SOME IDENTITIES FOR DEGENERATE FROBENIUS-EULER NUMBERS ARISING FROM NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we derive nonlinear differential equations from the generating function of degenerate Frobenius-Euler numbers which originate from Carlitz degenerate Euler numbers. In addition, we give some explicit identities for degenerate Frobenius-Euler numbers arising from our nonlinear differential equations.

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1. Introduction

For $u\,(\neq 1) \in \mathbb{C}$, the Frobenius-Euler numbers are defined by the generating function

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n\,(u)\,\frac{t^n}{n!}, \quad (\text{see [2], [12], [13]})\,. \tag{1.1}$$

In particular, $u=-1$, we have

$$\sum_{n=0}^{\infty} H_n\,(-1)\,\frac{t^n}{n!} = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \tag{1.2}$$

where E_n are the Euler numbers (see [6], [9]).

Thus, by (1.2), we get $H_n\,(-1) = E_n (n \geq 0)$. In [3], L. Carlitz considered the degenerate Euler numbers which are given by the generating function

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda} \frac{t^n}{n!}. \tag{1.3}$$

Note that

$$\lim_{\lambda\to 0} \mathcal{E}_{n,\lambda} = E_n \;\; (n \geq 0)\,.$$

Recently, Kim and Kim derived nonlinear differential equations from the generating function of degenerate Euler numbers and gave some new and explicit identities for Carlitz degenerate Euler numbers arising from those nonlinear differential equations (see [9]).

In [6], the nonlinear Changhee differential equations, which are related to Euler and degenerate Euler numbers, were introduced by Kim and Kim. They also gave some new explicit relationship between Changhee and Euler numbers arising from those differential equations.

Now, we consider the degenerate Frobenius-Euler numbers which originates from Carlitz degenerate Euler numbers and are given by the generating function:

$$\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u} = \sum_{n=0}^{\infty} h_{n,\lambda}\,(u)\,\frac{t^n}{n!}, \quad (u\,(\neq 1) \in \mathbb{C})\,, \quad (\text{see [7]})\,. \tag{1.4}$$

In the special case $u=-1$, we note that

$$h_{n,\lambda}\,(-1) = \mathcal{E}_{n,\lambda},\; (n \geq 0).$$

It is not difficult to show that

$$\lim_{\lambda\to 0} h_{n,\lambda}\,(u) = H_n\,(u)\,,\;\; (n \geq 0)\,.$$

Recently, several authors have studied degenerate Euler numbers and degenerate Frobenius-Euler numbers (see [1]–[14]).

In this paper, we derive nonlinear differential equations from the generating function of degenerate Frobenius-Euler numbers. In addition, we give some explicit identities for degenerate Frobenius-Euler numbers arising from our nonlinear differential equations.

2. Some identities for degenerate Frobenius-Euler numbers arising from nonlinear differential equations

Let

$$F = F(t;\lambda,u) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} - u}, \quad (u \neq 1). \tag{2.1}$$

Then, by (2.1), we get

$$F^{(1)} = \frac{dF}{dt}(t;\lambda,u) = \frac{(-1)}{1+\lambda t}\left(F + uF^2\right). \tag{2.2}$$

Thus, by (2.2), we have

$$uF^2 = -F - (1+\lambda t)F^{(1)}. \tag{2.3}$$

Taking the derivative of (2.3) with respect to t, we get

$$2!u^2F^3 = (-1)^2\left\{2F + (3+\lambda)(1+\lambda t)F^{(1)} + (1+\lambda t)^2 F^{(2)}\right\}, \tag{2.4}$$

and

$$\begin{aligned} 3!u^3F^4 &= (-1)^3\left\{6F + (\lambda^2+6\lambda+11)(1+\lambda t)F^{(1)}\right. \\ &\quad \left. + (3\lambda+6)(1+\lambda t)^2F^{(2)} + (1+\lambda t)^3F^{(3)}\right\}. \end{aligned} \tag{2.5}$$

So, we are led to put

$$N!u^NF^{N+1} = (-1)^N\sum_{i=0}^{N} a_i(N,\lambda)(1+\lambda t)^i F^{(i)}, \quad (N = 0,1,2,\ldots), \tag{2.6}$$

where $F^{(i)} = \left(\frac{d}{dt}\right)^i F(t;\lambda,u)$.

Let us take the derivative of (2.6) with respect to t. Then we have

$$\begin{aligned} (N+1)!u^NF^NF^{(1)} &= (-1)^N\sum_{i=0}^{N} a_i(N,\lambda)\, i\,(1+\lambda t)^{i-1}\lambda F^{(i)} \\ &\quad + (-1)^N\sum_{i=0}^{N} a_i(N,\lambda)(1+\lambda t)^i F^{(i+1)}. \end{aligned} \tag{2.7}$$

By (2.2) and (2.7), we easily get

$$\begin{aligned} &(N+1)!u^NF^N\left(F+uF^2\right) \\ &= (-1)^{N+1}\left\{\sum_{i=0}^{N} i\lambda a_i(N,\lambda)(1+\lambda t)^iF^{(i)} + \sum_{i=0}^{N} a_i(N,\lambda)(1+\lambda t)^{i+1}F^{(i+1)}\right\}. \end{aligned} \tag{2.8}$$

