ON MARGINAL AUTOMORPHISM OF INFINITE GROUPS

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Abstract. Let W be a non-empty subset of a free group and G be any group. An automorphism α of G is called marginal automorphism if $g^{-1}\alpha(g) \in W^*(G)$ for any $g \in G$ where $W^*(G)$ is marginal subgroup of G. In this paper we study some properties of marginal automorphism of infinite group G.

Keywords: marginal automorphism, purely non-abelian group, finitely generated group.

1. Introduction

Let F be a free group on a countably infinite set $\{x_1, x_2, ...\}$ and let W be a nonempty subset of F. If $w = {x_1}^{l_1}...{x_r}^{l_r} \in W$ and $g_1, ..., g_r$ are elements of a arbitrary group G, the *value* of the word w at $(g_1...., g_r)$ is $w(g_1, ..., g_r) = {g_1}^{l_1}...{g_r}^{l_r}$. We define

$$W(G) =< w(g_1, g_2, ...)|g_i \in G, w \in W > .$$

$$W^*(G) = \{g \in G|w(g_1, ..., g_{i-1}g_ig, g_{i+1}, ..., g_r) = w(g_1, ..., ..., g_r) \text{ for all } g_i \in G, \\ a \in N \text{ and all } w(x_1, x_2, ..., x_r) \in W\}.$$

$W(G)$ and $W^*(G)$ are the verbal subgroup and marginal subgroup of G with respect to W, respectively. Clearly that $W(G)$ is fully-invariant and $W^*(G)$ is always characteristic subgroup of G (see [9] for more information).

An automorphism σ of G is called marginal with respect to W if $g^{-1}\sigma(g) \in W^*(G)$ for all $g \in G$. The set of all marginal automorphism of G is normal subgroup of $\mathrm{Aut}(G)$ which is denoted by $\mathrm{Aut}_{W^*}(G)$.

An automorphism α of G is called verbal automorphism with respect to W if $g^{-1}\sigma(g) \in W(G)$ for all $g \in G$. The set of all verbal automorphism of G, denoted by $\mathrm{Aut}_W(G)$, is a normal subgroup of $\mathrm{Aut}(G)$.

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A variety is an equationally defined class of groups. More precisely the class of all groups G such that $W(G) = 1$, or equivalently $W^*(G) = G$ is called the variety ν determined by W. In particular, if $W = \{[x_1, x_2]\}$ where $[x_1, x_2] = {x_1}^{-1}{x_2}^{-1}x_1x_2$, then variety ν is the class of abelian groups and $W^*(G) = Z(G)$ and $W(G) = G'$. In abelian variety, $\mathrm{Aut}_W(G) = \mathrm{IA}(G)$, and it is called IA-automorphism of G, also $\mathrm{Aut}_{W^*}(G) = \mathrm{Autcent}(G)$, and it is called central automorphism of G, see also [1]-[3], [6], [7].

Franciosi, De Giovanni and M.L. Newell in [3], proved that if G be a purely non-abelian such that $Z(G)$ or $\dfrac{G}{G'}$ is torsion with finite abelian section rank, then $\gamma : \mathrm{Autcent}(G) \longrightarrow \mathrm{Hom}(G, Z(G))$ is a bijection map. Also they characterized such groups that have nilpotent central automorphism.

Rai in [8] showed that for a finite p-group G, $C_{IA(G)}Z(G) = \mathrm{Autcent}\ (G)$ if and only if $G' = Z(G)$. ($C_{IA(G)}Z(G)$ is the set of all IA-automorphism of G fixing $Z(G)$ element-wise.)

Jamali and Mosavi in [6] using the concept of isoclinic obtained the relation between central automorphism of two isoclinic groups. They proved that if G is a group such that $Z(G) \leq G'$ and H be a group isoclinic to G, then there is a monomorphism from $\mathrm{Aut}_c(G)$ into $\mathrm{Aut}_c(H)$.

In the present paper, we generalize the above result to marginal automorphism and prove the following theorems:

Theorem 1.1 *Let ν be a variety and $\emptyset \neq W \subseteq F$ and let G, H be two ν-isologism groups such that $W^*(G) \leq W(G)$ and $W^*(H) \leq W(H)$. Then*

$$Aut_{W^*}(G) \simeq Aut_{W^*}(H).$$

Theorem 1.2 *Let G be a purely non-abelian group and $\emptyset \neq W \subseteq F$ such that $W^*(G) \leq Z(G)$. If*

(i) *$W^*(G)$ is torsion group with finite abelian section rank, or*

(ii) *$\dfrac{G}{W(G)}$ is torsion group with finite abelian section rank and $W(G) \leq G'$.*

Then there is a bijection map between $Aut_{W^*}(G)$ and Hom $\left(\dfrac{G}{W(G)}, W^*(G)\right)$.

