ON THE ORDER AND LOWER ORDER OF LAPLACE-STIELTJES TRANSFORMATIONS WITH INDEX PAIR \((p, q)\)

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Abstract. In this paper, in order to study the precise growth of entire functions represented by Laplace-Stieltjes transformations, we have introduced the concept of \((p, q)\)-order and lower \((p, q)\)-order and obtained their coefficient characterizations.

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1. Introduction

Consider the Laplace-Stieltjes transformation defined by

\[
G(s) = \int_0^\infty \exp(-sx) \, d\alpha(x) 
\]

where \(\alpha(x)\) is a function of bounded variation on any finite interval \([0, X]\), \((0 < X < +\infty)\), \(s = \sigma + it\), \(\sigma\) and \(t\) are real variables. We choose a monotonic increasing sequence of real numbers \(\{\lambda_n\}\) satisfying the following conditions:

\[
0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \uparrow +\infty, \tag{1.2}
\]

\[
\limsup_{n \to \infty} (\lambda_{n+1} - \lambda_n) < +\infty, \quad \limsup_{n \to \infty} \frac{\ln n}{\lambda_n} = D < +\infty. \tag{1.3}
\]

We put

\[
K^*_n = \sup_{\lambda_n < y \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^x e^{-ity} \, d\alpha(y) \right|. 
\]

In [2], Jiarong obtained the following Valiron-Knopp-Bohr formula:
Theorem A. Suppose that the Laplace-Stieltjes transformation (1.1) satisfies

\[
\limsup_{n \to \infty} (\lambda_{n+1} - \lambda_n) < +\infty \quad \text{and} \quad \limsup_{n \to \infty} \frac{\ln n}{\lambda_n} < +\infty,
\]

and \( \sigma^G_\mu \) denotes the abscissa of uniform convergence of the integral in (1.1). Then

(1.4) \[
\limsup_{n \to \infty} \frac{\ln K^*_n}{\lambda_n} \leq \sigma^G_\mu \leq \limsup_{n \to \infty} \frac{\ln K^*_n}{\lambda_n} + \limsup_{n \to \infty} \frac{\ln n}{\lambda_n}.
\]

Suppose that

(1.5) \[
\limsup_{n \to \infty} \frac{\ln K^*_n}{\lambda_n} = 0.
\]

If \( D = 0 \), then, by (1.4) and (1.5), it follows that \( \sigma^G_\mu = 0 \) and \( F(s) \) is analytic in the right half plane \( \sigma > 0 \).

Various authors have obtained the growth properties of the analytic function defined by (1.1). Yinying and Daochun [6] introduced a type function using the proximate orders. Similarly, Hong et al [1] also obtained some growth properties using the proximate order and type function. Kong and Yang [7] considered the Laplace-Stieltjes transformations given by (1.1) converging uniformly in the whole complex plane \( Re(s) > -\infty \) and studied their growth properties. Yingying and Hong [8] have also studied the properties of Laplace-Stieltjes transformations. In 2012, Luo Xi and Kong Yinying [5] defined Laplace-Stieltjes transformations in a different manner by taking positive exponents in the integral in (1.1). Thus they defined Laplace-Stieltjes transformations as given below:

(1.6) \[
F(s) = \int_0^{+\infty} \exp (sy) \alpha(y) \, dy, \quad (s = \sigma + it),
\]

where \( \alpha(y) \) satisfies same conditions as stated earlier. Let the sequence \( \{\lambda_n\} \) satisfy both conditions stated in (1.3). We put

\[
A^*_n = \sup_{\lambda_n < \lambda \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{ity} \alpha(y) \, dy \right|.
\]

Then the result of Theorem A can be proved for (1.6) also. We assume that the function \( F(s) \) given by (1.6) satisfies the condition \( \liminf_{n \to \infty} \frac{\ln (A^*_n)}{\lambda_n} = +\infty \) i.e. \( F(s) \) represents an entire function.

