ON UPPER AND LOWER WEAKLY $\mathcal{I}$-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper, we introduce and study a new generalization of $\mathcal{I}$-continuous multifunctions called weakly $\mathcal{I}$-continuous multifunctions in topological spaces.

Keywords: topological spaces, $\mathcal{I}$-open sets, $\mathcal{I}$-closed sets, weakly $\mathcal{I}$-continuous multifunctions.
2000 Mathematics Subject Classification: 54C60, 54C08.

1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [2], [9], [12], [13], [14]. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [8] and Vaidyanathaswamy [17]. An ideal $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [17] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau,\mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau,\mathcal{I})$ called the $*$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau,\mathcal{I})$ when there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal topological space. In 1990, Jankovic and Hamlett [6] introduced the notion of $\mathcal{I}$-open sets in topological spaces. In 1992, Abd El-Monsef et al. [1] further investigated $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions. Several characterizations and properties of $\mathcal{I}$-open sets

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were provided in [1, 10]. Recently, Akdag [2] introduced and studied the concept of $\mathcal{I}$-continuous multifunctions in topological spaces. In this paper we introduce and study a new generalization of $\mathcal{I}$-continuous multifunctions called weakly $\mathcal{I}$-continuous multifunctions in topological spaces.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-open [6] if $S \subset \mathrm{Int}(S^*)$. The complement of an $\mathcal{I}$-closed set is said to be an $\mathcal{I}$-open set. The $\mathcal{I}$-closure and the $\mathcal{I}$-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\mathcal{I}\,\mathrm{Cl}(A)$ and $\mathcal{I}\,\mathrm{Int}(A)$, respectively. The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed, regular open, regular closed) sets of (X, τ) is denoted by $\mathcal{I}O(X)$ (resp. $\mathcal{I}C(X)$, $RO(X)$, $RC(X)$). We set $\mathcal{I}O(X, x) = \{A : A \in \mathcal{I}O(X)$ and $x \in A\}$. By a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the upper and lower inverse of any subset A of Y by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$.

Definition 1 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be:

1. upper weakly continuous [14], [15] if for each x of X and each open set V of Y containing $F(x)$, there exists an open set U containing x such that $F(U) \subset \mathrm{Cl}(V)$.
2. lower weakly continuous [14], [15] if for each x of X and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$;
3. upper $\mathcal{I}$-continuous [2] if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset F^+(V)$;
4. lower $\mathcal{I}$-continuous [2] if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset F^-(V)$;
5. $\mathcal{I}$-continuous [2] if it is both upper $\mathcal{I}$-continuous and lower $\mathcal{I}$-continuous;
6. lower almost $\mathcal{I}$-continuous [4] if for each $x \in X$ and for each open set V of Y such that $x \in F^-(V)$, there exists a $U \in \mathcal{I}O(X, x)$ such that $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$,
7. upper almost $\mathcal{I}$-continuous [4] if for each $x \in X$ and for each open set V of Y such that $x \in F^+(V)$, there exists a $U \in \mathcal{I}O(X, x)$ such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$;

8. almost $\mathcal{I}$-continuous [4] if it is both upper almost $\mathcal{I}$-continuous and lower almost $\mathcal{I}$-continuous;
9. lower weakly continuous [15] for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists an open set U containing x such that $U \subset F^-(\mathrm{Cl}(V))$;
10. upper weakly continuous [15] for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists an open set U containing x such that $U \subset F^+(\mathrm{Cl}(V))$;
11. upper slightly $\mathcal{I}$-continuous [5] at $x \in X$ if for each clopen set V of Y containing $F(x)$, there exists $U \in \mathcal{I}O(X)$ containing x such that $F(U) \subset V$;
12. lower slightly $\mathcal{I}$-continuous [5] at $x \in X$ if for each clopen set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \mathcal{I}O(X)$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$;
13. upper (lower) slightly $\mathcal{I}$-continuous [5] if it has this property at each point of X.

3. Weakly $\mathcal{I}$-continuous multifunctions

Definition 2 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be:

1. upper weakly $\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset F^+(\mathrm{Cl}(V))$;
2. lower weakly $\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset F^-(\mathrm{Cl}(V))$;
3. weakly $\mathcal{I}$-continuous if it is both upper weakly $\mathcal{I}$-continuous and lower weakly $\mathcal{I}$-continuous.

The proof of the following Propositions are follows from their respective definitions and hence omitted.

Proposition 3.1 *Every upper* (*lower*) *almost $\mathcal{I}$-continuous multifunction is upper* (*lower*) *weakly $\mathcal{I}$-continuous.*

Proposition 3.2 *Every upper* (*lower*) *weakly $\mathcal{I}$-continuous multifunction is upper* (*lower*) *slightly $\mathcal{I}$-continuous.*

The following examples show that the converse of Propositions 3.1 and 3.2 is not true in general.

Example 3.3 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, c\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the multifunction $F : (X, \tau, \mathcal{I}) \to (X, \sigma)$ defined by $F(x) = \{x\}$ for all $x \in X$ is upper weakly $\mathcal{I}$-continuous but is not upper almost $\mathcal{I}$-continuous.