Theorem 1.3 *Let G be a finitely generated purely non-abelian group and $\emptyset \neq W \subseteq F$ such that $G' \leq W(G) \leq W^*(G) \leq Z(G)$ and $W^*(G)$ is torsion with finite abelian section rank. Then $C_{Aut_W(G)}(W^*(G)) = Aut_{W^*}(G)$ if and only if $W^*(G) = W(G)$.*

Theorem 1.4 *Let G be a purely non-abelian group such that $W^*(G)$ or $\dfrac{G}{W(G)}$ is finite and $W^*(G) \leq Z(G)$. Then $Aut_{W^*}(G)$ is a nilpotent group of finite exponent.*

2. Preliminary Lemmas

Notation in this paper is standard. Let G be a group, by $Z(G)$, G',Aut(G) we denoted the center, the commutator subgroup and the group of all automorphism, respectively. Also a non abelian group G that has no non-trivial abelian direct factor is said to be purely non-abelian group.

Let G be an abelian group, the element of finite order of G form a subgroup $T(G)$, the torsion subgroup of G. Moreover, elements with order some power of a fixed prime p likewise form a subgroup G_p, the p-primary component of G. By The Primary Decomposition Theorem, the torsion-subgroup T is the direct sum of the primary components of G.

In this section, we collect all results that will be used in Section 3. We need the following results of Attar [7] that are useful in our investigations.

Lemma 2.5 ([7]) *Let G be a group and $\emptyset \neq W \subseteq F$. Every marginal automorphism of G leaves fixed every element of the verbal subgroup $W(G)$.*

Proposition 2.6 ([7]) *Let G be a group and and $\emptyset \neq W \subseteq F$. If $W^*(G)$ is abelian, then $Hom\big(G, W^*(G)\big) \simeq Hom\big(\frac{G}{W(G)}, W^*(G)\big)$.*

The next theorem, determines structure of $\mathrm{Aut}_W(G)$ and the set of all verbal automorphism of G fixing $W^*(G)$ element-wise.

Theorem 2.7 *Let G be a group and $\emptyset \neq W \subseteq F$ such that $W(G) \leq W^*(G) \cap Z(G)$. Then*

(i) $Aut_W(G) \simeq Hom\Big(\dfrac{G}{W(G)}, W(G)\Big)$.

(ii) $C_{Aut_W(G)}(W^*(G)) \simeq Hom\Big(\dfrac{G}{W^*(G)}, W(G)\Big)$.

Proof. (i) Let $\sigma : \mathrm{Aut}_W(G) \longrightarrow \mathrm{Hom}\Big(\dfrac{G}{W(G)}, W(G)\Big)$ defined by $\sigma(f) = \sigma_f$, where $\sigma_f\big(gW(G)\big) = g^{-1}f(g)$, for each $f \in \mathrm{Aut}_W(G)$ and $g \in G$. Since $g^{-1}f(g) \in W(G) \leq Z(G)$, it is clear that $\sigma_f \in \mathrm{Hom}\Big(\dfrac{G}{W(G)}, W(G)\Big)$, i.e., σ is well-defined.

Let $f_1, f_2 \in \mathrm{Aut}_W(G)$ since here $W(G) \leq W^*(G)$, so $\mathrm{Aut}_W(G) \leq \mathrm{Aut}_{W^*}(G)$ and by Lemma 2.5,

$$\sigma_{f_1 f_2}\big(gW(G)\big) = g^{-1}f_1\big(gg^{-1}f_2(g)\big) = g^{-1}f_1(g)g^{-1}f_2(g)$$

Therefore, $\sigma(f_1 f_2) = \sigma(f_1)\sigma(f_2)$ and σ is a homomorphism. Clearly, σ is injective, so it remains to show that σ is surjective. Let $h \in Hom\Big(\dfrac{G}{W(G)}, W(G)\Big)$, we define the map $f : G \longrightarrow G$ by $f(g) = gh\big(gW(G)\big)$ for all $g \in G$. It is obvious that f is a monomorphism. Now, $g = gh\big(gW(G)\big)\big(h(gW(G))\big)^{-1}$ for every $g \in G$. Since

$h\big(gW(G)\big) \in W(G)$, so $f\big(h(gW(G))\big)^{-1} = \big(h(gW(G))\big)^{-1}$, which implies that f is surjective. Also we have $g^{-1}f(g) = g^{-1}gh\big(gW(G)\big) \in W(G)$, so $f \in Aut_W(G)$ such that $\sigma(f) = h$. Therefore, σ is an isomorphism.

