$(\in, \in\vee q)$-OSMOTIC VALUES WITH APPLICATIONS IN BCK/BCI-ALGEBRAS

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Abstract. The notions of $(\in, \in\vee q)$-osmotic S-value and $(\overline{\in}, \overline{\in}\vee \overline{q})$-osmotic S-value are introduced, and related properties are investigated. Relations between osmotic S-value and $(\in, \in\vee q)$-osmotic S-value ($(\overline{\in}, \overline{\in}\vee \overline{q})$-osmotic S-value) are considered. Conditions for a number in the unit interval $[0, 1]$ to be an osmotic S-value (resp., $(\in, \in\vee q)$-osmotic S-value and $(\overline{\in}, \overline{\in}\vee \overline{q})$-osmotic S-value) are provided. Conditions for level sets to be S-energetic set and/or subalgebra are discussed.

Keywords: $(\in, \in\vee q)$-osmotic S-value, $(\overline{\in}, \overline{\in}\vee \overline{q})$-osmotic S-value, S-energetic set.

2010 Mathematics Subject Classification: 06F35, 03G25, 08A72.

1. Introduction

To make the computers simulate beings in dealing with certainty and uncertainty in information is an important task of artificial intelligence. The role of logic in mathematics and computer science is twofold – as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of classical logic to handle information with various facets of uncertainty (see [8] for generalized theory of uncertainty), such as fuzziness, randomness etc. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and

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uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [7]. Murali [5] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [6], played a vital role to generate some different types of fuzzy subsets. Jun et al. [3] introduced the notions of energetic (resp. right vanished, right stable) subsets and (anti) permeable values in BCK/BCI-algebras, and investigated several properties.

In this paper, we introduce the notions of $(\in, \in \vee q)$-osmotic S-value and $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value, and investigate related properties. We consider relations between osmotic S-value and $(\in, \in\vee q)$-osmotic S-value ($(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value), and provide conditions for a number in the unit interval $[0, 1]$ to be an osmotic S-value (resp., $(\in, \in\vee q)$-osmotic S-value and $(\overline{\in}, \overline{\in}\vee\overline{q})$-osmotic S-value). We discuss conditions for level sets to be S-energetic set and/or subalgebra.

2. Preliminaries

The BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and has been extensively investigated by many researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following axioms:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x*y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [1, 4] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [6] introduced the symbol $x_t\,\alpha\,\mu$, where $\alpha \in \{\in, q\}$. To say that $x_t \in \mu$ (resp. $x_t\,q\,\mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $x_t \in\vee\, q\,\mu$, we mean $x_t \in \mu$ or $x_t\,q\,\mu$. To say that $x_t\,\overline{\alpha}\,\mu$, we mean $x_t\,\alpha\,\mu$ does not hold, where $\alpha \in \{\in, \mathrm{q}\}$.

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y)\})\,. \tag{2.5}$$

A fuzzy set μ in a BCK/BCI-algebra X is called an $(\in, \in\vee\, q)$-*fuzzy subalgebra* (see [2]) of X if it satisfies: for all $x, y \in X$ and $t_1, t_2 \in (0,1]$,

$$x_{t_1} \in \mu,\ y_{t_2} \in \mu\ \Rightarrow (x * y)_{\min\{t_1,t_2\}} \in\vee\, q\,\mu. \tag{2.6}$$

3. $(\in, \in\vee\, q)$-osmotic values for a fuzzy set

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

For a fuzzy set μ in X and $t \in [0,1]$, the (strong) upper (resp. lower) t-level sets of μ are defined as follows:

$$U(\mu;t) := \{x \in X \mid \mu(x) \geq t\},\ \ U^*(\mu;t) := \{x \in X \mid \mu(x) > t\},$$

$$L(\mu;t) := \{x \in X \mid \mu(x) \leq t\},\ \ L^*(\mu;t) := \{x \in X \mid \mu(x) < t\}.$$