Example 3.4 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the multifunction $F : (X, \tau, \mathcal{I}) \to (X, \sigma)$ defined by $F(x) = \{x\}$ for all $x \in X$ is upper slightly $\mathcal{I}$-continuous but is not upper weakly $\mathcal{I}$-continuous.

Corollary 3.5 *Every $\mathcal{I}$-continuous multifunction is weakly $\mathcal{I}$-continuous.*

Theorem 3.6 *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent :*

1. *F is upper weakly $\mathcal{I}$-continuous;*
2. *$F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$ for any open set V of Y;*
3. *$\mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$ for any closed subset K of Y;*
4. *for each $x \in X$ and each open set V containing $F(x)$, there exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V)$.*

Proof. (1)⇒(2): Let V be any open subset of Y and $x \in F^+(V)$. By (1), there exists an $\mathcal{I}$-open set U containing x such that $U \subset F^+(\mathrm{Cl}(V))$. Hence, $x \in \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$.
(2)⇒(1): Let V be any open subset of Y such that $x \in F^+(V)$. By (2), $x \in F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+((\mathrm{Cl}(V))) \subset F^+(\mathrm{Cl}(V))$. Take $U = \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. Thus, we obtain that F is upper weakly $\mathcal{I}$-continuous.
(2)⇔(3): Let K be any closed subset of Y. By (2), we have $F^+(Y \backslash K) = X \backslash F^-(K) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(Y \backslash K))) = \mathcal{I}\,\mathrm{Int}(F^+(Y \backslash \mathrm{Int}(K))) = X \backslash \mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Int}(K)))$. Thus, $\mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$. The converse is similar.
(1)⇔(2): Obvious. ■

Theorem 3.7 *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent :*

1. *F is lower weakly $\mathcal{I}$-continuous;*
2. *$F^-(V) \subset \mathcal{I}\,\mathrm{Int}(F^-(\mathrm{Cl}(V)))$ for any open set V of Y;*
3. *$\mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^+(K)$ for any closed subset K of Y;*
4. *for each $x \in X$ and each open set V of Y such that $f(x) \cap V \neq \emptyset$, there exists an $\mathcal{I}$-open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$.*

Proof. It is similar to the proof of the Theorem 3.6. ■

Theorem 3.8 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is an open subset of Y for each $x \in X$. Then F is lower $\mathcal{I}$-continuous if and only if lower weakly $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be an open set of Y such that $F(x) \cap V \neq \emptyset$. Then there exists an $\mathcal{I}$-open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and hence F is lower $\mathcal{I}$-continuous. The converse follows by Corollary 3.5. ■

Theorem 3.9 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then F is upper weakly $\mathcal{I}$-continuous if and only if F is upper $\mathcal{I}$-continuous.*

Proof. Suppose that F is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and G be an open subset of Y containing $F(x)$. Since $F(x)$ is closed in Y and Y is normal, there exist open sets V and W such that $F(x) \subset V$, $X\backslash G \subset W$ and $V \cap W = \emptyset$. We have $F(x) \subset V \subset \mathrm{Cl}(V) \subset \mathrm{Cl}(X\backslash W) = X\backslash W \subset G$. Since F is upper weakly $\mathcal{I}$-continuous, there exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V) \subset G$. This shows that F is upper $\mathcal{I}$-continuous. The converse follows by Corollary 3.5. ∎

Definition 3 A subset A of a topological space (X, τ) is said to be

1. α-regular [7] if for each $a \in A$ and any open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$;
2. α-paracompact [7] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 3.10 [7] *If A is an α-paracompact α-regular set of a topological space (X, τ) and U an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, by $\mathrm{Cl}(F) : X \to Y$ we denote a multifunction as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, we define $\mathcal{I}\,\mathrm{Cl}\,F$.

The proof of the following two Lemmas are obvious and thus omitted.

Lemma 3.11 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then we have the following*

1. *$G^+(V) = F^+(V)$ for each open set V of Y,*
2. *$G^-(V) = F^-(V)$ for each closed set V of Y, where G denotes $\mathrm{Cl}\,F$ or $\mathcal{I}\,\mathrm{Cl}\,F$.*

Lemma 3.12 *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, we have the following*

1. *$G^-(V) = F^-(V)$ for each open set V of Y,*
2. *$G^+(V) = F^+(V)$ for each open set V of Y, where G denotes $\mathrm{Cl}\,F$ or $\mathcal{I}\,\mathrm{Cl}\,F$.*

Theorem 3.13 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for every $x \in X$. Then the following statements are equivalent:*

1. *F is upper weakly $\mathcal{I}$-continuous;*
2. *$\mathrm{Cl}\,F$ is upper weakly $\mathcal{I}$-continuous;*
3. *$\mathcal{I}\,\mathrm{Cl}\,F$ is upper weakly $\mathcal{I}$-continuous.*