(ii) If we utilize the above method for $C_{Aut_W(G)}(W^*(G))$, the result is obtained. ∎

Similar to Theorem 2.7, we obtain the following lemma, used in proof of Theorem 1.4.

Lemma 2.8 *Let G be a group and $\emptyset \neq W \subseteq F$ such that $W^*(G) \leq Z(G)$. Then*

(i) $C_{Aut_{W^*(G)}}(W^*(G)) \simeq Hom\Big(\dfrac{G}{W^*(G)W(G)}, W^*(G)\Big)$.

(ii) $C_{Aut_{W^*(G)}}\Big(\dfrac{G}{W(G)}\Big) \simeq Hom\Big(\dfrac{G}{W(G)}, W(G) \cap W^*(G)\Big)$.

The items of the following lemma are well-known and easily verified.

Lemma 2.9 *Let U, V and W be abelian groups. Then*

(i) $Hom(U \times V, W) \simeq Hom(U, W) \times Hom(V, W)$.

(ii) $Hom(U, V \times W) \simeq Hom(U, V) \times Hom(U, W)$.

(iv) *If U is torsion-free of rank n, then $Hom(U, V) \simeq V^n$.*

(v) *If U is a torsion and V is a torsion-free group, then $Hom(U, V) = 1$.*

The following concept was first introduced by Hall [4].

Definition 2.10 Let ν be a variety of groups defined by the set of laws W and G and H be groups. A ν-isologism between G and H is a pair of isomorphisms (α, β) with $\alpha : \dfrac{G}{W^*(G)} \longrightarrow \dfrac{H}{W^*(H)}$ and $\beta : W(G) \longrightarrow W(H)$, such that for all $s > 0$, all $w(x_1, ..., x_s) \in V(F_s)$ and all $g_1, ..., g_s \in G$, it holds that $\beta\big(w(g_1, ..., g_s)\big) = w(h_1 ..., h_s)$, whenever $h_i \in \alpha\big(g_i W^*(G)\big)$ $(i = 1, ..., s)$. We write $G \underset{\nu}{\sim} H$ and we will say that G and H are ν-isologic.

In abelian variety, two group G and H in above definition are said to be isoclinic.

Lemma 2.11 ([5]) *Let G and H be two groups and $\emptyset \neq W \subseteq F$. Let (α, β) be a ν-isologism between G and H. Let $v \in W(G)$. Then the following hold.*

(i) $\alpha\big(vW^*(G)\big) = \beta(v)W^*(H)$.

(ii) *If $g \in G$ and $h \in \alpha(gW^*(G))$, then $\beta(v^g) = \beta(v)^h$.*

Corollary 2.12 *Let ν be a variety and $\emptyset \neq W \subseteq F$ and let (α, β) be a ν-isologism between G and H. Then β induces an isomorphism from $W^*(G) \cap W(G)$ onto $W^*(H) \cap W(H)$.*

3. Main results

Proof of Theorem 1.1. Let (α, β) be a ν-isologism between G and H. Suppose $\sigma \in Aut_{W^*}(G)$ and $h \in H$, since α is surjective, there exists $g \in G$ such that $\alpha(gW^*(G)) = hW^*(H)$. Define a map $f_\sigma : H \longrightarrow H$ by $f_\sigma(h) = h\beta(g^{-1}\sigma(g))$ for any $h \in H$.

(i) f_σ is well-defined:
Let $h_1, h_2 \in H$ and $\alpha(g_iW^*(G)) = h_iW^*(H)$ for $i = 1, 2$. If $h_1 = h_2$ then $\alpha(g_1W^*(G)) = \alpha(g_2W^*(G))$ so $g_1{g_2}^{-1} \in W^*(G) \leq W(G)$, since σ fixes $W(G)$ element-wise hence ${g_1}^{-1}\sigma(g_1) = {g_2}^{-1}\sigma(g_2)$ and $f_\sigma(h_1) = f_\sigma(h_2)$

(ii) f_σ is homomorphism:
Let $h_1, h_2 \in H$ and $\alpha(g_iW^*(G)) = h_iW^*(H)$ for $i = 1, 2$. Then,

$$\begin{aligned} f_\sigma(h_1h_2) &= h_1h_2\beta\big((g_1g_2)^{-1}\sigma(g_1g_2)\big) \\ &= h_1h_2\beta\big({g_2}^{-1}{g_1}^{-1}\sigma(g_1)g_2{g_2}^{-1}\sigma(g_2)\big) \\ &= h_1\Big(\beta\big(({g_1}^{-1}\sigma(g_1))^{g_2}\big)\Big)^{{h_2}^{-1}} h_2\beta\big({g_2}^{-1}\sigma(g_2)\big). \end{aligned}$$

Applying Lemma 2.11, we obtain $\beta\big(({g_1}^{-1}\sigma(g_1))^{g_2}\big) = \beta\big({g_1}^{-1}\sigma(g_1)\big)^{h_2}$ so $f_\sigma(h_1h_2) = h_1\beta\big({g_1}^{-1}\sigma(g_1))h_2\beta({g_2}^{-1}\sigma(g_2)\big) = f_\sigma(h_1)f_\sigma(h_2)$.

