QUANTUM CONDITIONAL LOGICAL ENTROPY OF DYNAMICAL SYSTEMS

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Abstract. In this paper, we will define the notion of conditional logical entropy of dynamical systems on a quantum logic. We will prove some of its properties. The relation between this measure and the notion of logical entropy of quantum dynamical systems which was defined in my paper (Open Physics, 14:1-5, 2016), will be presented.

Keywords: logical entropy, quantum logic, quantum dynamical system, conditional logical entropy dynamical system.

2010 MSC: 03-XX, 05A18, 28D20.

1. Introduction

Birkhoff and Von Neumann in [1] have introduced the quantum logic approach. Entropy serves as a measure of information of the considered experiment and has an important application in dynamical systems ([4], [6], [7], [16]). Yuan, Khare and Roy, using the notion of state of quantum logic, introduced Shannon's entropy of finite partitions on a quantum logic ([11], [12], [20]). In my previous paper [3] the notion of relative entropy with countable partitions on a quantum logic was presented. Logical entropy is a measure on set of ordered pairs [8]. In 1982, Rao, Good, Patil and Taillie defined and studied the notion of logical entropy ([13], [14], [15]). Rao introduced precisely this concept as quadratic entropy [15] and in the years 2009 and 2013, this concept was discussed by Ellerman in ([8], [9], [10]). In [4], Ebrahimzadeh defined and studied the notion of logical entropy of quantum dynamical systems. Huls [2] and Zang [21] introduced the notion of conditional entropy of dynamical systems in the non-fuzzy sence. After that, Tok defined and studied the fuzzy conditional entropy of dynamical systems in ([17], [18]).

In the present paper, the notion of conditional logical entropy of quantum dynamical systems will be defined and some of its results will be proved.

2. Basic notions

We first, present some basic definitions and properties that will be useful in further considerations.

Definition 2.1. [20] A quantum logic QL is a σ-orthomodular lattice, i.e., a lattice L $(L, \leq, \vee, \wedge, 0, 1)$ with the smallest element 0 and the greatest element 1, an operation $' : L \to L$ such that the following properties hold for all $a, b, c \in L$:

(i) $a^{''} = a$, $a \leq b \Rightarrow b^{'} \leq a^{'}$, $a \vee a^{'} = 1$, $a \wedge a^{'} = 0$;

(ii) Given any finite sequence $(a_i)_{i\in I}$, $a_i \leq a_{j'}$, $i \neq j$, the join $\vee_{i\in\mathbb{N}} a_i$ exists in L;

(iii) L is orthomodular: $a \leq b \Rightarrow b = a \vee (b \wedge a^{'})$.

Two elements $a, b \in QL$ are called orthogonal if $a \leq b^{'}$ and denoted by $a \perp b$. A sequence $(a_i)_{i\in I}$ is said orthogonal if $a_i \perp a_j$, $\forall i \neq j$.

Definition 2.2. [20] Let L be a QL. A map $s : L \to [0, 1]$ is a state iff $s(1) = 1$ and for any orthogonal sequence $(a_i)_{i\in I}$, $s(\vee_{i\in I} a_i) = \sum_{i\in I} s(a_i)$.

Definition 2.3. [20] Let $P = \{a_1, ..., a_n\}$ be a finite system of elements of a QL. P is called to be a $\vee$-orthogonal system iff $\vee_{i=1}^{k} a_i \perp a_{k+1}$, $\forall k$.

Definition 2.4. [20] A system $P = \{a_1, ..., a_n\} \subset L$ is said to be a partition of L corresponding to a state s if:

(i) P is a $\vee$-orthogonal system;

(ii) $s(\vee_{i=1}^{n} a_i) = 1$.

Note that from Definition 2.2, we obtain $\sum_{i=1}^{n} s(a_i) = 1$.

Definition 2.5. [20] Let the system $(b_1, ..., b_m)$ be any partition corresponding to a state s and $a \in L$. The state s is said has Bayes' Property if $s(\vee_{j=1}^{m}(a \wedge b_j)) = s(a)$.

Lemma 2.6. [20] *Let $Q = (b_1, ..., b_m)$ be a partition on L, and $a \in L$, and the state s has Bayes' Property. Then* $\sum_{j=1}^{m} s(a \wedge b_j) = s(a)$.

