varepsilon) \subset V$. By Definition 1.4, there exist $j_0 \in \mathbb{N}$ so that, for all $j \geq j_0$, $||T_{1,n_j}T_{2,n_j}x|| \leq \varepsilon$, $||S_j(y)|| \leq \varepsilon$, and $||T_{1,n_j}T_{2,n_j}S_j(y) - y|| \leq \varepsilon$. Then, for each $j \geq j_0$, we have $z_j = x + S_j y \in B(x,\varepsilon) \subset U$ and $T_{1,n_j}T_{2,n_j}z_j \in B(y,\varepsilon) \subset V$. That is, $T_{1,n_j}T_{2,n_j}(U) \cap V \neq \emptyset$, $\forall j \geq j_0$. Hence, $\{(T_{1,n_j}, T_{2,n_j})\}_{n\in\mathbb{Z}_+}$ is topologically mixing. ■

If $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a sequence of $2-$tuples of continuous self-maps on a X, we denote

$$\mathbb{J} := \{(x,y) \in X \times X;\ \exists (u_n)_{n\in\mathbb{N}} \subset X : u_n \to x \text{ and } T_{1,n}T_{2,n}u_n \to y\}.$$

Proposition 2.6. *Let $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a sequence of $2-$tuples of continuous self-maps on a X, such that $\mathbb{J}$ is dense in $X \times X$. Then $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ are topologically mixing.*

Proof. Let U and V are two nonempty open sets in X. Since $\mathbb{J}$ is dense in $X \times X$, we can find $x \in U$ and $y \in V$ such that $(x,y) \in \mathbb{J}$. By definition of $\mathbb{J}$, there is a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that $u_n \to x$ and $T_{1,n}T_{2,n}u_n \to y$. Then, there exists $k_0 \in \mathbb{N}$ such that $u_n \in U$ and $T_{1,n}T_{2,n}u_n \in V$, $\forall k \geq k_0$. Hence $T_{1,n}T_{2,n}(U) \cap V \neq \emptyset$, $\forall k \geq k_0$. That is, $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ are topologically mixing. ■

We finish this paper by the following questions:

Question 1. Let $T_1, T_2, ..., T_N$ $(n \geq 2)$ be d-hypercyclic and invertible. Must for every $x \in X$, d-$J_{(T_1,T_2,...,T_N)}(x) = X^N$?

Question 2. For a infinite-dimensional separable complex Banach space X, let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. Suppose there exists a vector $x \in X$ such that d-$J_{(T_1,T_2,...,T_N)}(x)^\circ \neq \emptyset$. It is true that d-$J_{(T_1,T_2,...,T_N)}(x) = X^N$?

Question 3. For a infinite-dimensional separable complex Banach space X, let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. Suppose that d-$J_{(T_1,T_2,...,T_N)}(x)^\circ \neq \emptyset$ for every $x \in X$. Does it follow that $T_1, T_2, ..., T_N$ is d-topologically transitive?

Question 4. Is there a relation between d-hypercyclicity see [10, Definition 1.1] and d-$J_{(T_1,T_2,...,T_N)}$-sets?

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