(iii) f_σ is injective:
Let $f_\sigma(h) = 1$ where $\alpha(gW^*(G)) = hW^*(H)$, so $\beta(g^{-1}\sigma(g)) = h^{-1}$. By Lemma 2.11, $\alpha\big(g^{-1}\sigma(g)W^*(G)\big) = \beta\big(g^{-1}\sigma(g)\big)W^*(H) = h^{-1}W^*(H)$,

$$g^{-1}\sigma(g)W^*(G) = \alpha^{-1}(h^{-1}W^*(H)) = \alpha^{-1}(hW^*(H))^{-1} = g^{-1}W^*(G).$$

Hence $\sigma(g) \in W^*(G)$, since $W^*(G)$ is characteristic in G, $g \in W^*(G)$ that Lemma 2.5, implies $\sigma(g) = g$, so $h = 1$.

(iv) f_σ is surjective:
Let $h \in H$ where $\alpha(gW^*(G)) = hW^*(H)$. $h = \underbrace{h\beta(g^{-1}\sigma(g))}_{\in Imf_\sigma}(\beta(g^{-1}\sigma(g)))^{-1}$.
By Lemma 2.11, $\beta(g^{-1}\sigma(g))W^*(H) = \alpha\Big(\big(g^{-1}\sigma(g)\big)W^*(G)\Big)$.
Since $g^{-1}\sigma(g) \in W^*(G) \leq W(G)$ and σ fixes $W(G)$ element-wise.

$$\begin{aligned} f_\sigma\big(\beta(g^{-1}\sigma(g))\big) &= \beta(g^{-1}\sigma(g))\beta(g^{-1}\sigma(g))^{-1}\beta(\sigma(g^{-1}\sigma(g))) \\ &= \beta\big(\sigma(g^{-1}\sigma(g))\big) \\ &= \beta\big(g^{-1}\sigma(g)\big). \end{aligned}$$

Hence $H \leq Imf_\sigma$.

(v) $h^{-1}f_\sigma(h) = \beta(g^{-1}\sigma(g))$, for any $h \in H$ where $\alpha(gW^*(G)) = hW^*(H)$. Since $g^{-1}\sigma(g) \in W^*(G)$ and $W^*(G) \cap W(G) = W^*(G)$ by Lemma 2.12, $\beta(g^{-1}\sigma(g)) \in W^*(H) \cap W(H) = W^*(H)$. Hence, $f_\sigma \in Aut_{W^*}(H)$.

Let $\gamma : Aut_{W^*}(G) \longrightarrow Aut_{W^*}(H)$ such that $\gamma(\sigma) = f_\sigma$. We show that γ is an isomorphism. Suppose $\sigma_1, \sigma_2 \in \mathrm{Aut}_{W^*}(G)$ and $h \in H$ such that $\alpha(gW^*(G)) = hW^*(H)$. Then,

$$\begin{aligned}\gamma(\sigma_1\sigma_2) &= f_{\sigma_1\sigma_2}(h) = h\beta(g^{-1}\sigma_1\sigma_2(g)) \\ &= h\beta\big(g^{-1}\sigma_2(g)(\sigma_2(g))^{-1}\sigma_1(\sigma_2(g))\big) \\ &= h\beta\big(g^{-1}\sigma_2(g)\big)\beta\big((\sigma_2(g)^{-1}\sigma_1(\sigma_2(g))\big).\end{aligned}$$

Since $g^{-1}\sigma_2(g) \in W^*(G) \leq W(G)$, with use of Lemma 2.11, we have

$$\begin{aligned}\alpha^{-1}\Big(h\beta(g^{-1}\sigma_2(g))W^*(H)\Big) &= \alpha^{-1}\big(hW^*(H)\big)\alpha^{-1}\Big(\beta(g^{-1}\sigma_2(g))W^*(H)\Big) \\ &= gW^*(G)\alpha^{-1}\Big(\alpha\big(g^{-1}\sigma_2(g)W^*(G)\big)\Big) \\ &= \sigma_2(g)W^*(G).\end{aligned}$$

Therefore,

$$\gamma(\sigma_1\sigma_2) = f_{\sigma_1}\big(h\beta\big(g^{-1}\sigma_2(g)\big)\big) = f_{\sigma_1}f_{\sigma_2}(h) = \gamma(\sigma_1)\gamma(\sigma_2)$$

It remains to prove that γ is a bijection. Notice that $(\alpha^{-1}, \beta^{-1})$ is a ν-isologism between H and G. Let $f \in \mathrm{Aut}_{W^*}(H)$, we define $\sigma_f(g) = g\beta^{-1}(h^{-1}f(h))$ for any $g \in G$, where $\alpha^{-1}(hW^*(H)) = gW^*(G)$.