We now give
**Definition 1.** The maximum modulus, the maximum term and central index of $F(s)$ given by (1.6) are defined as

$$M(\sigma, F) = \sup_{-\infty < t < +\infty} |F(\sigma + it)|,$$

$$M_\mu(\sigma, F) = \sup_{0 < x < +\infty} \left| \frac{1}{\log x} \int_0^x e^{\eta y} d\alpha (y) \right|, s = \sigma + it, \sigma > 0,$$

$$\mu(\sigma, F) = \max_{1 \leq n < N} \{ A_n^* e^{\lambda_n \sigma} \}, \sigma > 0,$$

$$N(\sigma, F) = \max \{ n; \mu(\sigma, F) = A_n^* e^{\lambda_n \sigma} \}.$$  

Using Lemma 1 [5], we can easily show for our case that for $\varepsilon > 0$ and $\sigma$ sufficiently large,

\begin{equation}
\frac{1}{2} \mu(\sigma, F) \leq M_\mu(\sigma, F) \leq \mu(\sigma + D + \varepsilon, F). \tag{1.7}
\end{equation}

In [7], Kong and Yang have studied the growth properties of L-S transform representing entire function using generalized functions. In this paper we have defined $(p, q)$-order and lower $(p, q)$-order of Laplace-Stieltjes transformations and obtained their coefficient characterizations. These results are more explicit and depict the relation between growth parameters and behaviour of the sequence $\{A_n^*\}.$

Following the definitions given by Juneja et al [4] for classical Dirichlet series, we define the index pair $(p, q)$ of entire functions represented by L-S transformation and their order and lower order. We use the following notations: $\exp[0]x = \ln[0]x = x,$ $\exp[m]x = \ln[-m]x = \exp(\exp[-m+1]x) = \ln(\ln[-m-1]x),$ $m = 1, 2, \ldots,$ $E_m(x) = \prod_{i=0}^{m-1} \exp[i]x,$ $\Lambda_{m}(x) = \prod_{i=0}^{m-1} \ln[i]x,$ $E_{-m}(x) = \frac{x}{\Lambda_{m-1}(x)},$ $\Lambda_{-m}(x) = \frac{x}{E_{m-1}(x)},$ $m = 0, \pm 1, \ldots$

We define the $(p, q)$-order and lower $(p, q)$-order with index pair $(p, q)$ of entire functions represented by Laplace-Stieltjes transformations $F(s)$ given by (1.6) above. Hence we put

\begin{equation}
\rho(\lambda) = \limsup_{\sigma \to \infty} \left( \inf_{\sigma} \frac{\ln[p] M_\mu(\sigma, F)}{\ln[q] \sigma} \right), \tag{1.8}
\end{equation}

where $p$ and $q$ are integers such that $p > q \geq 0.$ It can be easily seen that $0 \leq \rho(p, q) \leq \infty$ if $p > q + 1$ and $1 \leq \rho(p, q) \leq \infty$ if $p = q + 1.$ Further, in view of (1.7) we have the equivalent formula for order $\rho$ and lower order $\lambda$ i.e.,

\begin{equation}
\rho(\lambda) = \limsup_{\sigma \to \infty} \left( \inf_{\sigma} \frac{\ln[p] \mu(\sigma, F)}{\ln[q] \sigma} \right). \tag{1.9}
\end{equation}

We also put for $\alpha \geq 0,$

$$P_\alpha(x) = \begin{cases} 
  x & \text{if } q + 1 < p < \infty \\
  x + \alpha & \text{if } p = q + 1 = 2 \\
  \max(1, x) & \text{if } 3 \leq q + 1 = p < \infty \\
  \infty & \text{if } p = q = \infty.
\end{cases}$$

We put $P_1(x) = P(x).$ For more details about the notations, we shall refer to [4].
2. Main results

We now prove

**Theorem 1.** Let \( F(s) = \int_0^\infty \exp(sy)d\alpha(y) \) be Laplace-Stieltjes transformation and let for a pair of integers \((p, q)\), \(p \geq 2, q \geq 0\), the \((p, q)\)-order \(\rho\) of \(F(s)\) be defined by (1.8). Then

\[
\rho = P(L),
\]

where

\[
L \equiv L(p, q) = \lim_{n \to \infty} \sup \frac{\ln^{[p-1]} \lambda_n}{\ln[q] \{(1/\lambda_n) \ln (A_n^*)^{-1}\}}.
\]

**Proof.** Let us assume that \(\rho = \rho(p, q) < \infty\), then from the definition of \(\rho\), for any \(\epsilon > 0\) and all \(\sigma > \sigma_0(\epsilon)\), we get

\[
\ln M_\mu(\sigma, F) < \exp^{[p-1]} \left\{ (\rho + \varepsilon) \ln^{[q]} \sigma \right\}.
\]