Proof. We put $G = \mathrm{Cl}(F)$ or $\mathcal{I}\,\mathrm{Cl}\,F$ in the sequel. Suppose that F is upper weakly $\mathcal{I}$-continuous. Then it follows by Theorem 3.6 and Lemmas 3.11 and 3.12 that for every open set V of Y containing $F(x)$, $G^+(V) = F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) = \mathcal{I}\,\mathrm{Int}(G^+(\mathrm{Cl}(V)))$. By Theorem 3.6, G is upper weakly $\mathcal{I}$-continuous. Conversely, Suppose that G is upper weakly $\mathcal{I}$-continuous. Then it follows by Theorem 3.6 and Lemmas 3.11 and 3.12 that for every open set V of Y containing $G(x)$, $F^+(V) = G^+(V) \subset \mathcal{I}\,\mathrm{Int}(G^+(\mathrm{Cl}(V))) = \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. It follows by Theorem 3.6 that F is upper weakly $\mathcal{I}$-continuous. ∎

Theorem 3.14 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ be α-regular α-paracompact for every $x \in X$. Then the following properties are equivalent:*

1. *F is lower weakly $\mathcal{I}$-continuous;*
2. *$\mathrm{Cl}\,F$ is lower weakly $\mathcal{I}$-continuous;*
3. *$\mathcal{I}\,\mathrm{Cl}\,F$ is lower weakly $\mathcal{I}$-continuous.*

Proof. Similar to the proof of Theorem 3.27. ∎

Definition 4 Let A be a subset of a topological space (X, τ). The $\mathcal{I}$-frontier of A, denoted by $\mathcal{I}Fr(A)$, is defined by $\mathcal{I}Fr(A) = \mathcal{I}\,\mathrm{Cl}(A) \cap \mathcal{I}\,\mathrm{Cl}(X \backslash A)$.

Theorem 3.15 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction. The set of all points x of X such that F is not upper weakly $\mathcal{I}$-continuous (resp. lower weakly $\mathcal{I}$-continuous) is identical with the union of the $\mathcal{I}$-frontiers of the upper (resp. lower) inverse images of the closure of open sets containing (resp. meeting) $F(x)$.*

Proof. Let x be a point of X at which F is not upper weakly $\mathcal{I}$-continuous. Then there exists an open set V containing $F(x)$ such that $U \cap (X \backslash F^+(\mathrm{Cl}(V))) \neq \emptyset$ for every $\mathcal{I}$-open set U containing x. Therefore, $x \in \mathcal{I}\,\mathrm{Cl}(X \backslash F^+(\mathrm{Cl}(V)))$. Since $x \in F^+(V)$, we have $x \in \mathcal{I}\,\mathrm{Cl}(F^+(\mathrm{Cl}(V)))$ and hence $x \in \mathcal{I}Fr(F^+(\mathrm{Cl}(V)))$. Conversely, if F is upper weakly $\mathcal{I}$-continuous at x, then for every open set V of Y containing $F(x)$ and there exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V)$; hence $U \subset F^+(\mathrm{Cl}(V))$. Therefore, we obtain $x \in U \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. This contradicts that $x \in \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$.

The case when F is lower weakly $\mathcal{I}$-continuous is similarly shown. ∎

Definition 5 A topological space (X, τ) is said to be strongly normal if for every disjoint closed sets V_1 and V_2 of X, there exist two disjoint open sets U_1 and U_2 such that $V_1 \subset U_1$, $V_2 \subset U_2$ and $\mathrm{Cl}(U_1) \cap \mathrm{Cl}(U_2) = \emptyset$.

Recall that a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be a point closed multifunction if $F(x)$ is a closed subset of Y for all $x \in X$.

Theorem 3.16 *Suppose that the product of two $\mathcal{I}$-open sets is $\mathcal{I}$-open. If Y is a strongly normal space and $F_i : X_i \to Y$ an upper weakly $\mathcal{I}$-continuous multifunction such that F_i is point closed for i = 1, 2, then the set $\{(x_1, x_2) \in X_1 \times X_2$: $F_1(x_1) \cap F_2(x_2) \neq \varnothing\}$ is $\mathcal{I}$-closed in $X_1 \times X_2$.*

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2\colon\ F_1(x_1) \cap F_2(x_2) \neq \varnothing\}$ and $(x_1, x_2) \in (X_1 \times X_2)\backslash A$. Then $F_1(x_1) \cap F_2(x_2) = \varnothing$. Since Y is strongly normal and F_i is point closed for $i = 1$, 2, there exist disjoint open sets V_1, V_2 such that $F_i(x_i) \subset V_i$ for $i = 1$, 2. We have $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F_i is upper weakly $\mathcal{I}$-continuous, there exist $\mathcal{I}$-open sets U_1 and U_2 containing x_1 and x_2, respectively such that $F_i(U_i) \subset \mathrm{Cl}(V_i)$ for $i = 1$, 2. Put $U = U_1 \times U_2$, then U is an $\mathcal{I}$-open set and $(x_1, x_2) \in U \subset (X_1 \times X_2) \backslash A$. This shows that $(X_1 \times X_2) \backslash A$ is $\mathcal{I}$-open; hence A is $\mathcal{I}$-closed in $X_1 \times X_2$. ∎

