SOME RESULTS ON n-CAMINA PAIRS OF GROUPS

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Abstract. In [6], Lewis found a bound for $|Z(G)|$ in terms of $[G : Z(G)]$ where $(G, Z(G))$ is an Camina pair of groups. In this paper we generalize this concept to n-Camina pairs of groups with respect to the variety of nilpotent groups of class at most n, and find a bound for $\left|\dfrac{Z_n(G)}{Z_{n-1}(G)}\right|$ where $(G, Z_n(G))$ is an n-Camina pair of groups and G is a finite p-group.

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1. Introduction

Let F be a free group freely generated by a countable set $\{x_1, x_2, ...\}$. Let $\mathcal{V}$ be the variety of groups defined by a set of laws V. It will be assumed that the reader is familiar with the notations of verbal subgroup, $V(G)$, and marginal subgroup, $V^*(G)$, associated to a variety of groups and a given group G. See, also, [4] for more information on varieties of groups.

In this paper, we always assume that $\mathcal{V}$ is the variety of groups defined by the set of laws V.

Let G be any group with a normal subgroup N, then we define $[NV^*G]$ to be the subgroup of G generated by the following set:

$$\{\nu(g_1, ..., g_{i-1}, g_i n, g_{i+1}, ..., g_r)\nu(g_1, ..., g_r)^{-1}|,\ 1 \le i \le r,\ \nu \in V,\ g_i \in G,\ n \in N\}.$$

It is easy to check that $[NV^*G]$ is the smallest normal subgroup T of G contained in N, such that

$$\frac{N}{T} \subseteq V^*\left(\frac{G}{T}\right). \qquad \text{(See [1]-[4], [5].)}$$

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We define the upper $\mathcal{V}$- marginal series of G to be:

$$1 = V_0^*(G) \leq V_1^*(G) = V^*(G) \leq \cdots \leq V_r^*(G) \leq \cdots ,$$

where

$$\frac{V_r^*(G)}{V_{r-1}^*(G)} = V^*\left(\frac{G}{V_{r-1}^*(G)}\right), \quad \text{for } r > 0.$$

The corresponding lower $\mathcal{V}$- marginal series of G is given by

$$G = V_0(G) \geq V_1(G) = V(G) \geq \cdots \geq V_r(G) \geq \cdots ,$$

where

$$V_r(G) = [V_{r-1}(G)V^*G], \quad \text{for } r > 0.$$

If H is a subgroup of G, we set $[H,_1 G] = [H, G]$ and by induction

$$[H,_n G] = [[H,_{n-1} G], G] \quad (n \geq 1).$$

In the special case when $\mathcal{V}$ is the variety of nilpotent groups of class at most n, $\mathcal{N}_n$, then $[NV^*G] = [N,_n G]$, $V_r(G) = \gamma_{rn+1}(G)$ and $V_r^*(G) = Z_{rn}(G)$, where $\gamma_i(G)$ is i-th term of lower central series and $Z_i(G)$ is i-th term of upper central series (see [8]).

Now, we have the following lemma:

Lemma 1.1. *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V and G be a group and $B, N \trianglelefteq G$ and $N \leq B$, then $[BV^*G] \leq N$ if and only if $\frac{B}{N} \leq V^*\left(\frac{G}{N}\right)$.*

Proof. See Lemma 2.2 of [9]. ∎

Corollary 1.2. *Let G be a group and $B, N \trianglelefteq G$ and $N \leq B$, then $[B,_n G] \leq N$ if and only if $\frac{B}{N} \leq Z_n\left(\frac{G}{N}\right)$.*

Proof. The assertion follows immediately from Lemma 1.1 where $\mathcal{V}=\mathcal{N}_n$. ∎

Definition 1.3. A group G is said to be $\mathcal{V}$-nilpotent if there exists a series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_r = G \qquad (*)$$

such that

$$G_i \trianglelefteq G \ \text{ and } \ \frac{G_{i+1}}{G_i} \leq V^*\left(\frac{G}{G_i}\right), \quad \text{for } 0 \leq i \leq r-1.$$

The length of the shortest series $(*)$ is the $\mathcal{V}$-nilpotent class of G.