In a similar proof as above, $\sigma_f \in Aut_{W^*}(G)$.

Now, we define $\delta : Aut_{W^*}(H) \longrightarrow Aut_{W^*}(G)$ as $\delta(f) = \sigma_f$.

Let $g \in G$ where $\alpha(gW^*(G)) = hW^*(H)$. Then

$$\delta\gamma(\sigma)(g) = \delta(f_\sigma)(h) = g\beta^{-1}(h^{-1}f_\sigma(h)) = g\beta^{-1}(h^{-1}h\beta(g^{-1}\sigma(g)) = \sigma(g).$$

So $\delta\gamma(\sigma) = \sigma$ i.e. $\delta\gamma = 1$, similarly $\gamma\delta = 1$. Thus γ is an isomorphism. ∎

Let $\sigma \in \mathrm{Autcent}(G)$ and $x \in G$ so the map $x \longrightarrow x^{-1}\sigma(x)$ defines a homomorphism from G into $Z(G)$. On the other hand the map $x \longrightarrow xf(x)$ defines an endomorphism of G for every f in $\mathrm{Hom}(G, Z(G))$. If G is a finite group, this endomorphism is a central automorphism if and only if $f(x) \neq x^{-1}$ for every $1 \neq x \in G$. In [1], Adney and Yen show that a necessary and sufficient condition for a finite group G to have such a bijection between $\mathrm{Autcent}(G)$ and $\mathrm{Hom}(G, Z(G))$ is that G has no abelian direct factors. Franciosi, De Giovanni and M.L. Newell in [3], with use of following lemma extend this result to a wider class containing all periodic groups with finite abelian section rank.

Lemma 3.13 ([3]) *Let G be a group and $\alpha \in Hom(G, Z(G))$ such that $\alpha(g) = g^{-1}$ for some $g \neq 1$. If $Im\alpha$ is the direct product of a torsion group with finite abelian section rank and a free abelian group, then G has a non-trivial abelian direct factor.*

Here we want to obtain the corresponding statement of Franciosi and De Giovanni theorem in arbitrary variety of group. If $W^*(G)$ is contained in the center of G then we have same result of Lemma 3.13 for $\alpha \in \mathrm{Hom}(G, W^*(G))$.

Proof of Theorem 1.2. By Proposition 2.6, it is enough to show that there is a bijection map between $\mathrm{Aut}_{W^*}(G)$ and $\mathrm{Hom}\big(G, W^*(G)\big)$. Let $\alpha \in \mathrm{Aut}_{W^*}(G)$ and $g \in G$ now we define $\alpha^* : G \longrightarrow W^*(G)$ such that $\alpha^*(g) = g^{-1}\alpha(g)$. From $g^{-1}\alpha(g) \in W^*(G) \leq Z(G)$ is obtained that α^* is a homomorphism so the map

$$\begin{aligned} \psi : \mathrm{Aut}_{W^*}(G) &\longrightarrow \mathrm{Hom}\big(G, W^*(G)\big) \\ \alpha &\longrightarrow \alpha^* \\ &\qquad \alpha^*(g) = g^{-1}\alpha(g) \end{aligned}$$

is well-defined. It is clear that ψ is a one-to-one map. Our map is surjective, for this let $\gamma \in \mathrm{Hom}\big(G, W^*(G)\big)$, we prove the map $\alpha : G \longrightarrow G$ where $\alpha(g) = g\gamma(g)$ is a marginal automorphism of G such that $\psi(\alpha) = \gamma$. Since $\gamma(g) \in W^*(G) \leq Z(G)$, α is an endomorphism of G. With let composition of the γ and incidence mapping $W^*(G) \longrightarrow Z(G)$, we have $\dfrac{G}{ker\gamma} \simeq Im\gamma \leq W^*(G) \leq Z(G)$ and $G' \leq ker\gamma$. Here by hypothesis $Im\gamma$ is torsion with finite abelian section rank. If $g \in ker\alpha$ then $\alpha(g) = 1$ so $\gamma(g) = g^{-1}$ that by Lemma 3.13, $g = 1$. Now, we show that α is onto and this concludes the proof of theorem. Let $T = Im\gamma$, the p-component T_p of T satisfies the minimal condition on subgroups for each prime p. Since $T_p \simeq \alpha(T_p)$, it follows that $\alpha(T_p) = T_p$, and T is contained in $Im\alpha$. Now $g = g\gamma(g)(\gamma(g))^{-1} \in Im\alpha$ for any $g \in G$, so $Im\alpha = G$. ∎