Theorem 3.17 *Let F and G be point closed multifunctions from a topological space (X, τ) to a strongly normal space (Y, σ). If F is upper weakly $\mathcal{I}$-continuous and G is upper weakly continuous, then the set $K = \{x : F(x) \cap G(x) \neq \varnothing\}$ is $\mathcal{I}$-closed in X.*

Proof. Let $x \in X\backslash K$. Then $F(x) \cap G(x) = \varnothing$. Since F and G are point closed multifunctions and Y is a strongly normal space, there exist disjoint open sets U and V containing $F(x)$ and $G(x)$, respectively we have $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$. Since F and G are upper weakly $\mathcal{I}$-continuous functions, then there exist $\mathcal{I}$-open set U_1 containing x and open set U_2 containing x such that $F(U_1) \subset \mathrm{Cl}(V)$ and $F(U_2) \subset \mathrm{Cl}(V)$. Now set $H = U_1 \cap U_2$, then H is an $\mathcal{I}$-open set containing x and $H \cap K = \varnothing$; hence K is $\mathcal{I}$-closed in X. ∎

Lemma 3.18 [1] *Let A and B be subsets of a space (X, τ).*

1. *If $A \in \mathcal{I}O(X, \tau)$ and $B \in \tau$, then $A \cap B \in \mathcal{I}O(B, \tau_B)$;*
2. *If $A \in \mathcal{I}O(B, \tau_B)$ and $B \in \mathcal{I}O(X, \tau)$, then $A \in \mathcal{I}O(X, \tau)$.*

Theorem 3.19 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction and U an open subset of X. If F is a lower (upper) weakly $\mathcal{I}$-continuous multifunction, then $F_{|_U} : U \to Y$ is a lower (upper) weakly $\mathcal{I}$-continuous multifunction.*

Proof. Let V be any open set of Y. Let $x \in U$ and $x \in F^-_{|_U}(V)$. Since F is lower weakly $\mathcal{I}$-continuous multifunction, then there exists an $\mathcal{I}$-open set G containing x such that $G \subset F^-(\mathrm{Cl}(V))$. Then $x \in G \cap U \in \mathcal{I}O(X, \tau)$ and $G \cap U \subset F^-_{|_U}(\mathrm{Cl}(V))$. This shows that $F_{|_U}$ is a lower weakly $\mathcal{I}$-continuous. The proof of the upper weakly $\mathcal{I}$-continuity of $F_{|_U}$ can be done by the similar manner. ∎

Theorem 3.20 *Let $\{A_i\}_{i \in I}$ be an open cover of a topological space X. Then a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper (lower) weakly $\mathcal{I}$-continuous if and only if $F_{|_{A_i}}\colon A_i \to Y$ is a upper (lower) weakly $\mathcal{I}$-continuous for each $i \in I$.*

Proof. Let $i \in I$ and V be any open set of Y. Since F is lower weakly $\mathcal{I}$-continuous, $F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. We obtain $(F|A_i)^+(V) = F^+(V) \cap A_i \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) \cap A_i = \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)) \cap A_i) \subset \mathcal{I}\,\mathrm{Int}_{A_i}(F^+(\mathrm{Cl}(V)))$. Hence $F_{|_A}\colon\ A \to Y$ is an upper weakly $\mathcal{I}$-continuous for each $i \in I$. Conversely, Let V be any open set of Y. Since F_i is lower weakly $\mathcal{I}$-continuous for each $i \in I$, from Theorem 3.6, $F_i^+(V) \subset \mathcal{I}\,\mathrm{Int}_{A_i}(F_i^+(\mathrm{Cl}(V)))$ and since A_i is open, we have

$F^+(V) \cap A_i \subset \mathcal{I}\,\mathrm{Int}_{A_i}(F^+(\mathrm{Cl}(V)) \cap A_i)$ and $F^+(V) \cap A_i \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) \cap A_i$. Since $\{A_i\}_{i\in I}$ is an open cover of X, it follows that $F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. Hence, from Theorem 3.6, we obtain that F is upper weakly $\mathcal{I}$-continuous. The proof of the lower weakly $\mathcal{I}$-continuity of $F_{|A}$ can be done by the similar manner.■

Definition 6 For a multifunction, $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the graph multifunction of F and is denoted by $G(x)$.

Lemma 3.21 [9] *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following holds:*

1. $G_F^+(A \times B) = A \cap F^+(B)$;
2. $G_F^-(A \times B) = A \cap F^-(B)$

for any subset A of X and B of Y.