From (1.7) and the definition of \(\mu(\sigma, F)\), we have

\[
\ln A_n^* + \lambda_n \sigma \leq \ln \mu(\sigma, F) \leq \ln M_\mu(\sigma, F) + o(1).
\]

Hence

\[
\ln A_n^* < \exp^{[p-1]} \left\{ (\rho + \varepsilon) \ln^{[q]} \sigma \right\} - \lambda_n \sigma.
\]

For \((p, q) \neq (2, 1)\), we choose

\[
\sigma = \exp^{[q-1]} \left\{ \ln^{[p-2]} (\lambda_n) \right\}^{1/(p+\varepsilon)}.
\]

For \(p \geq 2\) and \(q \geq 0\), from (2.3) we have

\[
\ln A_n^* < \lambda_n - \lambda_n \exp^{[q-1]} \left\{ \ln^{[p-2]} \lambda_n \right\}^{1/(p+\varepsilon)}.
\]

Now, if \(p = q + 1\), then \(\rho \geq 1\). Therefore, for \(p = q + 1 \geq 3\), the above inequality gives

\[
\frac{1}{\lambda_n} \ln (A_n^*)^{-1} > \exp^{[q-1]} \left\{ \ln^{[p-2]} \lambda_n \right\}^{1/(p+\varepsilon)} - 1,
\]

or

\[
\ln^{[q]} \left\{ \frac{1}{\lambda_n} \ln (A_n^*)^{-1} \right\} > \frac{1}{\rho + \varepsilon} \left\{ \ln^{[p-1]} \lambda_n \right\} (1 + o(1)),
\]

or

\[
\rho + \varepsilon > \frac{\ln^{[p-1]} \lambda_n}{\ln^{[q]} \{(1/\lambda_n) \ln (A_n^*)^{-1}\}} (1 + o(1)).
\]
By proceeding to limits, we get

\[(2.4) \quad \rho (p, p - 1) \geq \max \{1, L (p, p - 1)\}.\]

From the above inequality, we also have for \(p > q + 1\) and \((p, q) \neq (2, 1)\),

\[\rho (p, q) \geq L (p, q).\]

Thus \((2.1)\) is proved for all pairs \((p, q)\) except for \((p, q) = (2, 1)\).

Now, for \((p, q) = (2, 1)\), we choose

\[\sigma = \lambda_n^{1/(\rho + \varepsilon)}.\]

After substituting this value of \(\sigma\) in \((2.3)\), we get for \((p, q) = (2, 1)\)

\[\ln A_n^* < \lambda_n - \lambda_n (\lambda_n)^{1/(\rho + \varepsilon)},\]

or

\[\ln \left\{ \frac{1}{\lambda_n} \ln (A_n^*)^{-1} \right\} > \frac{1}{(\rho + \varepsilon)} \ln (\lambda_n) + o (1).\]

By proceeding to limits, we get

\[\limsup_{n \to \infty} \frac{\ln \lambda_n}{\ln \left\{ (1/\lambda_n) \ln (A_n^*)^{-1} \right\}} \leq \rho.\]

Hence, for all index pair \((p, q)\),

\[(2.5) \quad \rho (p, q) \geq L (p, q).\]

To prove the reverse inequality, we assume that \(L(p, q) < \infty\). For any \(\varepsilon > 0\), there is a positive integer \(n_o\) such that for \(n > n_o\), we have

\[(2.6) \quad A_n^* < \exp \left\{ -\lambda_n \exp^{[q-1]} \left( \ln^{[p-2]} \lambda_n \right)^{1/(L+\varepsilon)} \right\}.\]

Also, for a number \(\sigma_o > 1\) and \(\sigma > \sigma_o\), we can find an integer \(S\) such that

\[\lambda_S < \exp^{[p-2]} \left( \ln^{[q]} 2e^\sigma \right)^{(L+\varepsilon)} < \lambda_{S+1}.\]

