ON THE OVERGROUPS OF $\mathrm{SL}(1, K)$ IN $\mathrm{GL}(r, F)$

Xin Hou
Shangzhi Li

LMIB and School of Mathematics and Systems Science
Beihang University
Beijing, 100191
China
e-mails: houge19870512@126.com
lisz@buaa.edu.cn

Abstract. For some pair of division rings K and F with $K \supset F$ and $\dim_F K = r$, we want to determine the overgroups of $K^* = \mathrm{SL}(1, K)$ in $\mathrm{GL}(r, F)$ and obtain the maximal subgroups of $\mathrm{GL}(r, F)$. Let R stand for real number field, C for complex one and Q for the skew-field of quaternions. All the overgroups of $C^* = \mathrm{SL}(1, C)$ in $\mathrm{GL}(2, R)$ and $Q^* = \mathrm{SL}(1, Q)$ in $\mathrm{GL}(2, C)$ are found in this paper.

Keywords: classical group, maximal subgroup, overgroup, quaternion.
2010 Mathematics Subject Classification: 20G15, 20E28.

1. Introduction

The subgroup structure of classical groups, especially maximal subgroups in classical groups, is one of the most important topics of group theory. By a theorem of Aschbacher(see [2]), the maximal subgroups of a classical group over finite field must be either a member of one of the classes $C_1 \sim C_8$, or an almost simple group. Under the guidance of this theorem, series of works have been done to approach the complete classification of the maximal subgroups(see [3], [8]). The second author of this paper has done much work(see [4], [5], [6], [7]) on the maximality of the subgroups in Aschbacher's classes. However, the results are for the classical groups over arbitrary fields, not necessarily finite, or sometimes over arbitrary division rings.

Let K, F be two division rings, with $K \supset F$, and $\dim_F K = r < \infty$ We can regard K as a left F-space. Write n-dimensional left K-space as $V(n, K)$, it can be regarded as an nr-dimensional left space $V = V(nr, F)$ over F. Thus the $\mathrm{GL}(n, K)$ acting on $V(n, K)$ is a subgroup of $\mathrm{GL}(nr, F)$ acting on $V(nr, F)$. In the article [5], when $n \geq 2$, the overgroups of $\mathrm{SL}(n, K)$ in $\mathrm{GL}(nr, F)$ and the overgroups of $\mathrm{Sp}(n, K, f)$ in $\mathrm{GL}(nr, F)$ were determined. As an application of the main result of the paper to the case F is a finite field and r is prime, the maximality of the subgroups in Aschbacher's class C_3 was obtained. However, when $n = 1$, it is remained to determine the overgroups of $\mathrm{SL}(1, K)$ in $\mathrm{GL}(r, F)$. In this paper,we shall determine the overgroups of C^* in $\mathrm{GL}(2, R)$ and the overgroups of Q^* in $\mathrm{GL}(2, C)$ and obtain the maximal subgroups of $\mathrm{GL}(2, R)$ and $\mathrm{GL}(2, C)$. Our main results are the following two theorems.

Theorem 1.1 *Let R be the real number field and C the complex one. The group $\mathrm{SL}(1, C)$ is just the multiplicative group C^*, which can be written as the group $\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}(2, R) \mid a, b \in R\right\}$. Let X be the overgroup of C^* in $\mathrm{GL}(2, R)$, $C^* = \mathrm{SL}(1, C) < X < \mathrm{GL}(2, R)$, then one of the following holds.*

- $X = C^* \rtimes \mathrm{Aut}C/R = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad or \quad \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \mathrm{GL}(2, R) \mid a, b \in R\right\}$, *which is the normalizer of C^* in $\mathrm{GL}(2, R)$.*
- $X = \{A \in \mathrm{GL}(2, R) \mid \det A > 0\}$, *which is the group made up of all the elements in $\mathrm{GL}(2, R)$ whose determinant are positive real numbers.*