Theorem 3.22 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction. If the graph multifunction of F is upper (lower) weakly $\mathcal{I}$-continuous, then F is upper (lower) weakly $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be any open subset of Y and containing $F(x)$. Since $X \times V$ is an open set relative to $X \times Y$ and $G_F(x) \subset X \times V$, there exists an $\mathcal{I}$-open set U containing x such that $G_F(U) \subset \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. By Lemma 3.21, we have $U \subset G_F^+(X \times \mathrm{Cl}(V)) = F^+(\mathrm{Cl}(V))$ and $F(U) \subset \mathrm{Cl}(V)$. Thus, F is upper weakly $\mathcal{I}$-continuous. The proof of the lower weakly $\mathcal{I}$-continuity of F can be done by the similar manner. ■

Theorem 3.23 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is upper weakly $\mathcal{I}$-continuous if and only if $G_F : X \to X \times Y$ is upper weakly $\mathcal{I}$-continuous.*

Proof. Suppose that $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each point $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$ and $F(x)$ is compact. Therefore, there exist a finite number of points, say, $y_1, y_2, .., y_n$ in $F(x)$ such that $F(x) \subset \cup\{V(y_i) : 1 \leq i \leq n\}$. Set $U = \cap\{U(y_i) : 1 \leq i \leq n\}$ and $V = \cup\{V(y_i) : 1 \leq i \leq n\}$. Then U and V are open sets of X and Y, respectively, and $G_F(x) = \{x\} \times F(x) \subset U \times V \subset W$. Since F is upper weakly $\mathcal{I}$-continuous, there exists $U_0 \in \mathcal{I}O(X, x)$ such that $F(U_0) \subset \mathrm{Cl}(V)$. By Lemma 3.18, $U \cap U_0 \in \mathcal{I}O(X, x)$. By Lemma 3.21, $U \cap U_0 \subset \mathrm{Cl}(U) \cap F^+(\mathrm{Cl}(V)) = G_F^+(\mathrm{Cl}(U) \times \mathrm{Cl}(V)) \subset G_F^+(\mathrm{Cl}(W))$. Therefore, we obtain $U \cap U_0 \in \mathcal{I}O(X, x)$ and $G_F(U \cap U_0) \subset \mathrm{Cl}(W)$. This shows that G_F is upper weakly $\mathcal{I}$-continuous. Conversely, suppose that $G_F : X \to X \times Y$ is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \mathcal{I}O(X, x)$ such that $G_F(U) \subset \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. By Lemma 3.21, we have $U \subset G_F^+(X \times \mathrm{Cl}(V)) = F^+(\mathrm{Cl}(V))$. This shows that F is upper weakly $\mathcal{I}$-continuous. ■

Theorem 3.24 *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is lower weakly $\mathcal{I}$-continuous if and only if $G_F : X \to X \times Y$ is lower weakly $\mathcal{I}$-continuous.*

Proof. Suppose that F is lower weakly $\mathcal{I}$-continuous. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_{F^-}(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets $U \subset X$ and $V \subset Y$. Since F is lower weakly $\mathcal{I}$-continuous and $F(x) \cap V \neq \emptyset$, there exists $U_0 \in \mathcal{I}O(X, x)$ such that $U_0 \subset F^-(\mathrm{Cl}(V))$. By Lemma 3.21, we have $U \cap U_0 \subset U \cap F^-(\mathrm{Cl}(V)) = G_{F^-}(U \times \mathrm{Cl}(V)) \subset G_{F^-}(\mathrm{Cl}(W))$. Moreover, we have $U \cap U_0 \in \mathcal{I}O(X, x)$; and hence G_F is lower weakly $\mathcal{I}$-continuous. Conversely, suppose that G_F is lower weakly $\mathcal{I}$-continuous. Let $x \in X$ and V be an open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower weakly $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset G_{F^-}(\mathrm{Cl}(X \times V))$. By Lemma 3.21, we obtain $U \subset G_{F^-}(\mathrm{Cl}(X \times V)) = F^-(\mathrm{Cl}(V))$. This shows that F is lower weakly $\mathcal{I}$-continuous. ∎

Definition 7 A topological space (X, τ) is said to $\mathcal{I}$-T_2 [10] if for each pair of distinct points x and y in X, there exist disjoint $\mathcal{I}$-open sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 3.25 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an upper weakly $\mathcal{I}$-continuous injective multifunction and point closed from a topological space X to a strongly normal space Y, then X is an $\mathcal{I}$-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \varnothing$ since F is injective. Since Y is strongly normal, there exist disjoint open sets U and V containing $F(x)$ and $F(y)$, respectively such that $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$. Thus, there exist disjoint $\mathcal{I}$-open sets $F^+(U)$ and $F^+(V)$ containing x and y, respectively such $G \subset F^+(\mathrm{Cl}(U))$ and $W \subset F^+(\mathrm{Cl}(V))$. Therefore, we obtain $G \cap W = \emptyset$; hence X is $\mathcal{I}$-T_2. ∎