Definition 3.1 ([3])**.** A non-empty subset A of X is said to be *S-energetic* if it satisfies:

$$(\forall a, b \in X)\,(a * b \in A\ \Rightarrow\ \{a, b\} \cap A \neq \emptyset)\,. \tag{3.1}$$

Proposition 3.2. *If A and B are S-energetic subsets of X, then so are $A \cup B$ and $A \cap B$.*

Proof. Straightforward. ∎

Lemma 3.3 ([2])**.** *A fuzzy set μ in X is an $(\in, \in\vee\, q)$-fuzzy subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y), 0.5\})\,. \tag{3.2}$$

Definition 3.4. Let μ be a fuzzy set in X. A number $t \in [0,1]$ is called an *osmotic S-value* for μ (it is called an anti-permeable S-value for μ in [3]) if the following assertion is valid:

$$(\forall a, b \in X)\,(\mu(a * b) \leq t \;\Rightarrow \min\{\mu(a), \mu(b)\} \leq t)\,. \tag{3.3}$$

Theorem 3.5. *Let $t \in \alpha \subseteq [0,1]$. For a fuzzy set μ in X, if the nonempty strong upper t-level set of μ is a subalgebra of X, then t is an osmotic S-value for μ.*

Proof. Let $a, b \in X$ be such that $\mu(a * b) \leq t$ for $t \in \alpha \subseteq [0,1]$. If

$$\min\{\mu(a), \mu(b)\} \nleq t,$$

then $\mu(a) > t$ and $\mu(b) > t$, that is, $a, b \in U^*(\mu; t)$. Since $U^*(\mu; t)$ is a subalgebra of X, it follows that $a * b \in U^*(\mu; t)$, i.e., $\mu(a * b) > t$ which is a contradiction. Hence $\min\{\mu(a), \mu(b)\} \leq t$ and t is an osmotic S-value for μ. ∎

Definition 3.6. Let μ be a fuzzy set in X. A number $t \in [0,1]$ is called an $(\in, \in\!\vee q)$-*osmotic S-value* for μ if the following assertion is valid:

$$(\forall x, y \in X)\,(x * y \in L(\mu; t) \;\Rightarrow\; \min\{\mu(x), \mu(y), 0.5\} \leq t)\,. \tag{3.4}$$

We note that every osmotic S-value for μ is an $(\in, \in\!\vee q)$-osmotic S-value for μ.

Example 3.7. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	1
3	3	3	3	0	3
4	4	4	4	4	0

Let μ be a fuzzy set in X defined by

$$\mu = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.3 & 0.6 & 0.7 & 0.2 \end{pmatrix}.$$

It is routine to verify that if $t \in [0,1] \setminus [0.3, 0.6)$, then t is an $(\in, \in\!\vee q)$-osmotic S-value for μ. If $t \in [0.3, 0.5)$, then t is not an $(\in, \in\!\vee q)$-osmotic S-value for μ since $2 * 3 = 1 \in L(\mu; t)$ but $\min\{\mu(2), \mu(3), 0.5\} > t$.

Theorem 3.8. *Let μ be a fuzzy set in X and $t \in \alpha \subseteq [0,1]$. If the nonempty lower t-level set of μ is an S-energetic subset of X, then t is an osmotic S-value for μ.*

Proof. For $t \in \alpha \subseteq [0,1]$, let $x, y \in X$ be such that $\mu(x * y) \leq t$. Then $x * y \in L(\mu; t)$. Assume that $\min\{\mu(x), \mu(y)\} \nleq t$. Then $\mu(x) > t$ and $\mu(y) > t$, which imply that $x, y \notin L(\mu; t)$. Hence $\{x, y\} \cap L(\mu; t) = \emptyset$, a contradiction. Thus $\min\{\mu(x), \mu(y)\} \leq t$ and t is an osmotic S-value for μ. ∎

Corollary 3.9. *Let μ be a fuzzy set in X and $t \in \alpha \subseteq [0,1]$. If the nonempty lower t-level set of μ is an S-energetic subset of X, then t is an $(\in, \in\vee q)$-osmotic S-value for μ.*

We consider a condition for the converse of Theorem 3.8 to be true.