From the definition, we have \(M_\mu (\sigma, F) \leq \sum_{n=0} \infty A_n^* \exp (\sigma \lambda_n)\). Then

\[M_\mu (\sigma, F) \leq O(1) + \sum_{n=n_o+1}^S A_n^* \exp (\sigma \lambda_n)\]

\[+ \sum_{n=S+1}^\infty \exp \left\{ \sigma \lambda_n - \lambda_n \exp^{[q-1]} \left( \ln^{[p-2]} \lambda_n \right)^{1/(L+\varepsilon)} \right\}.\]
Hence, using (2.6) we obtain

\[
\sum_{n=n_0+1}^{S} A_n \exp(\sigma \lambda_n) < \exp(\sigma \lambda_S) \sum_{n=n_0+1}^{S} \exp \left\{ -\lambda_n \exp^{[q-1]} \left( \ln^{[p-2]} \lambda_n \right)^{1/(L+\varepsilon)} \right\}
\]

\[
< \exp(\sigma \lambda_S) \sum_{n=n_0+1}^{\infty} \exp \left\{ -\lambda_n \exp^{[q-1]} \left( \ln^{[p-2]} \lambda_n \right)^{1/(L+\varepsilon)} \right\}
\]

\[
< B \exp(\sigma \lambda_S)
\]

since the series on right hand side is convergent. Similarly, we have

\[
\sum_{n=S+1}^{\infty} \exp \left\{ \sigma \lambda_n - \lambda_n \exp^{[q-1]} \left( \ln^{[p-2]} \lambda_S \right)^{1/(L+\varepsilon)} \right\}
\]

\[
\leq \sum_{n=S+1}^{\infty} \exp \{ \sigma \lambda_n - \lambda_n \ln(2e^{\sigma}) \} \leq \sum_{n=0}^{\infty} 2^{-\lambda_n} = C < \infty.
\]

Combining the estimates obtained in (2.8) and (2.9), we get

\[
M_\mu(\sigma, F) \leq A + B \exp \left\{ \sigma \exp^{[p-2]} \left( \ln^{[q]} 2e^{\sigma} \right)^{(L+\varepsilon)} \right\} + C
\]

or

\[
\ln^{[p]} M_\mu(\sigma, F) \leq \exp^{[p-3]} \left( \ln^{[q]} 2e^{\sigma} \right)^{(L+\varepsilon)} (1 + o(1))
\]

or

\[
\ln^{[p]} M_\mu(\sigma, F) = \ln \left( \ln^{[q]} 2e^{\sigma} \right)^{(L+\varepsilon)} (1 + o(1))
\]

\[
\simeq (L + \varepsilon) \ln^{[q]}(\sigma).
\]

Proceeding to limits, we obtain

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\ln^{[p]} M_\mu(\sigma, F)}{\ln^{[q]}(\sigma)} = \rho(p, q) \leq L(p, q).
\]

Thus the above inequality is true for all pairs of \((p, q)\). From (2.5) and (2.10), we get (2.1) and the proof of the Theorem 1 is complete.

Next, we prove

**Lemma 1.** Let \( F(s) = \int_0^\infty \exp(sy) d\alpha(y) \) be Laplace-Stieltjes transformation having \((p, q)\)-order \( \rho \) and lower \((p, q)\)-order \( \lambda \). Then

\[
\limsup_{\sigma \to \infty} \frac{\ln^{[p-1]} \lambda_{N(\sigma, F)}}{\ln^{[q]}(\sigma)} = \rho(\lambda).
\]
Proof. From a result of Jingjing [3, p.7], we have

$$\ln \mu(\sigma, F) = \ln \mu(\sigma', F) + \int_{\sigma'}^{\sigma} \lambda_{N(x,F)} \, dx,$$

where $0 < \sigma' < \sigma < \infty$.

Since $\lambda_{N(\sigma,F)}$ is nondecreasing, therefore we have

$$\ln \mu(\sigma, F) \leq \ln \mu(\sigma', F) + \lambda_{N(\sigma,F)}(\sigma - \sigma')$$

or

$$(1 - o(1)) \ln \ln \mu(\sigma, F) \leq (1 + o(1)) \ln \lambda_{N(\sigma,F)},$$

or

$$\ln^{[p]} \mu(\sigma, F) \leq \ln^{[p-1]} \lambda_{N(\sigma,F)}.$$  

Hence,

$$\limsup_{\sigma \to \infty} \frac{\ln^{[p]} \mu(\sigma, F)}{\ln^{[q]} \sigma} \leq \limsup_{\sigma \to \infty} \frac{\ln^{[p-1]} \lambda_{N(\sigma,F)}}{\ln^{[q]} \sigma}.$$ 