Theorem 1.4. *Let $1 = G_0 \leq G_1 \leq ... \leq G_r = G$ be a series in a $\mathcal{V}$-nilpotent group G, then the following properties hold*

(i) $V_r(G) = 1$,

(ii) $V_r^*(G) = G$.

Proof. See Theorem 2.4 of [9]. ∎

Definition 1.5. A group G is said to be n-nilpotent if there exists a series

$$1 = G_0 \leq G_1 \leq ... \leq G_r = G \qquad (*)$$

such that

$$G_i \trianglelefteq G \text{ and } \frac{G_{i+1}}{G_i} \leq Z_n(\frac{G}{G_i}), \quad \text{for } 0 \leq i \leq r-1.$$

The length of the shortest series $(*)$ is the n-nilpotent class of G.

The next corollary is a result of Theorem 1.4, where $\mathcal{V}$=$\mathcal{N}_n$.

Corollary 1.6. *Let $1 = G_0 \leq G_1 \leq \cdots \leq G_r = G$ be a series in a n-nilpotent group G, then $\gamma_{rn+1}(G) = 1$ and $Z_{rn}(G) = G$.*

Proof. It follows immediately from Theorem 1.4. ∎

Lemma 1.7. *Let G be a $\mathcal{V}$-nilpotent group of class c, then $V_r(G) \leq V_{c-r}^*(G)$ for $0 \leq r \leq c$.*

Proof. We use induction on n, we know $V_0(G) = G$ and $V_c^*(G) = G$. Now, let $V_r(G) \leq V_{c-r}^*(G)$, so

$$V_{r+1}(G) = [V_r(G)V^*G] \leq [V_{c-r}^*(G)V^*G] \leq V_{c-r-1}^*(G). \quad ∎$$

Corollary 1.8. *Let G be a n-nilpotent group of class c, then $\gamma_{rn+1}(G) \leq Z_{(c-r)n}(G)$ for $0 \leq r \leq c$.*

Proof. This follows immediately from Lemma 1.7, where $\mathcal{V}$=$\mathcal{N}_n$. ∎

Lemma 1.9. *Let G be a group and $A, B \trianglelefteq G$, then $[AB,_n G] = [A,_n G][B,_n G]$.*

Proof. It is easily obtained by induction on n. ∎

2. n-Camina pairs

Let $\mathcal{N}_n$ be the variety of nilpotent groups of class at most n and $n \geq 2$, in this section, at first we define n-Camina pairs of groups and mention some properties of them. The main theorem shows that where (G, M) is an n-Camina pair of groups, there exists a connection between M and n-lower and n-upper central series of G.

Definition 2.1. A pair (G, M) is called a n-Camina pair of groups if $1 < M < G$ is a normal subgroup of G and for every elements $g_1, ..., g_n \in G - M$ and for every element $m \in M$, there exists element $y \in G$ such that $[y, [g_1, ..., g_n]] = m$.

In the following lemma, we mention the first and the most applied property of n-Camina pairs of groups:

Lemma 2.2. *Let (G, M) be a n-Camina pair of groups, then*

$$Z_n(G) \leq M \leq \gamma_{n+1}(G).$$

Proof. By definition, we have $M \leq \gamma_{n+1}(G)$ and if we assume that $Z_n(G) \nleq M$, then there exists element $g \in Z_n(G) - M$, so, for every elements $g_1, ..., g_{n-1} \in G - M$ and for every element $m \in M$, there exists element $y \in G$ such that $[y, [g_1, ..., g_{n-1}, g]] = m$, but $m = 1$ since $[g_1, ..., g_{n-1}, g] \in [\gamma_{n-1}(G), Z_n(G)] \leq Z_1(G) = Z(G)$, which is a contradiction. ∎

Lemma 2.3. *Let (G, M) be a n-Camina pair of groups and $H \trianglelefteq G$, $H < M$, then $\left(\frac{G}{H}, \frac{M}{H}\right)$ is an n-Camina pair of groups.*

Proof. It is clear by the definition. ∎

In the next theorem, we will prove that M is a term in both n-upper and n-lower central series, where (G, M) is a n-Camina pair of groups and G is n-nilpotent of class c .