Theorem 3.10. *For a fuzzy set μ in X, let $t \in [0,1]$ be an $(\in,\in\vee q)$-osmotic S-value for μ. If $t < 0.5$, then the nonempty lower t-level set of μ is an S-energetic subset of X.*

Proof. Assume that $x*y \in L(\mu;t)$ for $x,y \in X$. Then $\min\{\mu(x),\mu(y),0.5\} \leq t$ by (3.4). Since $t < 0.5$, it follows that $\mu(x) \leq t$ or $\mu(y) \leq t$. Hence $\{x,y\}\cap L(\mu;t) \neq \emptyset$, and therefore $L(\mu;t)$ is an S-energetic subset of X. ∎

Since every osmotic S-value for μ is an $(\in,\in\vee q)$-osmotic S-value for μ, we have the following theorem by Theorem 3.5.

Theorem 3.11. *Let $t \in \alpha \subseteq [0,1]$. For a fuzzy set μ in X, if the nonempty strong upper t-level set of μ is a subalgebra of X, then t is an $(\in,\in\vee q)$-osmotic S-value for μ.*

The following example shows that the converse of Theorem 3.11 may not be true, that is, there is an $(\in,\in\vee q)$-osmotic S-value t for μ such that the strong upper t-level set of μ is not a subalgebra of X.

Example 3.12. Consider the fuzzy set μ in Example 3.7. We can check that the number $t = 0.55$ is an $(\in,\in\vee q)$-osmotic S-value for μ, but $U^*(\mu;t) = \{0,2,3\}$ is not a subalgebra of X.

We now consider conditions for the converse of Theorem 3.11 to be true.

Theorem 3.13. *For a fuzzy set μ in X, let $t \in [0,1]$ be an $(\in,\in\vee q)$-osmotic S-value for μ. If $t < 0.5$, then the strong upper t-level set of μ is a subalgebra of X.*

Proof. Let $x,y \in U^*(\mu;t)$ for $x,y \in X$ and assume that $x*y \notin U^*(\mu;t)$. Then $\mu(x) > t$, $\mu(y) > t$ and $\mu(x*y) \leq t$, that is, $x*y \in L(\mu;t)$. It follows from (3.4) that $\min\{\mu(x),\mu(y),0.5\} \leq t$. Since $t < 0.5$, we have $\mu(x) \leq t$ or $\mu(y) \leq t$ which imply that $x \notin U^*(\mu;t)$ or $y \notin U^*(\mu;t)$. This is a contradiction, and so $x*y \in U^*(\mu;t)$. ∎

Definition 3.14. Let μ be a fuzzy set in X. A number $t \in [0,1]$ is called an $(\overline{\in}, \overline{\in}\vee\overline{q})$-*osmotic S-value* for μ if the following assertion is valid:

$$(3.5)\qquad (\forall x,y \in X)\,(x,y \in U^*(\mu;t) \;\Rightarrow\; \min\{\mu(x*y),0.5\} > t)\,.$$

Example 3.15. Let $X = \{0, 1, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

Let μ be a fuzzy set in X defined by

$$\mu = \begin{pmatrix} 0 & 1 & a & b & c \\ 0.6 & 0.7 & 0.2 & 0.4 & 0.2 \end{pmatrix}.$$

It is easy to verify that every number $t < 0.6$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ. If $t \in [0.6, 0.7)$, then t is not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ since $1 \in U^*(\mu; t)$ but $\min\{\mu(1 * 1), 0.5\} = 0.5 < t$.