Let $P = \{a_1, ..., a_n\}$ and $Q = \{b_1, ..., b_m\}$ be two finite partitions of a QL corresponding to a state s. The common refinement of these partitions is

$$P \vee Q = \{a_i \wedge b_j : a_i \in P, b_j \in R\}.$$

Definition 2.7. [4] Let $P = \{a_1, ..., a_n\}$ be a partition of a QL corresponding to a state s. The logical entropy of P with respect to state s is defined by

$$h_s^l(P) = \sum_{i=1}^{n} s(a_i)(1 - s(a_i)).$$

Definition 2.8. [4] Let $P = \{a_1, ..., a_n\}$ and $Q = \{b_1, ..., b_m\}$ be two partitions of a QL corresponding to a state s. The conditional logical entropy of P given R with respect to state s is defined as

$$h_s^l(P|Q) = \sum_{i=1}^{n} \sum_{j=1}^{m} s(a_i \wedge b_j)(s(b_j) - s(a_i \wedge b_j)).$$

Proposition 2.9. *Let P_1, P_2 and Q be finite partitions of a QL corresponding to a state s having Bayes' Property. Then*

(i) $h_s^l(P_1|Q) \geq 0$,

(ii) $h_s^l(P_1 \vee P_2|Q) = h_s^l(P_1|Q) + h_s^l(P_2|P_1 \vee Q)$,

(iii) $h_s^l(P_1 \vee Q) = h_s^l(Q) + h_s^l(P_1|Q)$,

(iv) $h_s^l(P_1|Q) \leq h_s^l(P_1)$.

Proof. See [4]. ∎

Definition 2.10. [4] Let $P = \{a_1, ..., a_n\}$ and $Q = \{b_1, ..., b_m\}$ be two partitions of a QL corresponding to a state s. We say Q is a s-refinement of P, denoted by $P \preceq_s Q$, if there exists a partition $I(1), ..., I(n)$ of the set $\{1, ..., m\}$ such that $a_i = \vee_{j \in I(i)} b_j$ for every $i = 1, ..., n$.

Proposition 2.11. *Let $P = \{a_1, ..., a_n\}, Q = \{b_1, ..., b_m\}$ and $R = \{c_1, ..., c_r\}$ be partitions of a QL corresponding to a state s. Then*

(i) *$P \preceq_s Q$ implies that $h_s^l(P) \leq h_s^l(Q)$;*

(ii) *If $P \preceq_s Q$ and the QL be distributive then $h_s^l(P|R) \leq h_s^l(Q|R)$.*

Proof. See [4]. ∎

3. Conditional Logical entropy of quantum dynamical systems

The assertion of the following proposition will be proved that will be useful in further propositions.

Proposition 3.1. *Let P_1, P_2 and Q be finite partitions of a QL corresponding to a state s having Bayes' Property. Then*

(i) $h_s^l(P_1|P_2 \vee Q) \leq h_s^l(P_1|P_2)$,

(ii) $h_s^l(P_1 \vee P_2|Q) \leq h_s^l(P_1|Q) + h_s^l(P_2|Q)$.

Proof. (i) Let $P_1 = \{a_1, ..., a_n\}$, $P_2 = \{b_1, ..., b_m\}$ and $Q = \{c_1, ..., c_r\}$. Since $P_1 \vee P_2$ is a partition, by Lemma 2.6, we have

$$\sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k) = s(a_i \wedge b_j)$$

Thus

$$\begin{aligned} h_s^l(P_1|P_2) &= \sum_{i=1}^{n}\sum_{j=1}^{m} s(a_i \wedge b_j)(s(b_j) - s(a_i \wedge b_j)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k)(\sum_{k=1}^{r} s(b_j \wedge c_k) - \sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k)) \\ &\geq \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k)(s(b_j \wedge c_k) - s(a_i \wedge b_j \wedge c_k)) \\ &= h_s^l(P_1|P_2 \vee Q). \end{aligned}$$

(ii) From the first part of this proposition and Proposition 2.9 (ii), we may write

$$\begin{aligned} h_s^l(P_1 \vee P_2|Q) &= h_s^l(P_1|Q) + h_s^l(P_2|P_1 \vee Q) \\ &\leq h_s^l(P_1|Q) + h_s^l(P_2|Q). \end{aligned}$$ ■

Definition 3.2. [12] Let L be a QL and $\varphi : L \to L$ be a map with the following properties:

(i) $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$, $\forall a, b \in B$;

(ii) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, $\forall a, b \in B$;

(iii) $\varphi(a^{'}) = (\varphi(a))^{'}$, $\forall a \in B$.

$\varphi : L \to L$ with respect to a state s is called state preserving if $s(\varphi(a)) = s(a)$ for every $a \in L$. Then the triple (L, s, φ) is said a quantum dynamical system where the state s having Bayes' Property.