Theorem 2.4. *Let (G, M) be a n-Camina pair of groups. If G is n-nilpotent of class c, then*

(i) $M = Z_{(c-r+1)n}(G)$,

(ii) $M = \gamma_{(r-1)n+1}(G)$,

for some r where $1 < r \leq c$.

Proof. (i) By assumption, we have (G, M) is a n-Camina pair of groups, so $Z_n(G) \leq M$. If $Z_n(G) = M$, then we obtain the conclusion. Thus, we may assume that $Z_n(G) < M$, by Lemma 2.3, $\left(\frac{G}{Z_n(G)}, \frac{M}{Z_n(G)}\right)$ is an n-Camina pair of groups, therefore,

$$Z_n\left(\frac{G}{Z_n(G)}\right) \leq \frac{M}{Z_n(G)}.$$

Hence,

$$\frac{Z_{2n}(G)}{Z_n(G)} \leq \frac{M}{Z_n(G)}.$$

Therefore, $Z_{2n}(G) \leq M$. By repetition of this process, it gives $M = Z_{(c-r+1)n}(G)$ for some r with $1 < r \leq c$. Hence, if for every $1 < r \leq c$, $Z_{(c-r+1)n}(G) < M$, then $Z_{(c-1)n}(G) < M$, but

$$\frac{G}{Z_{(c-1)n}(G)} = \frac{Z_{cn}(G)}{Z_{(c-1)n}(G)} = Z_n\left(\frac{G}{Z_{(c-1)n}(G)}\right).$$

Thus, $\gamma_{n+1}(G) \leq Z_{(c-1)n}(G) < M$, which is a contradiction.

(ii) By Corollary 1.8 and (i), we have

$$\gamma_{(r-1)n+1}(G) \leq Z_{(c-r+1)n}(G) = M.$$

We claim that $\gamma_{(r-1)n+1}(G) = M$. Hence, if $\gamma_{(r-1)n+1}(G) < M$, then

$$\left(\frac{G}{\gamma_{(r-1)n+1}(G)}, \frac{M}{\gamma_{(r-1)n+1}(G)}\right)$$

is an n-Camina pair of groups, so,

$$\frac{\gamma_{(r-2)n+1}(G)}{\gamma_{(r-1)n+1}(G)} \leq Z_n\left(\frac{G}{\gamma_{(r-1)n+1}(G)}\right) \leq \frac{M}{\gamma_{(r-1)n+1}(G)}.$$

Therefore, $\gamma_{(r-2)n+1}(G) \leq M = Z_{(c-r+1)n}(G)$. Hence $\gamma_{(c-1)n+1}(G) \leq Z_0(G) = 1$, which is a contradiction. ∎

Theorem 2.5. *Let (G, M) be a n-Camina pair of groups. If $M = Z_n(G)$ and G is n-nilpotent of class c, then $\frac{Z_{2n}(G)}{Z_n(G)}$ has exponent p.*

Proof. At first, we show that, if $H \trianglelefteq G$ and $H \subset Z_n(G)$, then

$$Z_n\left(\frac{G}{H}\right) = \frac{Z_n(G)}{H}.$$

By [4],

$$\frac{Z_n(G)H}{H} \subseteq Z_n\left(\frac{G}{H}\right)$$

and since $H \subset Z_n(G)$, we have

$$\frac{Z_n(G)}{H} \subseteq Z_n\left(\frac{G}{H}\right).$$

On the other hand, $(G, Z_n(G))$ is an n-Camina pair of groups and $H \subset Z_n(G)$, so

$$\left(\frac{G}{H}, \frac{Z_n(G)}{H}\right)$$

is an n-Camina pair of groups, hence $Z_n(\frac{G}{H}) \subseteq \frac{Z_n(G)}{H}$. By Lemma 1.9, we have

$$[(Z_{2n}(G))^p,_n G] = [Z_{2n}(G),_n G]^p \leq (Z_n(G))^p.$$

Therefore, by Corollary 1.2, we have

$$\frac{(Z_{2n}(G))^p}{(Z_n(G))^p} \leq Z_n\left(\frac{G}{(Z_n(G))^p}\right) = \frac{Z_n(G)}{(Z_n(G))^p},$$

that is, $(Z_{2n}(G))^p \leq Z_n(G)$. Thus, $\dfrac{Z_{2n}(G)}{Z_n(G)}$ has exponent p. ■

In closing this section, we mention two important results.