Theorem 1.2 *Let C be the complex number field and Q the skew-field of quaternions. The group $\mathrm{SL}(1, Q)$ is just the multiplicative group Q^*, which can be written as the group $\left\{\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{GL}(2, C) \mid \alpha, \beta \in C\right\}$. Let X be the overgroup of Q^* in $\mathrm{GL}(2, C)$, $Q^* = \mathrm{SL}(1, Q) < X < \mathrm{GL}(2, C)$, then one of the following holds.*

- $Q^* = \mathrm{SL}(1, Q) \lhd X \leq Q^* \rtimes \mathrm{Aut}Q/C$. *Let* $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, *Then* $Q^* \rtimes \mathrm{Aut}Q/C = \left\{\begin{pmatrix} \alpha & \beta \\ -\overline{\beta}e^{\mathrm{i}\theta} & \overline{\alpha}e^{\mathrm{i}\theta} \end{pmatrix} \in \mathrm{GL}(2, C) \mid \alpha, \beta \in C, \theta \in [0, 2\pi)\right\}$. *And* $Q^* \rtimes \mathrm{Aut}Q/C$ *is the normalizer of Q^* in $\mathrm{GL}(2, C)$.*
- $H \lhd X < \mathrm{GL}(2, C)$. *Here $H = \{A \in \mathrm{GL}(2, C) \mid 0 < \det A \in R\}$ which is the group made up of all the elements in $\mathrm{GL}(2, C)$ whose determinant are positive real numbers. $X = H \cdot \left\langle \left\{\begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}(\mathrm{k}\alpha)} \end{pmatrix} | \alpha \in I_\alpha\right\}\right\rangle$, where I_α is a set of angles in $[0, 2\pi)$.*

2. Preliminaries

While K can be regarded as a left F-space, we can take an F-basis $\{k_1, ..., k_r\}$ of K. The left K-basis of $V(n, K)$ is marked as $\{e_1, ..., e_n\}$, thus $\{e_{ij} = k_j e_i | 1 \leq i \leq n, 1 \leq j \leq r\}$ forms an F-basis of $V(nr, F)$. Now, we write all vectors in $V = V(nr, F)$ as nr-dimensional rows and write each $g \in \mathrm{GL}(nr, F)$ as a matrix in $\mathrm{Mat}_{nr}F$, sending each $x \in V(nr, F)$ to xg. Write K as $\vec{K}$ and denote each $\vec{x} = c_1\vec{k_1} + \cdots + c_r\vec{k_r}$with all $c_i \in F$ as $\vec{x} = (c_1, ..., c_r) \in \mathrm{Mat}_{1\times r}F$ when viewing K as a left F-space. For each $\theta \in K$ we can view it as an F-linear translation by

$\vec{x} \mapsto \vec{x\theta}$ on $\vec{K}$. This transformation can be identified with the matrix $\theta^{(r)} \in \mathrm{Mat}_r F$ relative to the basis $\{k_1, ..., k_r\}$. In this point of view, we have $\mathrm{Mat}_n K \subset \mathrm{Mat}_{nr} F$. For each $\sigma \in \mathrm{Aut} K/F = \{\sigma \in \mathrm{Aut} K \mid a^\sigma = a, \ \forall a \in F\}$, it can be written as an matrix $\theta^{(r)}$ of the F-linear translation $\vec{x} \mapsto \vec{x^\sigma}$ on $\vec{K}$ relative to the basis $\{k_1, ..., k_r\}$. We point out that the normalizer of K^* in $\mathrm{GL}(r, F)$ is $K^* \rtimes \mathrm{Aut} K/F$, and regard $\mathrm{Aut} K/F$ as a subgroup of $\mathrm{GL}(nr, F)$. Each $\sigma \in \mathrm{Aut} K/F$ sends the vector $\theta_1 e_1 + \cdots + \theta_r e_r \in V(n, K)$ to $\theta_1^\sigma e_1 + \cdots + \theta_r^\sigma e_r$ with all $\theta_i \in K$, having the matrix $\sigma^{(nr)} = \mathrm{diag}(\sigma^{(r)}, ..., \sigma^{(r)})$. One can see that the normalizer of $\mathrm{SL}(n, K)$ in $\mathrm{GL}(nr, F)$ is $\Gamma = \mathrm{GL}(n, K) \rtimes \mathrm{Aut} K/F$. When $n \geq 2$, the overgroups of $\mathrm{SL}(n, K)$ in $\mathrm{GL}(nr, F)$ have been determined in the following theorem.