Theorem 3.16. *For a fuzzy set μ in X, let $t \in [0, 1]$ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ. If $t \geq 0.5$, then the nonempty lower t-level set of μ is an S-energetic subset of X and the nonempty strong upper t-level set of μ is a subalgebra of X.*

Proof. Let $x*y \in L(\mu; t)$ for $x, y \in X$. If $\{x, y\} \cap L(\mu; t) = \emptyset$, then $x, y \in U^*(\mu; t)$ and so $\min\{\mu(x * y), 0.5\} > t$ by (3.5). Since $t \geq 0.5$, it follows that $\mu(x * y) > t$, that is, $x * y \notin L(\mu; t)$. This is a contradiction, and therefore $\{x, y\} \cap L(\mu; t) \neq \emptyset$. Hence $L(\mu; t)$ is an S-energetic subset of X. Now, let $x, y \in U^*(\mu; t)$ for $x, y \in X$. Then $\min\{\mu(x * y), 0.5\} > t$ by (3.5), and so $\mu(x * y) > t$ since $t \geq 0.5$. Hence $x * y \in U^*(\mu; t)$, and therefore $U^*(\mu; t)$ is a subalgebra of X. ∎

Lemma 3.17. *For $t \in \alpha \subseteq [0, 1]$, if t is an osmotic S-value for a fuzzy set μ, then the nonempty lower t-level set of μ is an S-energetic subset of X.*

Proof. Let $x*y \in L(\mu; t)$ for $x, y \in X$. Then $\mu(x*y) \leq t$, and so $\min\{\mu(x), \mu(y)\} \leq t$ by (3.3). Hence $\mu(x) \leq t$ or $\mu(y) \leq t$, that is, $x \in L(\mu; t)$ or $y \in L(\mu; t)$. Thus $\{x, y\} \cap L(\mu; t) \neq \emptyset$, and therefore $L(\mu; t)$ is an S-energetic subset of X. ∎

We know that any osmotic S-value may not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value as seen in the following example.

Example 3.18. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Let μ be a fuzzy set in X defined by

$$\mu = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.7 & 0.4 & 0.3 & 0.1 \end{pmatrix}.$$

If we take $t \in [0.5, 0.7)$, then t is an osmotic S-value for μ. But it is not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ since $0, 1 \in U^*(\mu; t)$ but $\min\{\mu(1 * 0), 0.5\} = 0.5 \leq t$.

We provide a condition for an osmotic S-value to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value.

Theorem 3.19. *For a fuzzy set μ in X, let $t \in \alpha \subseteq [0, 1]$ be an osmotic S-value for μ. If $t < 0.5$, then it is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ.*

Proof. Let $x, y \in U^*(\mu; t)$ for $x, y \in X$. Then $\mu(x) > t$ and $\mu(y) > t$, which imply that $x, y \notin L(\mu; t)$. Thus $\{x, y\} \cap L(\mu; t) = \emptyset$, and so $x * y \notin L(\mu; t)$ by Lemma 3.17. Thus $x * y \in U^*(\mu; t)$, and thus $\mu(x * y) > t$. Since $t < 0.5$, it follows that $\min\{\mu(x * y), 0.5\} > t$. Therefore t is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ. ∎

We consider conditions for a number t in $[0, 1]$ to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value.

Theorem 3.20. *For a fuzzy set μ in X and $t \in [0, 1]$, if $t < 0.5$ and the nonempty strong upper t-level set of μ is a subalgebra of X, then t is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ.*

Proof. Let $x, y \in U^*(\mu; t)$ and assume that $\min\{\mu(x * y), 0.5\} \leq t$ for $x, y \in X$. Then $\mu(x * y) \leq t$ since $t < 0.5$. Hence $x * y \in L(\mu; t)$ and so $x * y \notin U^*(\mu; t)$. This is impossible, and thus $\min\{\mu(x * y), 0.5\} > t$ whenever $x, y \in U^*(\mu; t)$. This shows that t is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ. ∎

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Accepted: 04.05.2016