Definition 3.3. [4] Let (L, s, φ) be a quantum dynamical system and P be a partition of (L, s). The logical entropy of T respect to P is defined by

$$h_s^l(\varphi, P) = \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=1}^{n} \varphi^i P).$$

The logical entropy of φ is defined by

$$h_s^l(\varphi) = \sup_{P} h_s^l(\varphi, P)$$

where the supremum is taken over all finite partitions of (L, s).

In the following proposition, the existence of the limit in Definition 3.5, is shown.

Proposition 3.4. *Let (L, s, φ) be a quantum dynamical system and P, Q be partitions of (L, s). Then* $\lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=0}^{n-1}\varphi^i P|Q)$ *exists.*

Proof. Let $a_n = h_s^l(\vee_{i=0}^{n-1}\varphi^i P|Q)$. We show that for $n, t \in \mathbf{N}$, $a_{n+t} \le a_n + a_t$ and then, by Theorem 4.9 in [19], $\lim_{n\to\infty} \frac{1}{n} a_n$ exists and equals $\inf_n \frac{a_n}{n}$. By Proposition 3.1, (ii) we have

$$\begin{aligned} a_{n+t} &= h_s^l(\vee_{i=0}^{n+t-1}\varphi^i P|Q) \\ &\le h_s^l(\vee_{i=0}^{n-1}\varphi^i P|Q) + h_s^l(\vee_{i=n}^{n+t-1}\varphi^i P|Q) \\ &= a_n + h_s^l(\vee_{i=0}^{t-1}\varphi^{n+i} P|Q) \\ &= a_n + h_s^l(\varphi^n(\vee_{i=0}^{t-1}\varphi^i P)|Q) \\ &= a_n + h_s^l(\vee_{i=0}^{t-1}\varphi^i P|Q) \\ &= a_n + a_t. \end{aligned}$$

■

The second stage and the final stage of the definition of the conditional logical entropy of a quantum dynamical system (L, s, φ) are given in the next definitions.

Definition 3.5. Let (L, s, φ) be a quantum dynamical system and P be a partition of (L, s). The quantum conditional logical entropy of P with respect to Q is defined by:

$$h_s^l(\varphi, P|Q) = \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=0}^{n-1}\varphi^i P|Q).$$

Definition 3.6. Let (L, s, φ) be a quantum dynamical system and Q be a partition of (L, s). The quantum conditional logical entropy of φ is defined as following:

$$h_s^l(\varphi|Q) = \sup_P h_s^l(\varphi, P|Q)$$

where the supremum is taken over all finite partitions P of (L, s).

Now, the notion $h_s^l(\varphi, P|Q)$ under the relation of s-refinement will be studied.

Proposition 3.7. *Let (L, s, φ) be a quantum dynamical system and P_1, P_2 and Q be partitions of (L, s). Then*

(i) $h_s^l(\varphi, P_1|Q) \ge 0$,

(ii) $Q = \{0, 1\} \Longrightarrow h_s^l(\varphi, P_1|Q) = h_s^l(\varphi, P_1)$,

(iii) $h_s^l(\varphi, P_1|Q) \le h_s^l(\varphi, P_1)$,

(iv) *If $P_1 \preceq_s P_2$ and L be distributive then* $h_s^l(\varphi, P_1|Q) \le h_s^l(\varphi, P_2|Q)$,

(v) *If φ is invertible then* $h_s^l(\varphi, \varphi^{-1}P_1|\varphi^{-1}Q) = h_s^l(\varphi, P_1|Q)$.

Proof. (i), (ii) are clear.

(iii) Since for each $n \in \mathbb{N}$, $\vee_{i=0}^{n-1}\varphi^i P_1$ is a partition of (L, s), from Proposition 2.9 (iv) we get $h_s^l(\vee_{i=0}^{n-1}\varphi^i P_1|Q) \le h_s^l(\vee_{i=0}^{n-1}\varphi^i P_1)$. So by Definitions 3.3 and 3.5 the assertion holds.

(iv) $P_1 \preceq_s P_2$ implies that for each $n \in \mathbb{N}$, $\vee_{i=0}^{n-1}\varphi^i P_1 \preceq_s \vee_{i=0}^{n-1}\varphi^i P_2$. So, by Proposition 2.11 (ii), we obtain the result.