Conversely, we have

$$\ln \mu(2\sigma, F) = \ln \mu(\sigma', F) + \int_{\sigma'}^{2\sigma} \lambda_{N(x,F)} \, dx \geq O(1) + \int_{\sigma}^{2\sigma} \lambda_{N(x,F)} \, dx \geq \sigma \lambda_{N(\sigma,F)}.$$  

Following as above, we obtain

$$\limsup_{\sigma \to \infty} \frac{\ln^{[p]} \mu(\sigma, F)}{\ln^{[q]} \sigma} \geq \limsup_{\sigma \to \infty} \frac{\ln^{[p-1]} \lambda_{N(\sigma,F)}}{\ln^{[q]} \sigma}.$$ 

Combining the two inequalities obtained above, we get (2.11).

Now, we obtain some lower bounds for the lower $(p, q)$-order $\lambda$.

Lemma 2. Let $F(s) = \int_0^\infty \exp(sy)d\alpha(y)$ be Laplace-Stieltjes transformation having lower $(p,q)$-order $\lambda$ and let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of natural numbers. Then

(2.12) \[ \lambda \geq P_\alpha (3) \]

where

$$\alpha \equiv \alpha(\{n_k\}) = \lim_{k \to \infty} \inf \ln \frac{\lambda_{n_k-1}}{\ln \lambda_{n_k}},$$

and

(2.13) \[ \Im \equiv \Im(\{n_k\}, p, q) = \lim_{k \to \infty} \inf \frac{\ln^{[p-1]} \lambda_{n_k-1}}{\ln^{[q]} \left\{ 1/\lambda_{n_k} \ln \left( A_{n_k}^* \right)^{-1} \right\}}. \]
**Proof.** We assume that $0 < \Im < \infty$. Then, for a given $\varepsilon$, $\Im > \varepsilon > 0$, we have from (2.13)

\begin{equation}
(2.14) \quad (A_{n_k}^*) > \exp \left[ -\lambda_{n_k} \exp^{[q]} \left\{ \ln^{[p-1]} \lambda_{n_{k-1}} / (\Im - \varepsilon) \right\} \right].
\end{equation}

Since addition of an exponential polynomial does not affect the growth of $F(s)$, we can assume that (2.14) holds for all $k$. Now for the pair $(p, q) \neq (2, 1)$, we choose a sequence

\begin{equation}
(2.15) \quad \sigma_k = 2 + \exp^{[q]} \left\{ \ln^{[p-1]} \lambda_{n_{k-1}} / (\Im - \varepsilon) \right\}, \quad k = 1, 2, \ldots.
\end{equation}

Then $\{\sigma_k\}$ is monotonic increasing and $\sigma_k \to \infty$ as $k \to \infty$. If $\sigma_k \leq \sigma < \sigma_{k+1}$, then for $(p, q) \neq (2, 1)$, from (2.14) we get

$$
\ln M_{\mu}(\sigma, F) \geq \ln A_{n_k}^* + \lambda_{n_k} \sigma_k > \sigma_k \lambda_{n_k} - \lambda_{n_k} \exp^{[q]} \left\{ \ln^{[p-1]} \lambda_{n_{k-1}} / (\Im - \varepsilon) \right\}.
$$

Now, substituting for $\sigma_k$, from (2.15), we get

$$
\ln M_{\mu}(\sigma, F) \geq 2 \lambda_{n_k} + 2 \exp^{[p-1]} \left\{ (\Im - \varepsilon) \ln^{[q]} (\sigma_{k+1} - 2) \right\} \geq 2 \exp^{[p-1]} \left\{ (\Im - \varepsilon) \ln^{[q]} (\sigma - 2) \right\}.
$$

For sufficiently large value of $\sigma$, we have

$$
\ln^{[p]} M_{\mu}(\sigma, F) \geq (\Im - \varepsilon) \ln^{[q]} \sigma + o(1).
$$

For $p = q + 1 \geq 3$, we have $\lambda \geq 1$, so we have, by the above inequality,

\begin{equation}
(2.16) \quad \lim_{\sigma \to \infty} \inf \frac{\ln^{[p]} M_{\mu}(\sigma, F)}{\ln^{[q]} \sigma} \geq \max (1, \Im).
\end{equation}