By the previous theorem, we can study the relation between $C_{Aut_W(G)}(W^*(G))$ and $\mathrm{Aut}_{W^*}(G)$. Let G be any group and $W^*(G) \leq Z(G)$ and $\dfrac{G}{W(G)}$ be torsion group so, for any $\sigma \in \mathrm{Aut}_{W^*}(G)$ and $g \in G$, there exsits $n \in \mathbb{N}$ such that $g^n \in W(G)$ and by Lemma 2.5, $\sigma(g^n) = g^n$. Since $g^{-1}\sigma(g) \in W^*(G) \leq Z(G)$, $(g^{-1}\sigma(g))^n = 1$. Now, if $W^*(G)$ is torsion-free then $\mathrm{Aut}_{W^*}(G) = 1$. Moreover, if $W(G) \leq W^*(G)$ then $\dfrac{G}{W^*(G)}$ is torsion and $W(G)$ is torsion-free so, by Lemmas 2.8 and 2.9, $C_{Aut_W(G)}(W^*(G)) = 1$. Therefore, we have Theorem 1.3.

Proof of Theorem 1.3. Let $W^*(G) = W(G)$ so, by Lemma 2.5, result is obtained clearly.

Conversely, suppose $C_{Aut_W(G)}(W^*(G)) = \mathrm{Aut}_{W^*}(G)$ and $W(G) \neq W^*(G)$ then $W(G) < W^*(G)$. By Theorems 1.2 and 2.7, there is a bijection between $\mathrm{Hom}\Big(\dfrac{G}{W^*(G)}, W(G)\Big)$ and $\mathrm{Hom}\,\Big(\dfrac{G}{W(G)}, W^*(G)\Big)$.

G is finitely generated and $G' \leq W(G) \leq W^*(G)$ and $W^*(G)$ is torsion so $W^*(G), W(G)$ are finite and $\dfrac{G}{W(G)}, \dfrac{G}{W^*(G)}$ are finitely generated abelian groups. Let $\dfrac{G}{W(G)} = T\big(\dfrac{G}{W(G)}\big) \times \mathbb{Z}^r$ and $\dfrac{G}{W^*(G)} = T\big(\dfrac{G}{W^*(G)}\big) \times \mathbb{Z}^s$ for $r, s \geqslant 0$.

By Lemma 2.9,

$$\left|\mathrm{Hom}\left(T\Big(\frac{G}{W^*(G)}\Big), W(G)\right)\right| \cdot |W(G)|^s = \left|\mathrm{Hom}\left(T\Big(\frac{G}{W(G)}\Big), W^*(G)\right)\right| \cdot |W^*(G)|^r$$

$$\frac{G/W(G)}{W^*(G)/W(G)} \simeq \frac{G}{W^*(G)} \text{ so } \frac{T\big(G/W(G)\big)}{W^*(G)/W(G)} \times \mathbb{Z}^r \simeq T\big(G/W^*(G)\big) \times \mathbb{Z}^s.$$

Hence $r = s$ and $\dfrac{|T\big(G/W(G)\big)|}{|T\big(G/W^*(G)\big)|} = \dfrac{|W^*(G)|}{|W(G)|}$ and, since $W(G) \neq W^*(G)$, we get $\left|T\left(\frac{G}{W^*(G)}\right)\right| < \left|T\left(\frac{G}{W(G)}\right)\right|$. Now, by Lemma 2-8 of [2],

$$\left|\mathrm{Hom}\left(T\left(\frac{G}{W^*(G)}\right), W(G)\right)\right| < \left|\mathrm{Hom}\left(T\left(\frac{G}{W(G)}\right), W^*(G)\right)\right|,$$

which is a contradiction. ■

To determine a group that its marginal automorphism is nilpotent, we need the following lemma of [3].