Theorem 2.1 ([5], Theorem 1) *Let K and F be division rings with $K \supset F$ and $\dim_F K = r < \infty$, $n \geq 2$, $N = \mathrm{SL}(n, K) \leq X \leq G = \mathrm{GL}(nr, F)$, then one of the following holds.*

- $\mathrm{SL}(nd, E) \lhd X < \Gamma = \mathrm{GL}(nd, E) \rtimes \mathrm{Aut} E/F$, *for an intermediate division ring E between F and K, where $d = \dim_E K$.*
- *$n = 2$, K is a field, $N = \mathrm{SL}(2, K) = \mathrm{Sp}(2, K, f)$ for any non-degenerate alternating K-form f, $X \trianglerighteq \mathrm{Sp}(2d, E, f_E)$ for an intermediate field E ($F \subseteq E \subseteq K$, $d = \dim_E K$) and an alternating E-form $f_E = \phi_E f$ with $0 \neq \phi_E \in \mathrm{Hom}_E(K, E)$.*
- *$N = \mathrm{SL}(2, 4) \cong A_5$ and $G = \mathrm{GL}(4, 2) \cong A_8$, $X = \mathrm{Sp}(4, 2)' \cong A_6$ or $X \cong A_7$.*

Let $1 \leq i, j \leq n$ be distinct, E_{ij} stands for $n \times n$ matrix whose (i, j)-entry is equal to 1 and zeros for all other positions.D enote identity matrix by I. $T_{ij}(a) = I + aE_{ij}$ with I $a \in F$. Then $T_{ij} = \{T_{ij}(a) | a \in F\}$ are subgroups of $\mathrm{GL}(n, F)$ which are called root subgroups of $\mathrm{GL}(n, F)$. To prove our main results,we use the following facts.

Remark 2.2 *Let F^+ be the addition group of F. Then each $T_{ij} \cong F^+$.*

$T_{ij}(a)T_{ij}(b) = T_{ij}(a + b)$, hence $T_{ij}(a)^{-1} = T_{ij}(-a)$.
With the map $t_{ij} : T_{ij}(a) \mapsto a$, we can easily get this remark.

Remark 2.3 ([1], Propositions 6.2, 6.3) $\mathrm{SL}(n, F)$ *is generated by the root subgroups T_{ij}. And the subgroups T_{ij} are conjugate in* $\mathrm{SL}(n, F)$.

Remark 2.4 *Let R stand for real number field,C for complex one and Q for the skew-field of quaternions. Then* $\mathrm{Aut}(C/R) \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$, *while* $\mathrm{Aut}(Q/C) \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$.

Take a basis $\{1, \mathrm{i}\}$ of C as a F-space, each element of C can be written as a 2-dimensional vector on $V(2, R)$. For each $a + b\mathrm{i} \in C$ with $a, b \in R$, $(a + b\mathrm{i})^\sigma = a \pm b\mathrm{i}$

with $\sigma \in \mathrm{Aut}(C/R)$. Then $\sigma : (a,b) \mapsto (a,b)\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}$, hence $\mathrm{Aut}(C/R) \cong \left\{\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}\right\}$. In the same way, take a basis $\{1, \mathrm{j}\}$ of Q as a C-space, the elements of Q can be written as $\alpha + \beta\mathrm{j}$ with $\alpha, \beta \in C$. For each $\sigma \in \mathrm{Aut}(Q/C), q_1, q_2 \in Q$, from $(q_1q_2)^\sigma = q_1^\sigma q_2^\sigma$ we can get

$$\mathrm{ij}^\sigma = -\mathrm{j}^\sigma\mathrm{i}, (\mathrm{j}^\sigma)^2 = -1.$$

Let $\mathrm{j}^\sigma = \alpha + \beta\mathrm{j}$. Then we can get $\alpha = 0, \beta = e^{\mathrm{i}\theta}$ and $\mathrm{j}^\sigma = e^{\mathrm{i}\theta}$. Therefore, $\mathrm{Aut}(Q/C) \cong \left\{\begin{pmatrix}1 & 0\\ 0 & e^{\mathrm{i}\theta}\end{pmatrix} \mid \theta \in [0, 2\pi)\right\}$.