(v) From [12], we have $\vee_{i=0}^{n-1}\varphi^i(\varphi^{-1}P_1) = \vee_{i=0}^{n-1}\varphi^{-1}(\varphi^i P_1) = \varphi^{-1}(\vee_{i=0}^{n-1}\varphi^i P_1)$. So

$$\begin{aligned} h_s^l(\varphi, \varphi^{-1}P_1|\varphi^{-1}Q) &= \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i(\varphi^{-1}P_1)|\varphi^{-1}Q) \\ &= \lim_{n\to\infty}\frac{1}{n}h_s^l(\varphi^{-1}(\vee_{i=0}^{n-1}\varphi^i P_1)|\varphi^{-1}Q) \\ &= \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i P_1|Q) \\ &= h_s^l(\varphi, P_1|Q). \end{aligned}$$

■

In the next proposition, we study the notion $h_s^l(\varphi, P|Q)$ under the concept of common refinement.

Proposition 3.8. *Let (L, s, φ) be a quantum dynamical system and P_1, P_2 and Q be partitions of (L, s). Then*

$$h_s^l(\varphi, P_1 \vee P_2|Q) \le h_s^l(\varphi, P_1|Q) + h_s^l(\varphi, P_2|Q).$$

Proof. We first show that $\vee_{i=0}^{n-1}\varphi^i(P_1 \vee P_2) = (\vee_{i=0}^{n-1}\varphi^i P_1) \vee (\vee_{i=0}^{n-1}\varphi^i P_2)$. We have

$$\begin{aligned} \varphi(P_1 \vee P_2) &= \{\varphi(a \wedge b) : a \in P_1, b \in P_2\} \\ &= \{\varphi(a) \wedge \varphi(b) : a \in P_1, b \in P_2\} \\ &= \varphi(P_1) \vee \varphi(P_2). \end{aligned}$$

Thus since $\varphi(P_1)$ and $\varphi(P_2)$ are partitions of (L, s) we get

$$\varphi^2(P_1 \vee P_2) = \varphi(\varphi(P_1 \vee P_2)) = \varphi(\varphi(P_1) \vee \varphi(P_2)) = \varphi^2(P_1) \vee \varphi^2(P_2).$$

So by induction we have $\varphi^n(P_1 \vee P_2) = \varphi^n(P_1) \vee \varphi^n(P_2)$. Hence we may write

$$\begin{aligned} \vee_{i=0}^{n-1}\varphi^i(P_1 \vee P_2) &= \varphi(P_1) \vee \varphi(P_2) \vee \varphi^2(P_1) \vee \varphi^2(P_2) \vee ...\varphi^n(P_1) \vee \varphi^n(P_2) \\ &= (\vee_{i=0}^{n-1}\varphi^i P_1) \vee (\vee_{i=0}^{n-1}\varphi^i P_2). \end{aligned}$$

Now, from Proposition 3.1 (ii), we have

$$\begin{aligned} h_s^l(\varphi, P_1 \vee P_2|Q) &= \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i(P_1 \vee P_2)|Q) \\ &= \lim_{n\to\infty}\frac{1}{n}h_s^l((\vee_{i=0}^{n-1}\varphi^i P_1) \vee (\vee_{i=0}^{n-1}\varphi^i P_2)|Q) \\ &\le \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i P_1)|Q) + \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i P_2)|Q) \\ &= h_s^l(\varphi, P_1|Q) + h_s^l(\varphi, P_2|Q). \end{aligned}$$

■

In the following proposition, some ergodic properties of $h_s^l(\varphi|Q)$ will be studied.

Proposition 3.9. *Let (L, s, φ) be a quantum dynamical system and Q be a partition of (L, s). Then*

(i) $h_s^l(id|Q) = 0$, *where id is the identity function,*

(ii) $h_s^l(\varphi|Q) \leq h_s^l(\varphi)$,

(iii) $Q = \{0, 1\} \Longrightarrow h_s^l(\varphi|Q) = h_s^l(\varphi)$,

(iv) $h_s^l(\varphi|Q_1 \vee Q_2) \leq h_s^l(\varphi|Q_1)$.

Proof. (i) Let P be an arbitrary partition of (L, s). Since $\vee_{i=0}^{n-1}(id)^i P = P$, we have

$$h_s^l(id|Q) = \sup_P h_s^l(id, P|Q) = \lim_{n\to\infty} \frac{1}{n} h_s^l(P|Q) = 0.$$

(ii), (iii) Follow from Proposition 3.7 (ii) and (iii).
(iv) From Proposition 3.1 (i), for each partition P of (L, s) we get

$$h_s^l(\varphi, P|Q_1 \vee Q_2) \leq h_s^l(\varphi, P|Q_1).$$

So, by Definition 3.6, the assertion holds. ∎

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Accepted: 17.05.2016