Lemma 3.14 ([3]) *Let G he a purely non-abelian group, and let α be an automorphism of G such that $\alpha^n = 1$ for some positive integer n. If M is a torsion central subgroup of G such that $[G, \alpha] \leq M$, then $[G, \alpha] \leq M_\pi$, where π is the set of prime divisors of n.*

Lemma 3.15 *Let G be a purely non-abelian group and $\emptyset \neq W \subseteq F$ such that $W^*(G) \leq Z(G)$. If $[G, Aut_{W^*}(G)]$ is torsion. Then every torsion subgroup of $Aut_{W^*}(G)$ is the direct product of its Sylow subgroups.*

Proof. Let $T = [G, \mathrm{Aut}_{W^*}(G)]$ that is an abelian torsion group so T is the direct sum of the primary components of G. Let $\alpha \in \mathrm{Aut}_{W^*}(G)$ and $g \in G$ and $\alpha^* : G \longrightarrow T$ be a map where $g \longrightarrow g^{-1}\alpha(g)$. Since $W^*(G) \leq Z(G)$, the map α^* is a homomorphism. We define $\alpha_p : G \longrightarrow G$ such that $\alpha_p(g) = g\pi_p\alpha^*(g)$ which $\pi_p : T \longrightarrow T_p$ be the natural projection. It is strightforward to see that $\alpha_p \in \mathrm{Aut}_{W^*}(G)$. Let $\theta : \mathrm{Aut}_{W^*}(G) \longrightarrow \mathrm{Aut}_{W^*}(G)$ defined by $\theta(\alpha) = \alpha_p$ for each $\alpha \in \mathrm{Aut}_{W^*}(G)$, now θ is a homomorphism: for if $\alpha, \beta \in \mathrm{Aut}_{W^*}(G)$ and $g \in G$, then

$$\begin{aligned}
\alpha_p\beta_p(g) &= \alpha_p\Big(g\pi_p(\beta^*(g))\Big) = g\pi_p(\beta^*(g))\pi_p\Big(\alpha^*\big(g\pi_p\beta^*(g)\big)\Big) \\
&= g\pi_p(\beta^*(g))\pi_p(\alpha^*(g))\pi_p\Big(\alpha^*\big(\pi_p\beta^*(g)\big)\Big) \\
&= g\pi_p(\alpha^*(g))\pi_p(\beta^*(g))\pi_p\Big(\alpha^*\big(\beta^*(g)\big)\Big) \\
&= g\pi_p(\alpha^*(g))\pi_p\Big(\beta^*(g)\alpha^*\big(\beta^*(g)\big)\Big) \\
&= g\pi_p(\alpha^*(g))\pi_p\Big(\alpha\big(\beta^*(g)\big)\Big) = g\pi_p\Big(g^{-1}\alpha(g)\alpha\big(g^{-1}\beta(g)\big)\Big) \\
&= g\pi_p\big(g^{-1}\alpha\beta(g)\big) = g\pi_p(\alpha\beta)^*(g) = (\alpha\beta)_p(g).
\end{aligned}$$

Let A be a torsion subgroup of $\mathrm{Aut}_{W^*}(G)$ so $\theta(A) = A_p$ is also torsion and since $[G, A_p] \in T_p$, with use of Lemma 3.14, A_p is a p-group. Let group C is generated by A_p's and prove C is an abelian group. Let p, q are distinct primes and show $[A_p, A_q] = 1$

$$\alpha_p(\alpha_q(g)) = \alpha_p(gg^{-1}\alpha_q(g)) = \alpha_p(g)\alpha_p\big(g^{-1}\alpha_q(g)\big) = gg^{-1}\alpha_p(g)\alpha_p\big(g^{-1}\alpha_q(g)\big).$$

Since $g^{-1}\alpha_q(g) \in T_q \leq W^*(G) \leq Z(G)$ and $[T_q, A_p] = [T_p, A_q] = 1$ so $\alpha_p(\alpha_q(g)) = g\alpha_p\big(g^{-1}\alpha_q(g)\big)g^{-1}\alpha_p(g) = \alpha_q(g)g^{-1}\alpha_p(g) = \alpha_q(g)\alpha_q\big(g^{-1}\alpha_p(g)\big) = \alpha_q(\alpha_p(g))$.

Now, C is a torsion abelian group so it is direct product of its Sylow subgroups. If we show A is contained in C, the proof is complete.