3. Proofs of the main results

From Theorem 2.1 we can know that when F is a maximal skew-subfield of K, then $\mathrm{SL}(n,K) \rtimes \mathrm{Aut}K/F$ is a maximal subgroup of $\mathrm{GL}(nr,K)$ in most cases. Therefore, we can guess that $\Gamma = C^* \rtimes \mathrm{Aut}C/R$ resp. $\Lambda = Q^* \rtimes \mathrm{Aut}Q/C$ may be a maximal subgroup of $\mathrm{GL}(2,R)$ resp. $\mathrm{GL}(2,C)$. To prove this, we need the following lemma.

Lemma 3.1 *Let X be an overgroup of $\Gamma = C^* \rtimes \mathrm{Aut}C/R$ in $\mathrm{GL}(2,R)$, $A \in X\backslash\Gamma$. $\langle\Gamma, A\rangle$ refers to the group generated by Γ and A. Then $\mathrm{SL}(2,R) \lhd \langle\Gamma, A\rangle \leq X$.*

Proof. Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $a,b,c,d \in R$, transform A with a matrix $\begin{pmatrix}a & b\\ -b & a\end{pmatrix} \in C^*$ as follows:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ -b & a\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ s & t\end{pmatrix},$$

where

$$s = \frac{ac+bd}{a^2+b^2}, t = \frac{ad-bc}{a^2+b^2} = \frac{\det A}{a^2+b^2}.$$

When $t = \pm 1, s \neq 0$. Or $A \in C^*$, in contradiction. Next, we give the following transformation:

$$\begin{pmatrix}1 & x\\ -x & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ s & t\end{pmatrix} = \begin{pmatrix}1+sx & tx\\ s-x & t\end{pmatrix},$$

$$\begin{pmatrix}1+sx & tx\\ s-x & t\end{pmatrix}\begin{pmatrix}1+sx & tx\\ -tx & 1+sx\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ f(x) & g(x)\end{pmatrix},$$

where

$$f(x) = \frac{s+(s^2+t^2-1)x-s^2x}{(1+sx)^2+(tx)^2}, g(x) = \frac{t(1+x^2)}{(1+sx)^2+(tx)^2}, x \in R.$$

Note that $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1}$, $\begin{pmatrix} 1 & x \\ -x & 1 \end{pmatrix}$, $\begin{pmatrix} 1+sx & tx \\ -tx & 1+sx \end{pmatrix}^{-1}$ are elements in C^*, then the matrices $\begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ f(x) & g(x) \end{pmatrix} \in \langle \Gamma, A \rangle$. Then, the commutator

$$\begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ f(x) & g(x) \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f(x) & g(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r(x) & 1 \end{pmatrix}$$
$$= T_{21}(r(x)) \in \langle \Gamma, A \rangle,$$

where

$$r(x) = \frac{[s^2 + (t-1)^2]x(1+t+sx)}{t^2(1+x^2)}, \ x \in R.$$

We can find that $r(x)$ is a continuous function for $x \in R$ because $t^2(1+x^2) > 0$.

$$r'(x) = \frac{[s^2 + (t-1)^2][(1+t)(1-x^2) + 2sx]}{t^2(1+x^2)^2}, \ x \in R.$$

While $s = 0$ and $t = \pm 1$,$r'(x) \equiv 0$. However, these two cases show that $A \in \Gamma$, in contradiction. Therefore, $r(x)$ has two Extreme values denoted a and b with $a < b$. Note that

$$\lim_{x \to \pm\infty} r(x) = \frac{s[s^2 + (t-1)^2]}{t^2},$$

we can get the range of the continuous function $r(x)$ is $[a, b]$.