For $p > q + 1$, the same inequality implies

\begin{equation}
(2.17) \quad \lim_{\sigma \to \infty} \inf \frac{\ln^{[p]} M_{\mu}(\sigma, F)}{\ln^{[q]} \sigma} \geq \Im.
\end{equation}

Now, for $(p, q) = (2, 1)$, as $1 \leq \Im < \infty$, we choose

\begin{equation}
(2.18) \quad \sigma_k = 2 \left( \frac{1}{\Im - \varepsilon} \right)^{1/(\Im - \varepsilon)}, \quad k = 1, 2, \ldots
\end{equation}

If $\sigma_k \leq \sigma < \sigma_{k+1}$, after proceeding as in the previous case, we get

$$
\ln M_{\mu}(\sigma, F) \geq \lambda_{n_k} \left( \lambda_{n_{k-1}} \right)^{1/(\Im - \varepsilon)},
$$

which gives

$$
\frac{\ln \ln M_{\mu}(\sigma, F)}{\ln \sigma} \geq \frac{\ln \lambda_{n_k}}{\ln \sigma_{k+1}} + \frac{1}{(\Im - \varepsilon)} \frac{\ln \lambda_{n_{k-1}}}{\ln \sigma_{k+1}}.
$$
After substituting the value of \( \sigma_{k+1} \), from (2.18) and proceeding to limits as \( k \to \infty \), we get
\[
(2.19) \quad \lambda \geq \Im + \chi.
\]
After combining (2.16), (2.17) and (2.19) we get the desired result for \( 0 < \Im < \infty \).
If \( \Im = 0 \) (2.12) is trivially true and if \( \Im = \infty \), then, repeating the above arguments with an arbitrarily large number in place of \( (\Im - \varepsilon) \), we get the desired result. This completes the proof of the Lemma 2.

In the next result, we characterize the lower \((p,q)\)-order in terms of the ratio \( \{A_{n-1}^*/A_n^*\} \). We prove

**Lemma 3.** Let \( F(s) = \int_0^\infty \exp(sy)d\alpha(y) \) be Laplace-Stieltjes transformation having lower \((p,q)\)-order \( \lambda \) and let \( \{n_k\}_{k=1}^{\infty} \) be an increasing sequence of natural numbers, then
\[
(2.20) \quad \lambda \geq C_\chi(\Im^*)
\]
where
\[
(2.21) \quad \Im^* = \Im^*(\{n_k\}, p, q) = \lim_{k \to \infty} \inf \left\{ \frac{\ln^{[p-1]} \lambda_{n_{k-1}}}{\ln^{[q]} \left\{ \left( \ln A_{n_{k-1}}^*/A_n^* \right) / (\lambda_{n_k} - \lambda_{n_{k-1}}) \right\} } \right\},
\]
and \( \chi \) is as defined in Lemma 2.