Theorem 3.26 *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where X_α is connected space for each $\alpha \in J$. Let $F : X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$. If F is upper (lower) weakly $\mathcal{I}$-continuous multifunction, then $P_\alpha \circ F$ is upper (lower) weakly $\mathcal{I}$-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be an open set in $(X_{\alpha 0}, \tau_{\alpha 0})$. Then $(P_{\alpha 0} \circ F)^+(V_{\alpha 0}) = F^+(P^+_{\alpha 0}(V_{\alpha 0})) = F^+(V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$. We take $x \in (P_{\alpha_0} \circ F)^+(V_{\alpha_0})$. Since F is upper weakly $\mathcal{I}$-continuous and $V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is an open set to $\prod_{\alpha \in J} X_\alpha$, there exists an $\mathcal{I}$-open set U containing x such $G \subset F^+(\mathrm{Cl}(V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha))$. Since $F^+(\mathrm{Cl}(V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)) = F^+(\mathrm{Cl}(V_{\alpha 0}) \times \prod_{\alpha \neq \alpha_0} X_\alpha) = (P_{\alpha 0} \circ F)^+(\mathrm{Cl}(V_{\alpha 0}))$, $P_\alpha \circ F$ is upper weakly $\mathcal{I}$-continuous. The proof of the lower weakly $\mathcal{I}$-continuity of F can be done by the same manner. ∎

Theorem 3.27 *Suppose that for each $\alpha \in J$, (X_α, τ_α), $(Y_\alpha, \sigma_\alpha)$ are topological spaces. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to the product space $\prod_{\alpha \in J} Y_\alpha$. If F is upper (lower) weakly $\mathcal{I}$-continuous multifunction, then each F_α is upper (lower) weakly $\mathcal{I}$-continuous multifunction for each $\alpha \in J$.*

Proof. Similar to the proof of Theorem 3.26. ∎

4. Additional properties

Theorem 4.1 *Let $F : X \to \prod_{i \in I} X_i$ be a multifunction from a topological space X to the product space $\prod_{i \in I} X_i$ and let $p_i : \prod_{i \in I} X_i \to X_i$ be the projection for each $i \in I$. If F is an upper (resp. lower) weakly $\mathcal{I}$-continuous multifunction, then $p_i oF$ is an upper (resp. lower) weakly $\mathcal{I}$-continuous multifunction for each $i \in I$.*

Proof. We shall prove this only for the upper continuity. Let V_{i0} be an open set in X_{i0}. We have $(p_{i0}oF)^+(V_{i0}) = F^+(p_{i0}^+(V_{i0})) = F^+(V_{i0} \times \prod_{i \in I} X_i)$. Take $x \in (p_{i0}oF)^+(V_{i0})$. Since F is upper weakly $\mathcal{I}$-continuous and $V_{i0} \times \prod_{i \in I} X_i$ is an open set, there exists an $\mathcal{I}$-open set U containing x such that $U \subset F^+(\mathrm{Cl}(V_{i0} \times \prod_{i \in I} X_i))$. Since $F^+(\mathrm{Cl}(V_{i0} \times \prod_{i \in I} X_i)) = F^+(\mathrm{Cl}(V_{i0}) \times \prod_{i \in I} X_i) = (p_{i0}oF)^+(\mathrm{Cl}(V_{i0}))$, $p_{i0}oF$ is upper weakly $\mathcal{I}$-continuous multifunction. ∎

Theorem 4.2 *Suppose that $F_1 : X \to Y$, $F_2 : X \to Z$ are multifunctions. Let $F_1 \times F_2 : X \to Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) weakly $\mathcal{I}$-continuous multifunction, then F_1 and F_2 are upper (resp. lower) weakly $\mathcal{I}$-continuous functions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be open sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so $F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper weakly $\mathcal{I}$-continuous multifunction, there exists an $\mathcal{I}$-open set U containing x such that $U \subset (F_1 \times F_2)^+(\mathrm{Cl}(K \times H))$. We obtain that $U \subset F_1^+(\mathrm{Cl}(K))$ and $U \subset F_2^+(\mathrm{Cl}(H))$. Thus, we obtain that F_1 and F_2 are upper weakly $\mathcal{I}$-continuous multifunctions. The proof of the lower weakly $\mathcal{I}$-continuity of F_1 and F_2 is similar to that presented above. ∎

Definition 8 Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction. The multigraph G_F is said to be $\mathcal{I}$-closed in $X \times Y$ if for each $(x, y) \notin G_F$, there exist $\mathcal{I}$-open set U and open set V containing x and y, respectively, such that $(U \times V) \cap G_F = \emptyset$.

Definition 9 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is called punctually α-paracompact [7] if $F(x)$ is α-paracompact for each point $x \in X$.

Theorem 4.3 *Let $F : (X,\tau) \to (Y,\sigma)$ be an upper weakly $\mathcal{I}$-continuous and punctually α-paracompact multifunction into a Hausdorff space (Y,σ). Then the multigraph G_F of F is an $\mathcal{I}$-closed in $X \times Y$.*