Corollary 2.6. *Let $(G, Z_n(G))$ be a n-Camina pair of groups. If G is n-nilpotent of class c, then $Z_n(G) = \gamma_{(c-1)n+1}(G)$.*

Proof. By Theorem 2.4, the result holds. ■

Corollary 2.7. *Let $(G, Z_n(G))$ be a n-Camina pair of groups. If G is n-nilpotent of class 2, then*

(i) $Z_n(G) = \gamma_{n+1}(G)$

(ii) $\dfrac{G}{Z_n(G)}$ *has exponent p.*

Proof. (i)By the previous corollary, the result holds.

(ii) Since $Z_{2n}(G) = G$, we obtain the assertion from Theorem 2.5. ■

3. Bounding $|Z_n(G)/Z_{n-1}(G)|$

Let $\mathcal{N}_n$ be the variety of nilpotent groups of class at most n ($n \geq 2$), and let $(G, Z_n(G))$ an n-Camina pair of groups. Then, by Lemma 2.2, we have $Z_n(G) \leq \gamma_{n+1}(G)$. Therefore, we have $Z_n(G) \leq G'$. Thus, we may assume that $Z_n(G) < G'$. In this section, we try to find a bound for $\left|\dfrac{Z_n(G)}{Z_{n-1}(G)}\right|$ in terms of $[G : Z_n(G)]$ and $[G : G']$ under some conditions.

Let G be a group. We define

$$D_G(g) = \{x \in G \mid [g, x] \in Z_n(G)\},$$

and

$$C_G(g) = \{x \in G \mid [g, x] \in Z_{n-1}(G)\},$$

which are subgroups of G and $C_G(g)$ is a subgroup of $D_G(g)$.

The following lemma is important in our study.

Lemma 3.1. *Let $(G, Z_n(G))$ be an n-Camina pair of groups. If we have $g_1, ..., g_n \in G - Z_n(G)$, then*

$$\frac{D_G([g_1, ..., g_n])}{C_G([g_1, ..., g_n])} \cong \frac{Z_n(G)}{Z_{n-1}(G)} = Z\left(\frac{G}{Z_{n-1}(G)}\right).$$

In particular, $(D_G([g_1, ..., g_n]))' \leq C_G([g_1, ..., g_n])$.

Proof. We define the map

$$f : D_G([g_1, ..., g_n]) \to \frac{Z_n(G)}{Z_{n-1}(G)},$$

where

$$f(d) = [d, [g_1, ..., g_n]]Z_{n-1}(G).$$

By definition of $D_G(g_1, ..., g_n)$, we know that f is well-defined and f is a homomorphism. Since $(G, Z_n(G))$ is an n-Camina pair of groups, for every element $Z_{n-1}(G) \in \dfrac{Z_n(G)}{Z_{n-1}(G)}$ there exists $y \in G$, so that

$$[y, [g_1, ..., g_n]] = z \in Z_n(G).$$

Thus, $y \in D_G([g_1, ..., g_n])$. By the definition of $D_G([g_1, ..., g_n])$, that is f is onto. Clearly, $ker(f) = C_G([g_1, ..., g_n])$. By the 1^{st} isomorphism theorem, we have

$$\frac{D_G([g_1, ..., g_n])}{C_G([g_1, ..., g_n])} \cong \frac{Z_n(G)}{Z_{n-1}(G)}.$$ ∎

Now, we find the first bound for $\left|\dfrac{Z_n(G)}{Z_{n-1}(G)}\right|$ in terms of $[G : Z_n(G)]$.

Theorem 3.2. *Let $(G, Z_n(G))$ be a n-Camina pair of groups and G be a p-group. If there exist $g_1, ..., g_n \in G$ such that $o([g_1, ..., g_n]Z_n(G)) = p^k$, where $k \geq 2$ and $\dfrac{Z_n(G)}{Z_{n-1}(G)}$ is an elementary abelian p-group, then*

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^k p^k \leq [G : Z_n(G)].$$

Proof. At first, we show that

$$C_G([g_1, ..., g_n]^{p^{i+1}}) \geq D_G([g_1, ..., g_n]^{p^i}) \text{ for } 0 \leq i \leq k-1.$$