**Proof.** If \( \Im^* \) is zero then (2.20) is trivially true, therefore it is sufficient to consider the case when \( 0 < \Im^* \leq \infty \). For any \( \varepsilon \) such that \( \Im^* > \varepsilon > 0 \) and for all \( k > k_1 = k_1(\varepsilon) \), from (2.21), we have
\[
\left( A_{n_{k-1}}^*/A_{n_k}^* \right) < \exp \left[ (\lambda_{n_k} - \lambda_{n_{k-1}}) \exp^{[q]} \left\{ \ln^{[p-1]} \lambda_{n_{k-1}} / (\Im^* - \varepsilon) \right\} \right].
\]
Writing the above inequality for \( m = k_1 + 1, ..., k \) and multiplying side by side, we get
\[
(2.22) \quad A_{n_{k-1}}^* > A_{n_{k_1}}^* \prod_{m=k_1+1}^k \exp \left[ - (\lambda_{n_m} - \lambda_{n_{m-1}}) \exp^{[q]} \left\{ \ln^{[p-1]} \lambda_{n_{m-1}} / (\Im^* - \varepsilon) \right\} \right].
\]
Let the sequence \( \{\sigma_k\} \) be chosen as in equations (2.15) and (2.18) for the two cases and let \( \sigma_k \leq \sigma < \sigma_{k+1} \). Then for all \( \sigma > \sigma_o = \sigma_o(\varepsilon) \), we have by (2.22)
\[
\ln M_\mu(\sigma, F) \geq \ln A_{n_k}^* + \lambda_n \sigma_k
\]
\[
> \ln A_{n_{k_1}}^* - \sum_{m=k_1+1}^k (\lambda_{n_m} - \lambda_{n_{m-1}}) \exp^{[q]} \left\{ 1/((\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_{m-1}}) \right\} + \lambda_n \sigma_k
\]
\[
= \ln A_{n_{k_1}}^* - \lambda_n \exp^{[q]} \left\{ 1/((\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_{m-1}}) \right\}
\]
\[
+ \sum_{m=k_1}^{k-1} \lambda_{n_m} \left\{ \exp^{[q]} \left\{ 1/((\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_{m-1}}) \right\} - \exp^{[q]} \left\{ 1/((\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_{m-1}}) \right\} \right\}
\]
\[
> \ln A_{n_{k_1}}^* - \lambda_n \exp^{[q]} \left\{ 1/((\Im^* - \varepsilon) \ln^{[p-1]} \lambda_{n_{k-1}}) \right\} + \lambda_n \sigma_k.
\]
Now, choosing the sequence \( \{\sigma_k\} \) for the two cases and proceeding as in Lemma 2 we get the desired result (2.20).

In the next result, we obtain the reverse estimates for the lower \((p,q)\) order. We have

**Lemma 4.** Let \( F(s) = \int_0^\infty \exp(st)\,d\alpha(y) \) be Laplace-Stieltjes transformation having lower \((p,q)\)-order \( \lambda \) such that \( \theta(n) \equiv \frac{\ln(A_n^*/A_{n+1}^*)}{\lambda_n-1} \) forms a nondecreasing function of \( n \) for \( n > n_o \). Then

\[
\lambda \leq C(\mathcal{S}_o)
\]

and

\[
\lambda \leq C(\mathcal{S}_o^*)
\]

where

\[
\mathcal{S}_o \equiv \mathcal{S}_o(p,q) = \lim_{n \to \infty} \frac{\ln^{[p-1]}\lambda_n}{\ln^{[q]}\{1/\lambda_n \ln (A_n^*)^{-1}\}},
\]

and

\[
\mathcal{S}_o^* \equiv \mathcal{S}_o^*(p,q) = \lim_{n \to \infty} \frac{\ln^{[p-1]}\lambda_n}{\ln^{[q]}\{1/(\lambda_n - \lambda_n)^{-1}\} \ln (A_n^*/A_{n+1}^*)}.
\]

**Proof.** We know that, if \( \mu(\sigma,F) = A_n^e^{\lambda_n^*} \) is the maximum term for a given \( \sigma \), then

\[
A_n^* e^{\lambda_n^*-1} \leq A_n^* e^{\lambda_n^*} \leq A_{n+1}^* e^{\lambda_{n+1}^*}
\]

or

\[
\theta(n-1) \leq \sigma < \theta(n).
\]

Hence, for infinitely many values of \( n \), \( \theta(n) > \theta(n-1) \) and \( \theta(n) \to \infty \) as \( n \to \infty \). When \( \theta(n) > \theta(n-1) \), we have \( \mu(\sigma,F) = \max_{n \in \mathbb{N}} \{A_n^* e^{\lambda_n^*}\} \), \( \sigma > 0 \) and \( N(\sigma,F) = n \) for \( \theta(n-1) \leq \sigma < \theta(n) \).

Now, let

\[
\phi = \lim_{\sigma \to \infty} \frac{\ln^{[p-1]}\lambda_N(\sigma,F)}{\ln^{[q]}\sigma}.
\]

First, we assume that \( \phi > 0 \). For any \( \varepsilon \) such that \( \phi > \varepsilon > 0 \), and for all \( \sigma > \sigma_o(\varepsilon) \), we have

\[
\lambda_{N(\sigma,F)} > \exp^{[p-1]}\left\{ (\phi - \varepsilon) \ln^{[q]}\sigma \right\}.
\]