Proof. Suppose that $(x_0, y_0) \notin G_F$. Then $y_0 \notin F(x_0)$. Since (Y,σ) is a Hausdorff space for each $y \in F(x_0)$, there exist open sets $V(y)$ and $W(y)$ containing y and y_0, respectively such that $V(y) \cap W(y) = \emptyset$. The family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus, it has a locally finite open refinement $\Phi = \{U_\beta : \beta \in I\}$ which covers $F(x_0)$. Let W_0 be an open neighborhood of y_0 such that W_0 intersects only finitely many members $U_{\beta 1}, U_{\beta 2}, ..., U_{\beta n}$ of Φ. Choose $y_1, y_2, .., y_n$ in $F(x_0)$ such that $U_{\beta i} \subset V(y_i)$ for each i=1,2,..n and $W = W_0 \cap (\bigcap\limits_{i=1}^{n} W(y_i))$. Then W is an open neighborhood of y_0 with $W \cap (\bigcup\limits_{\beta\in I} U_\beta) = \emptyset$, which implies that $W \cap \mathrm{Cl}(\bigcup\limits_{\beta\in I} U_\beta) = \emptyset$. By the upper weakly $\mathcal{I}$-continuity of F, there exists $U \in \mathcal{I}O(X, x_0)$ such that $F(U) \subset \mathrm{Cl}(\bigcup\limits_{\beta\in I} U_\beta)$. It follows that $(U \times W) \cap G_F = \emptyset$. Hence, the graph G_F is an $\mathcal{I}$-closed in $X \times Y$. ∎

Definition 10 A multifunction $F : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be almost $\mathcal{I}$-open if $F(U) \subset \mathrm{Int}(\mathrm{Cl}(F(U)))$ for every $\mathcal{I}$-open set U in X.

Theorem 4.4 *If a multifunction $F : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is upper weakly $\mathcal{I}$-continuous and almost $\mathcal{I}$-open, then F is upper almost $\mathcal{I}$-continuous.*

Proof. Let V be an arbitrary open set containing $F(x)$. There exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V)$. Since F is almost $\mathcal{I}$-open, we have $F(U) \subset \mathrm{Int}(\mathrm{Cl}(F(U))) \subset \mathrm{Int}(\mathrm{Cl}(V))$. It then follows that F is upper almost $\mathcal{I}$-continuous. ∎

Theorem 4.5 *Let $F : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ be a multifunction such that $F(x)$ is an open set of Y for each $x \in X$. Then the following assertions are equivalent:*

1. *F is lower $\mathcal{I}$-continuous,*
2. *F is lower almost $\mathcal{I}$-continuous,*
3. *F is lower weakly $\mathcal{I}$-continuous.*

Proof. (1)⇒(2), (2)⇒(3): Obvious.
(3)⇒(1): Let $x \in X$ and V be an open set of Y such that $F(x) \cap V \neq \emptyset$. Then there exists an $\mathcal{I}$-open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and hence F is lower $\mathcal{I}$-continuous. ∎

Theorem 4.6 *If $F : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is upper weakly $\mathcal{I}$-continuous and satisfies the condition $F^+(\mathrm{Cl}(V)) \subset F^+(V)$ for every open set V of Y, then F is upper $\mathcal{I}$-continuous.*

Proof. Let V be any open set. Since F upper weakly $\mathcal{I}$-continuous, $F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$ and hence $F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) \subset \mathcal{I}\,\mathrm{Int}(F^+(V))$. Then $F^+(V)$ is $\mathcal{I}$-open; hence F is upper $\mathcal{I}$-continuous. ∎

Theorem 4.7 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then the following assertions are equivalent:*

1. *F is upper weakly $\mathcal{I}$-continuous,*
2. *F is upper almost $\mathcal{I}$-continuous,*
3. *F is upper $\mathcal{I}$-continuous.*

Proof. (3)⇒(2), (2)⇒(1): Obvious.
(1)⇒(3): Suppose that F is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and G be an open set containing $F(x)$. Since $F(x)$ is closed in Y, by the normality of Y, there exists an open set V such that $F(x) \subset V \subset \mathrm{Cl}(V) \subset G$. Since F is upper weakly $\mathcal{I}$-continuous, there exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V) \subset G$. This shows that F is upper $\mathcal{I}$-continuous. ■

Definition 11 [16] A topological space X is said to be almost regular if for each $x \in X$ and each regular closed set F of X not containing x, there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$.

Lemma 4.8 *If A is an α-almost regular α-paracompact set of a topological space X and U is a regular open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

Theorem 4.9 *If a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly $\mathcal{I}$-continuous and $F(x)$ is an α-almost regular and α-paracompact set for each $x \in X$, then F is upper almost $\mathcal{I}$-continuous.*

Proof. Let V be a regular open set of Y and $x \in F^+(V)$. Then $F(x) \subset V$ and by Lemma 4.8 there exists an open set W of Y such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. Since F is upper weakly $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X, x)$ such that $F(U) \subset \mathrm{Cl}(W) \subset V$. Therefore, we have $x \in U \subset F^+(V)$. This shows that $F^+(V) \in \mathcal{I}O(X, x)$. Hence F is upper almost $\mathcal{I}$-continuous. ■

Corollary 4.10 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an upper weakly $\mathcal{I}$-continuous multifunction such that $F(x)$ is compact for each $x \in X$ and Y is almost regular. Then F is upper almost $\mathcal{I}$-continuous.*

Lemma 4.11 *If A is an α-almost regular set of a space X, then for every regular open set U which intersects A, there exists an open set G such that $A \cap G \neq \emptyset$ and $\mathrm{Cl}(G) \subset U$.*