From (2.6) and (2.8), we can derive the following equation:

$$
\begin{aligned}
&(N+1)!u^{N+1}F^{N+2}\\
&=-\left(N+1\right)N!u^{N}F^{N+1}\\
&+\left(-1\right)^{N+1}\left\{\sum_{i=0}^{N} i\lambda a_i\left(N,\lambda\right)\left(1+\lambda t\right)^{i}F^{(i)}+\sum_{i=0}^{N} a_i\left(N,\lambda\right)\left(1+\lambda t\right)^{i+1}F^{(i+1)}\right\}\\
&=\left(-1\right)^{N+1}\left\{\sum_{i=0}^{N}\left(N+1+i\lambda\right)a_i\left(N,\lambda\right)\left(1+\lambda t\right)^{i}F^{(i)}\right.\\
&\left.+\sum_{i=1}^{N+1} a_{i-1}\left(N,\lambda\right)\left(1+\lambda t\right)^{i}F^{(i)}\right\}.
\end{aligned} \tag{2.9}
$$

On the other hand, by replacing N by $N+1$ in (2.6), we get

$$(N+1)!u^{N+1}F^{N+2}=\left(-1\right)^{N+1}\sum_{i=0}^{N+1} a_i\left(N+1,\lambda\right)\left(1+\lambda t\right)^{i}F^{(i)}. \tag{2.10}$$

By comparing the coefficients on both sides of (2.9) and (2.10), we have

$$a_0\left(N+1,\lambda\right)=\left(N+1\right)a_0\left(N,\lambda\right),\quad a_{N+1}\left(N+1,\lambda\right)=a_N\left(N,\lambda\right), \tag{2.11}$$

and

$$a_i\left(N+1,\lambda\right)=a_{i-1}\left(N,\lambda\right)+\left(N+1+i\lambda\right)a_i\left(N,\lambda\right),\quad\left(1\le i\le N\right). \tag{2.12}$$

From $F=a_0\left(0,\lambda\right)F$ in (2.6), we have

$$a_0\left(0,\lambda\right)=1. \tag{2.13}$$

By (2.3) and (2.6), we easily get

$$-F-\left(1+\lambda t\right)F^{(1)}=uF^2=-a_0\left(1,\lambda\right)F-a_1\left(1,\lambda\right)\left(1+\lambda t\right)F^{(1)}. \tag{2.14}$$

Comparing the coefficients on both sides of (2.14), we obtain

$$a_0\left(1,\lambda\right)=1,\quad a_1\left(1,\lambda\right)=1. \tag{2.15}$$

From (2.11), we easily note that

$$
\begin{aligned}
&a_0\left(N+1,\lambda\right)\\
&=\left(N+1\right)a_0\left(N,\lambda\right)=\left(N+1\right)Na_0\left(N-1,\lambda\right)=\cdots=\left(N+1\right)!,
\end{aligned} \tag{2.16}
$$

and

$$
\begin{aligned}
&a_{N+1}\left(N+1,\lambda\right)\\
&=a_N\left(N,\lambda\right)=a_{N-1}\left(N-1,\lambda\right)=\cdots=a_1\left(1,\lambda\right)=a_0\left(0,\lambda\right)=1.
\end{aligned} \tag{2.17}
$$

For $i = 1, 2, 3$ in (2.12), we have

$$a_1 (N+1, \lambda) = \sum_{k=0}^{N} (N+1+\lambda)_k \, a_0 (N-k, \lambda), \tag{2.18}$$

$$a_2 (N+1, \lambda) = \sum_{k=0}^{N-1} (N+1+2\lambda)_k \, a_1 (N-k, \lambda), \tag{2.19}$$

and

$$a_3 (N+1, \lambda) = \sum_{k=0}^{N-2} (N+1+3\lambda)_k \, a_2 (N-k, \lambda), \tag{2.20}$$

where

$$(x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1), \quad (x)_0 = 1.$$

Continuing this process, we note that

$$a_i (N+1, \lambda) = \sum_{k=0}^{N-i+1} (N+1+i\lambda)_k \, a_{i-1} (N-k, \lambda). \tag{2.21}$$

Now, we give explicit expressions for $a_i (N+1)$.