Suppose that $a \in D_G([g_1, ..., g_n]^{p^i})$, so $[a, [g_1, ..., g_n]^{p^i}] \in Z_n(G)$, that is,

$$[a, [g_1, ..., g_n]^{p^i}],\ Z_{n-1}(G) \in \frac{Z_n(G)}{Z_{n-1}(G)}.$$

Since $\dfrac{Z_n(G)}{Z_{n-1}(G)}$ is an elementary abelian p-group, we have

$$([a, [g_1, ..., g_n]^{p^i}]Z_{n-1}(G))^p = Z_{n-1}(G).$$

Therefore, $([aZ_{n-1}(G), [g_1Z_{n-1}(G), ..., g_nZ_{n-1}(G)]^{p^i}])^p = Z_{n-1}(G)$.

Since

$$[aZ_{n-1}(G), [g_1Z_{n-1}(G), ..., g_nZ_{n-1}(G)]^{p^i}] \in Z\left(\frac{G}{Z_{n-1}(G)}\right),$$

we have

$$\begin{aligned}&([aZ_{n-1}(G), [g_1Z_{n-1}(G), ..., g_nZ_{n-1}(G)]^{p^i}])^p\\ &\quad= [aZ_{n-1}(G), [g_1Z_{n-1}(G), ..., g_nZ_{n-1}(G)]^{p^{i+1}}] = Z_{n-1}(G).\end{aligned}$$

So, $[a, [g_1, ..., g_n]^{p^{i+1}}]Z_{n-1}(G) = Z_{n-1}(G)$, that is, $[a, [g_1, ..., g_n]^{p^{i+1}}] \in Z_{n-1}(G)$. This implies that $a \in C_G([g_1, ..., g_n]^{p^{i+1}})$.

Since $[g_1, ..., g_n]^{p^i} \notin Z_n(G)$ for $0 \leq i \leq k-1$, by Lemma 3.1,

$$\frac{D_G([g_1, ..., g_n]^{p^i})}{C_G([g_1, ..., g_n]^{p^i})} \cong \frac{Z_n(G)}{Z_{n-1}(G)}.$$

Clearly, $[g_1, ..., g_n] \in C_G([g_1, ..., g_n])$, hence

$$[g_1, ..., g_n]Z_n(G) \in \frac{C_G([g_1, ..., g_n])}{Z_n(G)}.$$

It follows that $p^k | [C_G([g_1, ..., g_n]) : Z_n(G)]$. We have

$$\begin{aligned}[G : Z_n(G)] &= [G : C_G([g_1, ..., g_n])][C_G([g_1, ..., g_n]) : Z_n(G)]\\ &\geq [G : C_G([g_1, ..., g_n])]p^k.\end{aligned}$$

Now, we calculate $[G : C_G([g_1, ..., g_n])]$. Clearly,

$$\begin{aligned}&[G : C_G([g_1, ..., g_n])]\\ &\quad= [G : (C_G([g_1, ..., g_n])^{p^k}] \prod_{i=0}^{k-1} [C_G([g_1, ..., g_n]^{p^{i+1}}) : C_G([g_1, ..., g_n]^{p^i})].\end{aligned}$$

Since $[g_1, ..., g_n]^{p^k} \in Z_n(G)$, we have $C_G([g_1, ..., g_n]^{p^k}) = G$, so

$$\begin{aligned}[G : C_G([g_1, ..., g_n])] &= \prod_{i=0}^{k-1} [C_G([g_1, ..., g_n]^{p^{i+1}}) : C_G([g_1, ..., g_n]^{p^i})]\\ &\geq \prod_{i=0}^{k-1} [D_G([g_1, ..., g_n]^{p^i}) : C_G([g_1, ..., g_n]^{p^i})]\\ &= \prod_{i=0}^{k-1} \left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|\\ &= \left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^k.\end{aligned}$$

Finally, we have

$$[G : Z_n(G)] \geq \left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^k p^k. \qquad \blacksquare$$

Now, we mention a sufficient condition for $\frac{D_G([g_1, ..., g_n])}{Z_n(G)}$ to be abelian, where $g_1, ..., g_n \in G - Z_n(G)$.