Theorem 4.12 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a lower weakly $\mathcal{I}$-continuous multifunction such that $F(x)$ is an α-almost regular set of Y for every $x \in X$, then F is lower almost $\mathcal{I}$-continuous.*

Proof. Let V be a regular open set of Y and $x \in F^-(V)$. Since $F(x)$ is α-almost regular, by Lemma 4.11 there exists an open set W of Y such that $F(x) \cap W \neq \emptyset$ and $\mathrm{Cl}(W) \subset V$. Since F is lower weakly $\mathcal{I}$-continuous there exists $U \in \mathcal{I}O(X, x)$ such that $F(u) \cap \mathrm{Cl}(W) \neq \emptyset$; hence $F(u) \cap V \neq \emptyset$ for every $u \in U$. Therefore, we have $x \in U \subset F^-(V)$. This shows that $F^-(V) \in \mathcal{I}O(X, x)$. Hence F is lower almost $\mathcal{I}$-continuous. ■

Definition 12 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be complementary continuous if for each open set V of Y, $F^{-}(Fr(V))$ is a closed set of X, where $Fr(V)$ denotes the frontier of V.

Theorem 4.13 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly $\mathcal{I}$-continuous and complementary continuous, then it is upper $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. There exists $G \in \mathcal{I}O(X, x)$ such that $F(G) \subset \mathrm{Cl}(V)$. Put $U = G \cap (X \backslash F^{-}(Fr(V)))$. Since $F^{-}(Fr(V))$ is closed in X, $U \in \mathcal{I}O(X)$. Moreover, we have $F(x) \cap Fr(V) \subset [V \cap \mathrm{Cl}(V)] \cap (Y \backslash V) = \emptyset$; and hence $x \in X \backslash F^{-}(Fr(V))$. Thus, we obtain $U \in \mathcal{I}O(X, x)$ and $F(U) \subset V$ since $F(U) \subset F(G) \subset \mathrm{Cl}(V)$ and $F(u) \cap Fr(V) = \emptyset$ for each $u \in U$. Therefore, F is upper $\mathcal{I}$-continuous. ■

Definition 13 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be complementary $\mathcal{I}$-continuous if for each open set V of Y, $F^{-}(Fr(V))$ is an $\mathcal{I}$-closed set of X.

Theorem 4.14 *If multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly continuous and complementary $\mathcal{I}$-continuous, then it is upper $\mathcal{I}$-continuous.*

Proof. It is similar to the proof of Theorem 4.13. ■

Definition 14 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be injective if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Theorem 4.15 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an upper weakly $\mathcal{I}$-continuous injective multifunction into a Urysohn space (Y, σ) and $F(x)$ is compact in Y for each $x \in X$, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2.*

Proof. For any distinct points x_1, x_2 of X, we have $F(x_1) \cap F(x_2) = 0$, since F is injective. Since $F(x_i)$ is compact in a Urysohn space (Y, σ), there exist open sets V_i such that $F(x_i) \subset V_i$ for i=1,2 and $\mathrm{Cl}(V_i) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F is upper weakly $\mathcal{I}$-continuous there exists $U_i \in \mathcal{I}(X, x_i)$ such that $F(U_i) \subset \mathrm{Cl}(V_i)$ for $i = 1, 2$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$ and hence $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2. ■

Theorem 4.16 *If $F_1, F_2 : X \to Y$ be multifunctions into a Urysohn space (Y, σ) and $F_i(x)$ compact in Y for each $x \in X$ and each $i = 1, 2$. If $F(x_i) \cap F_2(X) \neq \emptyset$ for each $x \in X$ and F_1 is upper weakly $\mathcal{I}$-continuous, then a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, defined as follows $F(x) = F_1(x) \cap F_2(x)$ for each $x \in X$, is upper weakly $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $A = F_1(x) \backslash V$ and $B = F_2(x) \backslash V$ are disjoint compact sets. Then there exist open sets V_1 and V_2 such that $A \subset V_1$ and $B \subset V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F_1 is upper weakly $\mathcal{I}$-continuous, there exists $U_1 \in \mathcal{I}O(X, x)$ such that $F_1(U_1) \subset \mathrm{Cl}(V_i \cup V)$. Since F_2 is upper weakly $\mathcal{I}$-continuous, there exists an $\mathcal{I}$-open set U_2 containing x such that $F_2(U_2) \subset \mathrm{Cl}(V_2 \cap V)$. Set $U = U_1 \cap U_2$, then $U \in \mathcal{I}O(X, x)$. If $y \in F(x_0)$ for any $x \in U$, then $y \in \mathrm{Cl}(V_1 \cup V) \cap \mathrm{Cl}(V_1 \cup V) = (\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2)) \cup \mathrm{Cl}(V)$. Since $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$, we have $y \in \mathrm{Cl}(V)$ and hence $F(U) \subset \mathrm{Cl}(V)$. Therefore, F is upper weakly $\mathcal{I}$-continuous. ■

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Accepted: 07.06.2016