Form Remark 2.2 we know $T_{21} = \{T_{21}(x) \mid x \in R\} \cong R^+$. Because the additive group R^+ can be generated by all the elements in $[a, b]$, the multiplicative group $T_{21} = \{T_{21}(x) \mid x \in R\}$ can be generated by all the elements in $\{T_{21}(r(x)) \mid x \in R\} = \{T_{21}(x) \mid x \in [a, b]\}$. So, for all $x \in R$, the $T_{21}(x) \in \langle \Gamma, A \rangle$. From Remark 2.3, we can find a matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$, such that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix},$$

we can get $T_{12} < \langle \Gamma, A \rangle$. So, the special linear group $\mathrm{SL}(2, R)$ is a subgroup of $\langle \Gamma, A \rangle$ by the Remark 2.3. Note that $\mathrm{SL}(2, R)$ is just the kernel of det from $\langle \Gamma, A \rangle$ to the multiplicative group R^*, $\mathrm{SL}(2, R) \lhd \langle \Gamma, A \rangle$. ∎

From this lemma, we can get the following corollary.

Corollary 3.2 *Let X be an overgroup of C^* in $\mathrm{GL}(2, R)$, $A \in X \backslash \Gamma = C^* \rtimes \mathrm{Aut}\ C/R$. $\langle C^*, A \rangle$ refers to the group generated by C^* and A. Then, $\mathrm{SL}(2, R) \lhd \langle C^*, A \rangle \leq X$.*

Lemma 3.3 *Let X be an overgroup of $\Gamma = C^* \rtimes \mathrm{Aut} C/R$ in $\mathrm{GL}(2, R)$, $A \in X \backslash \Gamma$. $\langle \Gamma, A \rangle$ refers to the group generated by Γ and A. Then $\langle \Gamma, A \rangle = X = \mathrm{GL}(2, R)$.*

Proof. For each matrix $B \in \mathrm{GL}(2,R)$, denote $\det B = d$.We can give the following transformations:

$$\begin{cases} d>0, B\begin{pmatrix}\sqrt{d} & 0\\ 0 & \sqrt{d}\end{pmatrix}^{-1} \in \mathrm{SL}(2,R) \\ d<0, B\begin{pmatrix}\sqrt{-d} & 0\\ 0 & \sqrt{-d}\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in \mathrm{SL}(2,R)\end{cases}$$

Note that the matrices $\begin{pmatrix}\sqrt{d} & 0\\ 0 & \sqrt{d}\end{pmatrix}^{-1}, \begin{pmatrix}\sqrt{-d} & 0\\ 0 & \sqrt{-d}\end{pmatrix}^{-1}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ all are elements in Γ and $\mathrm{SL}(2,R) \lhd \langle \Gamma, A\rangle$, we can get $\mathrm{GL}(2,R) \le \langle \Gamma, A\rangle$. Hence $\mathrm{GL}(2,R) = \langle \Gamma, A\rangle = X$. ■

Lemma 3.4 *Let A be a matrix in $\mathrm{GL}(2,C)$ whose determinant is $\det A \in R$, $A \in X\backslash\Lambda = Q^* \rtimes \mathrm{Aut}Q/C$. $\langle C^*, A\rangle$ refers to the group generated by C^* and A. Then $\mathrm{SL}(2,C) \lhd \langle C^*, A\rangle$.*

Proof. Denote $A = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \notin \Lambda = Q^* \rtimes \mathrm{Aut}Q/C$. Science $\det A \in R$, we can transform B with a matrix $\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix} \in Q^*$ as follow:

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ \gamma_1 & t\end{pmatrix},$$

where $\gamma_1 \in C, t \in R$ as $\det\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix} \in R$. Science the complex number γ_1 can be written as $se^{\mathrm{i}\theta}$ where $s \in R, \theta \in [0, 2\pi), e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, we can transform the matrix $\begin{pmatrix}1 & 0\\ \gamma_1 & t\end{pmatrix}$ to $\begin{pmatrix}1 & 0\\ s & t\end{pmatrix}$ as follow:

$$\begin{pmatrix}e^{\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & e^{-\mathrm{i}\frac{\theta}{2}}\end{pmatrix}\begin{pmatrix}1 & 0\\ \gamma_1 & t\end{pmatrix}\begin{pmatrix}e^{-\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & e^{\mathrm{i}\frac{\theta}{2}}\end{pmatrix} = \begin{pmatrix}1 & 0\\ s & t\end{pmatrix}, s, t \in R.$$

Note that the matrices $\begin{pmatrix}e^{\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & e^{-\mathrm{i}\frac{\theta}{2}}\end{pmatrix}, \begin{pmatrix}e^{-\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & e^{\mathrm{i}\frac{\theta}{2}}\end{pmatrix} \in Q^*$, we can know $\begin{pmatrix}1 & 0\\ s & t\end{pmatrix} \notin \Lambda$. Then, from Corollary 3.2 we have $\mathrm{SL}(2,C) \lhd \langle C^*, A\rangle$. ■

Lemma 3.5 *Let X be an overgroup of $\Lambda = Q^* \rtimes \mathrm{Aut}Q/C$ in $\mathrm{GL}(2,C)$, $A \in X\backslash\Lambda$. $\langle \Lambda, A\rangle$ refers to the group generated by Λ and A. Then $\langle \Lambda, A\rangle = X = \mathrm{GL}(2,C)$.*

Proof. For an arbitrary matrix $A \in X\backslash\Lambda$, Q^* is not normalized by A. Hence, there exists a matrix $M \in Q^*$ with $AM \neq MA$. Let $B = AMA^{-1}$, then $B \notin Q^*$, and $\det B = \det M \in R$. From Lemma 3.4 we know $\mathrm{SL}(2,C) \lhd \langle C^*, B\rangle < \langle \Lambda, B\rangle \le X \le \mathrm{GL}(2,C)$.

Now, we just have to prove $X = \mathrm{GL}(2,C)$. For each matrix $D \in \mathrm{GL}(2,C)$, denote $\det D = \delta = de^{\mathrm{i}\theta}$ with $d > 0$ and $\theta \in [0, 2\pi)$.W e can give the following transformations:

$$D\begin{pmatrix}\sqrt{d} & 0\\ 0 & \sqrt{d}\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 0 & e^{\mathrm{i}\theta}\end{pmatrix}^{-1} \in \mathrm{SL}(2,C).$$

Note that the matrices $\begin{pmatrix}\sqrt{d} & 0\\ 0 & \sqrt{d}\end{pmatrix}^{-1}, \begin{pmatrix}1 & 0\\ 0 & e^{\mathrm{i}\theta}\end{pmatrix}^{-1}$ all are elements in Λ and $\mathrm{SL}(2,C)\lhd\langle\Lambda, D\rangle$, we can get $\mathrm{GL}(2,C) \leq \langle\Lambda, D\rangle$. Hence $\mathrm{GL}(2,C) = \langle\Lambda, D\rangle = X$. ∎

Proof of Theorem 1.1. The first item of Theorem 1.1 follows immediately from Lemma 3.1 and Lemma 3.3 as $C^* \rtimes \mathrm{Aut}C/R$ is a maximal subgroups of $\mathrm{GL}(2,R)$. For an arbitrary matrix $A \in X\backslash\Gamma$, C^* is not normalized by A. There exists a matrix $M \in C^*$ with $AM \neq MA$. Let $B = AMA^{-1}$, then $B \notin C^*$, and $\det B = \det M > 0$. From Corollary 3.2, $\langle C^*, B\rangle \rhd \mathrm{SL}(2,R)$. Then, for an arbitrary matrix D whose determinant is $d > 0$, we can give the following transformation: $D\begin{pmatrix}\sqrt{d} & 0\\ 0 & \sqrt{d}\end{pmatrix}^{-1} \in \mathrm{SL}(2,R)$. That means $\langle C^*, B\rangle = \mathrm{GL}_+(2,R) := \{A \in \mathrm{GL}(2,R) | \det A > 0\}$. So, the second item of Theorem 1.1 is established. ∎