Lemma 3.3. *Let $(G, Z_n(G))$ be an n-Camina pair of groups. If there exist $g_1, ..., g_n \in G - Z_n(G)$ such that $C_G([g_1, ..., g_n]) \cap G' = Z_n(G)$, then $\frac{D_G([g_1, ..., g_n])}{Z_n(G)}$ is abelian.*

Proof. By Lemma 3.1, $(D_G([g_1, ..., g_n]))' \leq C_G([g_1, ..., g_n])$.

On the other hand, $(D_G([g_1, ..., g_n]))' \leq G'$ so

$$(D_G([g_1, ..., g_n]))' \leq C_G([g_1, ..., g_n]) \cap G' = Z_n(G).$$

Thus, $\frac{D_G([g_1, ..., g_n])}{Z_n(G)}$ is abelian. ■

In the next lemma, we give a bound for $\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|$ in terms of the index of the derived subgroup.

Lemma 3.4. *Let $(G, Z_n(G))$ be an n-Camina pair of groups where $n \geq 2$. If there exist $g_1, ..., g_n \in Z_{n+1}(G) - Z_n(G)$, then*

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| \leq [G : G'Z_{n+1}(G)].$$

In particular,

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| \leq [G : G'].$$

Proof. For every element $a \in G$, we have

$$[[g_1, ..., g_n], a] \in [[Z_{n+1}(G), G] \leq Z_n(G),$$

that is, $a \in D_G([g_1, ..., g_n])$, so $G = D_G([g_1, ..., g_n])$.

By Lemma 3.1, $(D_G([g_1, ..., g_n]))' \leq C_G([g_1, ..., g_n])$ so $G' \leq C_G([g_1, ..., g_n])$. On the other hand, for every element $x \in Z_{n+1}(G)$, we have

$$[[g_1, ..., g_n], x] \in [\gamma_n(G), Z_{n+1}(G)] \leq Z_1(G) \leq Z_{n-1}(G).$$

Therefore, $x \in C_G([g_1, ..., g_n])$. This implies that $Z_{n+1}(G) \leq C_G([g_1, ..., g_n])$. So, $G'Z_{n+1}(G) \leq C_G([g_1, ..., g_n])$. Thus,

$$\begin{aligned}\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| &= [D_G([g_1, ..., g_n]) : C_G([g_1, ..., g_n])] \\ &= [G : C_G([g_1, ..., g_n])] \leq [G : G'Z_{n+1}(G)].\end{aligned} \qquad \blacksquare$$

Now, we show that $\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|$ can be bound by $[G : Z_n(G)]$ under some conditions.

Theorem 3.5. *Let $(G, Z_n(G))$ be an n-Camina pair of groups and G be a p-group. If there exist $g_1, ..., g_n \in Z_{n+1}(G) - Z_n(G)$ such that $\frac{D_G([g_1, ..., g_n])}{Z_n(G)}$ is abelian and $[G : D_G([g_1, ..., g_n])] = p$, then $\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^2 \leq [G : Z_n(G)]$.*

Proof. Since G is a finite p-group and $[G : D_G([g_1, ..., g_n])] = p$, we have

$$D_G([g_1, ..., g_n]) \trianglelefteq G$$

and there exists $b \in G - D_G([g_1, ..., g_n])$ such that $G = \langle b \rangle D_G([g_1, ..., g_n])$. Since $b \in D_G(b)$, we have $G = D_G(b)D_G([g_1, ..., g_n])$. It follows that

$$\frac{D_G([g_1, ..., g_n]) \cap D_G(b)}{Z_n(G)} \leq \frac{Z_{n+1}(G)}{Z_n(G)},$$

so

$$D_G([g_1, ..., g_n]) \cap D_G(b) \leq Z_{n+1}(G).$$

We have

$$\begin{aligned}
&[G : D_G([g_1, ..., g_n]) \cap D_G(b)] \\
&\quad = [G : D_G([g_1, ..., g_n])][D_G([g_1, ..., g_n]) : D_G([g_1, ..., g_n]) \cap D_G(b)] \\
&\quad = p[G : D_G(b)].
\end{aligned}$$

Since $b \in G - D_G([g_1, ..., g_n])$, we have $[C_G(b) : Z_n(G)] \geq p$. Also, by Lemma 3.4,