Let $\gamma \in A$ so γ has finite order n let $p_1, ..., p_r$ be prime divisors of n, hence by Lemma 3.14, $[G, \gamma] \leq T_{p_i}$ for $1 \leq i \leq r$ so $[G, \gamma] \leq T_{p_1} \times ... \times T_{p_r}$. Let g be any element of G with use of definition of maps γ_{p_i} and γ^* we have

$$\begin{aligned}\gamma(g) = gg^{-1}\gamma(g) = g\gamma^*(g) &= g\pi_{p_1}(\gamma^*(g))\pi_{p_2}(\gamma^*(g))...\pi_{p_r}(\gamma^*(g))\\ &= \gamma_{p_r}(\gamma_{p_{r-1}}(...\gamma_{p_1}(g))).\end{aligned}$$

Therefore, $\gamma = \gamma_{p_r}\big(\gamma_{P_{r-1}}(...(\gamma_{p_1}))\big) \in C$, which implies $A \leq C$. ∎

Lemma 3.16 *Let G be a purely non-abelian group and $\emptyset \neq W \subseteq F$ such that $W^*(G)$ or $\dfrac{G}{W(G)}$ is torsion and $W^*(G) \leq Z(G)$. Then every torsion subgroup of $Aut_{W^*}(G)$ is the direct product of its Sylow subgroups and the set $\pi(Aut_{W^*}G)$ is contained in $\pi(W^*(G))$. In particular, every finite subgroup of $Aut_{W^*}(G)$ is nilpotent.*

Proof. By use of Lemma 3.15, it is enough to show that $T = [G, \mathrm{Aut}_{W^*}(G)]$ is torsion. We first assume that $W^*(G)$ is torsion. Then $T \leq W^*(G)$ is torsion. Now, suppose that $\dfrac{G}{W(G)}$ is torsion. Let $\alpha \in \mathrm{Aut}_{W^*}(G)$ and define the map $f : G \longrightarrow [G, \alpha]$, it is clearly that f is an epimorphism with $C_G(\alpha)$ as its kernel so $\dfrac{G}{C_G(\alpha)} \simeq [G, \alpha]$. Let $\dfrac{G}{C_G(\alpha)}$ is torsion-free and $gC_G(\alpha)$ be any non trivial element of it, i.e., $g \notin C_G(\alpha)$ since by Lemma 2.5, $W(G) \leq C_G(\alpha)$, $gC_G(\alpha)$ is nontrival element of $\dfrac{G}{W(G)}$ so there exist $n \in N$ such that $g^n \in W(G) \leq C_G(\alpha)$ then $\big(gC_G(\alpha)\big)^n = 1$ that it is a contradiction. Hence $\dfrac{G}{C_G(\alpha)} \simeq [G, \alpha]$ is torsion and since α be arbitrary element of $\mathrm{Aut}_{W^*}(G)$ so T is torsion.

Now, if α be a marginal automorphism of order p, then by Lemma 3.14, $[G, \alpha] \in T_p$ so $1 \neq T_p \leq W^*(G)$ and $p \in \pi(W^*(G))$.

Since every finite subgroup of $\mathrm{Aut}_{W^*}(G)$ is the direct product of its Sylow subgroups, so it is nilpotent. ∎

As a consequence of Lemma 3.16, we obtain the following corollary.

Corollary 3.17 *Let G be a purely non-abelian p-group (p prime) and $\emptyset \neq W \subseteq F$ such that $W^*(G) \leq Z(G)$. Then every torsion subgroup of $Aut_{W^*}(G)$ is a p-group.*

Proof of Theorem 1.4. Let $W^*(G)$ is torsion. Since $W^*(G)$ is characteristic in G, there is a homomorphism $\mathrm{Aut}_{W^*}(G) \longrightarrow \mathrm{Aut}\ W^*(G)$ with kernel $C_{Aut_{W^*(G)}} W^*(G)$, that by Theorem 2.8, its kernel is an abelian group of finite exponent. Hence, $\mathrm{Aut}_{W^*}(G)$ is an abelian-by-finite group of finite exponent which its Sylow subgroups by Lemma 6-34 of [10] are nilpotent. Now, use of Lemma 3.16, implies that $\mathrm{Aut}_{W^*}(G)$ is nilpotent group.

Suppose now that $\dfrac{G}{W(G)}$ is finite, so there is a homomorphism

$$\begin{aligned} \psi : \mathrm{Aut}_{W^*}(G) \quad &\longrightarrow \quad \mathrm{Aut}\left(\frac{G}{W(G)}\right) \\ \alpha \quad &\longrightarrow \quad \alpha^* \\ &\quad\quad\ \alpha^*(gW(G)) = \alpha(g)W(G) \end{aligned}$$

Since α is an automorphism, α^* is epimorphism and if $gW(G) \in ker\alpha^*$ then $\alpha(g) \in W(G)$ but $W(G)$ is fully-invariant subgroup G so $g \in W(G)$ and α^* is one-to-one. Therefore, ψ is well-define. It is clear that ψ is homomorphism with kernel $C_{Aut_{W^*(G)}}\left(\dfrac{G}{W(G)}\right)$. The result with use of Lemma 2.8 is obtained in the previous case. ∎

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