From (2.18), we note that

$$\begin{aligned} a_1 (N+1, \lambda) &= \sum_{k_1=0}^{N} (N+1+\lambda)_{k_1} \, a_0 (N-k_1, \lambda) \\ &= \sum_{k_1=0}^{N} (N+1+\lambda)_{k_1} (N-k_1)!, \end{aligned} \tag{2.22}$$

$$\begin{aligned} a_2 (N+1, \lambda) &= \sum_{k_2=0}^{N-1} (N+1+2\lambda)_{k_2} \, a_1 (N-k_2, \lambda) \\ &= \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-1-k_2} (N+1+2\lambda)_{k_2} (N-k_2+\lambda)_{k_1} (N-k_2-k_1-1)!, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} a_3 (N+1, \lambda) &= \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-2-k_3} \sum_{k_1=0}^{N-k_3-k_2-2} (N+1+3\lambda)_{k_3} \\ &\quad \times (N-k_3+2\lambda)_{k_2} (N-k_3-k_2-1+\lambda)_{k_1} \\ &\quad \times (N-k_3-k_2-k_1-2)!. \end{aligned} \tag{2.24}$$

So we can deduce that, for $1 \leq i \leq N$,

$$\begin{aligned} & a_i (N+1, \lambda) \\ = & \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-i+1-k_i} \cdots \sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2} \prod_{j=1}^{i} \left(N+j+1-i-\sum_{l=j+1}^{i} k_l + j\lambda \right)_{k_j} \\ & \times \left(N+1-i-\sum_{l=1}^{i} k_l \right)!. \end{aligned} \tag{2.25}$$

Therefore, we obtain the following theorem.

Theorem 1. *The nonlinear differential equations*

$$N! u^N F^{N+1} = (-1)^N \sum_{i=0}^{N} a_i (N, \lambda) (1+\lambda t)^i F^{(i)}, \quad (N = 1, 2, \ldots),$$

have a solution

$$F = F(t; \lambda, u) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} - u},$$

where

$$\begin{aligned} a_0 (N, \lambda) &= N!, \\ a_i (N, \lambda) &= \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots k_2} \left(N-i-\sum_{l=1}^{i} k_l \right)! \\ & \times \prod_{j=1}^{i} \left(N+j-i-\sum_{l=j+1}^{i} k_l + j\lambda \right)_{k_j}. \end{aligned}$$

We recall that the degenerate Frobenius-Euler numbers, $h_{n,\lambda}(u)$, $(n \geq 0)$, are defined by the generating function

$$\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u} = \sum_{n=0}^{\infty} h_{n,\lambda}(u) \frac{t^n}{n!}.$$

Also, the degenerate Frobenius-Euler numbers, $h_{n,\lambda}^{(r)}(u)$, of order r are given by the genreating function

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u} \right)^r = \sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}(u) \frac{t^n}{n!}, \quad (u \in \mathbb{C}, u \neq 1).$$

From Theorem 1, we have

$$\begin{aligned} & N! u^N (1-u)^{N+1} F^{N+1} \\ & = (-1)^N (1-u)^N \sum_{i=0}^{N} a_i (N, \lambda) (1+\lambda t)^i \left(\frac{d}{dt} \right)^i \left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}} - u} \right). \end{aligned} \tag{2.26}$$

The LHS of (2.26) is given by

$$N!u^N\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^{N+1}=N!u^N\sum_{n=0}^{\infty}h_{n,\lambda}^{(N+1)}(u)\frac{t^n}{n!}. \tag{2.27}$$

It is easy to show that

$$\begin{aligned}
&(-1)^N(1-u)^N\sum_{i=0}^{N}a_i(N,\lambda)(1+\lambda t)^i\left(\frac{d}{dt}\right)^i\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)\\
&=(-1)^N(1-u)^N\sum_{i=0}^{N}a_i(N,\lambda)\sum_{k=0}^{\infty}(i)_k\lambda^k\frac{t^k}{k!}\sum_{l=0}^{\infty}h_{l+i,\lambda}(u)\frac{t^l}{l!}\\
&=(-1)^N(1-u)^N\sum_{i=0}^{N}a_i(N,\lambda)\sum_{n=0}^{\infty}\sum_{k=0}^{n}(i)_k\lambda^k h_{n-k+i,\lambda}(u)\binom{n}{k}\frac{t^n}{n!}\\
&=(-1)^N(1-u)^N\sum_{n=0}^{\infty}\sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}(i)_k\lambda^k a_i(N,\lambda)h_{n-k+i,\lambda}(u)\frac{t^n}{n!}.
\end{aligned} \tag{2.28}$$

By (2.26), (2.27) and (2.28), we get

$$\begin{aligned}
&N!u^N h_{n,\lambda}^{(N+1)}(u)\\
&=(-1)^N(1-u)^N\sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}(i)_k\lambda^k a_i(N,\lambda)h_{n-k+i,\lambda}(u).
\end{aligned} \tag{2.29}$$

Therefore, by (2.29), we obtain the following theorem.

Theorem 2. *For n, $N=0,1,2,\ldots$, we have*

$$\begin{aligned}
&N!u^N h_{n,\lambda}^{(N+1)}(u)\\
&=(-1)^N(1-u)^N\sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}(i)_k\lambda^k a_i(N,\lambda)h_{n-k+i,\lambda}(u),
\end{aligned}$$

where

$$\begin{aligned}
a_0(N,\lambda)&=N!\\
a_i(N,\lambda)&=\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-i-k_i}\cdots\sum_{k_1=0}^{N-i-k_i-\cdots-k_2}\left(N-i-\sum_{l=1}^{i}k_l\right)!\\
&\times\left(\prod_{j=1}^{i}\left(N+j-i-\sum_{l=j+1}^{i}k_l+j\lambda\right)_{k_j}\right).
\end{aligned}$$

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