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| \leq [G : G'Z_{n+1}(G)].$$

Hence, we may conclude that

$$\begin{aligned}
\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| &\leq [G : G'Z_{n+1}(G)] \leq [G : Z_{n+1}(G)] \leq [G : D_G([g_1, ..., g_n]) \cap D_G(b)] \\
&= p[G : D_G(b)] \leq [C_G(b) : Z_n(G)][G : D_G(b)].
\end{aligned}$$

By Lemma 3.1, we deduce that $[D_G(b) : C_G(b)] = \left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|$. This result implies that

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^2 \leq [G : Z_n(G)]. \qquad \blacksquare$$

As a corollary, we obtain our bound when $[G' : Z_n(G)] = p$ and there exist $g_1, ..., g_n \in Z_{n+1}(G) - Z_n(G)$ such that $C_G([g_1, ..., g_n]) \cap G' < G'$.

Corollary 3.6. *Let $(G, Z_n(G))$ be an n-Camina pair of groups. If $[G' : Z_n(G)] = p$ and there exist $g_1, ..., g_n \in Z_{n+1}(G) - Z_n(G)$ such that $C_G([g_1, ..., g_n]) \cap G' < G'$, then*

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^2 \leq [G : Z_n(G)].$$

Proof. Since $[G' : Z_n(G)] = p$, we have

$$\left(\frac{G}{Z_n(G)}\right)' = \frac{G'}{Z_n(G)} \leq Z(\frac{G}{Z_n(G)}).$$

For any element $g \in G$, we define the map $f : G \to \frac{G'}{Z_n(G)}$, where $f(x) = [g, x]Z_n(G)$ for every element $x \in G$. It is easy to show that f is a homomorphism and $Ker(f) = D_G(g)$, so $\left|\frac{G}{D_G(g)}\right| \leq |\frac{G'}{Z_n(G)}| = p$ for every $g \in G$. Clearly, $Z_n(G) \leq C_G([g_1, ..., g_n]) \cap G'$, so

$$p = [G' : Z_n(G)] = [G' : C_G([g_1, ..., g_n]) \cap G'][C_G([g_1, ..., g_n]) \cap G' : Z_n(G)].$$

Therefore, $C_G([g_1, ..., g_n]) \cap G' = Z_n(G)$. Thus, by Lemma 3.3, $\frac{D_G([g_1, ..., g_n])}{Z_n(G)}$ is abelian. Since $C_G([g_1, ..., g_n]) \cap G' < G'$, we have $[G : D_G([g_1, ..., g_n])] = p$. So, the assertion is obtained from Theorem 3.5. ∎

Now, we define the set $\mathcal{L}(G)$, where $(G, Z_n(G))$ is an n-Camina pair of groups

$$\mathcal{L}(G) = \{x \in G \mid C_G(x) \cap G' > Z_n(G)\}.$$

If $G' = Z_n(G)$, then $\mathcal{L}(G)$ is empty, so $\mathcal{L}(G)$ is nonempty if $Z_n(G) < G'$. Clearly, G' and $C_G(G')$ are subsets of $\mathcal{L}(G)$.

In the next lemma, we give a second characterization of $\mathcal{L}(G)$.

Lemma 3.7. *Let $(G, Z_n(G))$ be an n-Camina pair of groups with $Z_n(G) < G'$. Then, we have $\mathcal{L}(G) = \bigcup_{a \in G' - Z_n(G)} C(a)$.*

Proof. If $x \in \mathcal{L}(G)$, then $C_G(x) \cap G' > Z_n(G)$, so there exists $b \in C_G(x) \cap G' - Z_n(G)$. Hence, $x \in C_G(b) \subseteq \bigcup_{a \in G' - Z_n(G)} C(a)$.

By Lemma 3.1, we deduce that $\mathcal{L}(G) \subseteq \bigcup_{a \in G' - Z_n(G)} C(a)$. This result implies that, if $x \in \bigcup_{a \in G' - Z_n(G)} C(a)$, then there exists $b \in G' - Z_n(G)$ such that $x \in C_G(b)$. It follows that $b \in (C_G(x) \cap G') - Z_n(G)$ so $Z_n(G) < C_G(x) \cap G'$, therefore, $x \in \mathcal{L}(G)$. ∎

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