Now, let \( A_{n_1}^* \exp(\lambda_{n_1}s) \) and \( A_{n_2}^* \exp(\lambda_{n_2}s) \), \((n_1 > n_o, \theta(n_1 - 1) > \sigma_o)\), be two consecutive maximum terms of \( F(s) \) so that \( n_1 \leq n_2 - 1 \). Let \( n_1 < n < n_2 \). Since \( A_{n_1}^* \exp(\lambda_{n_1}s) \) is the maximum term we have \( N(\sigma,F) = n_1 \) for \( \theta(n_1 - 1) \leq \sigma < \theta(n_1) \). For \( \sigma \) in this interval,

\[
\lambda_{n_1} > \exp^{[p-1]}\left\{ (\phi - \varepsilon) \ln^{[q]}\sigma \right\}.
\]
Since \( \theta(n_1) = \theta(n_1 + 1) = \ldots = \theta(n - 1) \), we have
\[
\lambda_{n-1} \geq \lambda_{n_1} > \exp^{[p-1]} \left\{ (\phi - \varepsilon) \ln^{[q]} \left( \theta(n - 1) - \delta \right) \right\}
\]  
where \( \delta = \max \{1, |\theta(n_1) - \theta(n_1 - 1)|/2\} \). Since \( \theta(n) \) is nondecreasing, we have
\[
\ln |A_{n_0}^*/A_{n_0+1}^*| + \ldots + \ln |A_{n-1}^*/A_n^*| = \ln |A_{n_0}^*/A_{n}^*| = \sum_{k=n_0}^{n-1} (\lambda_{k+1} - \lambda_k) \theta(k) \
\leq (\lambda_n - \lambda_{n_0}) \theta(n - 1), \quad (n > n_0),
\]
and hence
\[
(2.26) \quad \ln^{[q]} \left( \frac{1}{\lambda_n} \ln (A_n^*)^{-1} \right) \leq \ln^{[q]} \theta(n - 1) + o(1).
\]
From (2.26), we have
\[
(2.27) \quad \ln^{[p-1]} \lambda_{n-1} \geq \left\{ (\phi - \varepsilon) \ln^{[q]} \left( \theta(n - 1) - \delta \right) \right\}.
\]
Combining (2.27) and (2.28), we get
\[
\ln^{[q]} \left( \frac{1}{\lambda_n} \ln (A_n^*)^{-1} \right) \leq \frac{1}{(\phi - \varepsilon)} + o(1).
\]
By passing to limits, the above inequality leads to
\[
(2.29) \quad \phi \leq \Im_o.
\]
When \( \phi = 0 \), the above inequality is evidently true. It follows from (2.29) that \( \lambda = P(\phi) \leq P(\Im_o) \) [using Lemma 1]. This proves (2.23). To prove (2.24), we find from (2.28) that for sufficiently large \( n \), and for any \( \varepsilon \) such that \( \phi > \varepsilon > 0 \),
\[
\frac{\ln^{[p-1]} \lambda_{n-1}}{\ln^{[q]} (\theta(n - 1) - \delta)} \geq \phi - \varepsilon.
\]
After substituting the value of \( \theta(n - 1) \) and passing to limits, we get
\[
\phi \leq \Im^*_o.
\]
The inequality is obvious if \( \phi = 0 \).

So, from the last inequality, we get \( \lambda = C(\phi) \leq C(\Im^*_o) \) [Lemma 1] and this completes the proof of Lemma 4.

Combining the results of Lemmas 2, 3 and 4, we obtain the following characterization of lower \((p, q)\) order.
Theorem 2. Let $F(s) = \int_0^\infty \exp(sy)d\alpha(y)$ be Laplace-Stieltjes transformation having lower $(p,q)$-order $\lambda$ such that $\theta(n) \equiv \frac{\ln(A_n^{*}/A_{n+1}^{*})}{\lambda_{n+1}-\lambda_{n}}$ forms a nondecreasing function of $n$ for $n > n_{o}$ and (1.3) holds. Then, for $(p,q) \neq (2,1)$,

$$\lambda = C(\Im_{o}) = C(\Im_{o}^{*}),$$

where $\Im_{o}$ and $\Im_{o}^{*}$ are defined as in Lemma 3.

Further, (2.30) holds for $(p,q) = (2,1)$ also if $\ln \lambda_{n} \simeq \ln \lambda_{n+1}$, as $n \to \infty$.

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References


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