Proof of Theorem 1.2. From Lemma 3.5,we can get $Q^* \rtimes \mathrm{Aut}Q/C$ is a maximal subgroup of $\mathrm{GL}(2,C)$. From Remark 2.4, the group $Q^* \rtimes \mathrm{Aut}Q/C = \left\{\begin{pmatrix}\alpha & \beta\\ -\overline{\beta}e^{\mathrm{i}\theta} & \overline{\alpha}e^{\mathrm{i}\theta}\end{pmatrix} \in \mathrm{GL}(2,C) \mid \alpha,\beta \in C, \theta \in [0,2\pi)\right\}$. Then, the first item of Theorem 1.2 is established.

For an arbitrary matrix $A \in X\backslash\Lambda$, Q^* is not normalized by A. There exists a matrix $M \in Q^*$ with $AM \neq MA$. Let $B = AMA^{-1}$, then $B \notin Q^*$, and $0 < \det B = \det M \in R$. From Lemma 3.4, $\langle Q^*, B\rangle > \langle C^*, B\rangle \rhd \mathrm{SL}(2,C)$. Then, for an arbitrary matrix D whose determinant is $d > 0$, we can give the following transformation: $D\begin{pmatrix}\sqrt{d} & 0\\ 0 & \sqrt{d}\end{pmatrix}^{-1} \in \mathrm{SL}(2,C)$. That means $\langle Q^*, B\rangle = \mathrm{GL}_+(2,C) := \{A \in \mathrm{GL}(2,C) | 0 < \det A \in R\}$. For the overgroup X between $\langle Q^*, B\rangle$ and $\mathrm{GL}(2,C)$, we can give a homomorphism

$$\Theta : X \to \left\{\begin{pmatrix}1 & 0\\ 0 & e^{\mathrm{i}\theta}\end{pmatrix} \mid \theta \in [0,2\pi)\right\}.$$

For each element $A \in X$, whose determinant is $\delta = te^{\mathrm{i}\theta}, \Theta(A) = \begin{pmatrix}1 & 0\\ 0 & e^{\mathrm{i}\theta}\end{pmatrix}$. The kernel of the homomorphism is just

$$H = \mathrm{GL}_+(2,C) := \{A \in \mathrm{GL}(2,C) \mid 0 < \det A \in R\}.$$

So

$$X = H \cdot \left\{\begin{pmatrix}1 & 0\\ 0 & e^{\mathrm{i}\theta}\end{pmatrix} \mid \theta \in [0,2\pi)\right\}.$$

The second item of Theorem 1.2 is finished. ∎

References

[1] Alperin, J.L., Bell, R.B., *Groups and representations*, Springer, New York, 1995.

[2] Aschbacher, M., *On the maximal subgroups of finite classical groups,* Invent. Math., 76 (1984), 469-514.

[3] Kleidman, P., Liebeck, M., *The subgroup structure of the finite classical groups*, Cambridge Univ. Press, Cambridge, 1989.

[4] Li, S., *Maximal subgroups in classical groups over arbitrary fields,* Proc. Sympos. Pure Math., 47 (1987), 487-493.

[5] Li, S., *Overgroups in* $\mathrm{GL}(nr, F)$ *of certain subgroups of* $\mathrm{SL}(n, K)$. I, J. Algebra, 125 (1989), 215-235.

[6] Li, S., *Overgroups in* $\mathrm{GL}(U \otimes W)$ *of certain subgroups of* $\mathrm{GL}(U) \otimes \mathrm{GL}(W)$. I, J. Algrbra, 137 (1991), 338-368.

[7] Li, S., *Overgroups of certain subgroups in the classical groups over arbitrary division rings,* Contemp. Math., 82 (1989), 53-57.

[8] Seitz, G., *The maximal subgroups of classical algebraic groups,* Mem. AMS, 67, no. 365 (1987).

Accepted: 17.05.2016