SOME ELEGANT PROOFS IN 2-METRIC SPACE AND G-METRIC SPACE

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Abstract. Elegant analytical proofs of some fixed point theorems in 2-metric space and G-metric space are presented through elementary set theoretical notions of real numbers and repeated use of the rectangle inequality of the G-metric, without an appeal to iterations. The unique fixed points obtained are shown to be contractive fixed points in these spaces.

Keywords: 2-metric space, G-metric space, infimum property, self-map, fixed point.

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1. Introduction

Let (X, ρ) be a metric space and $a = \rho(x, y)$, $b = \rho(y, z)$ and $c = \rho(z, x)$ be the sides of Δxyz with vertices x, y and z in the plane. Intuitively, the area of Δxyz defines a d-metric d on X, while the perimeter of Δxyz defines a G-metric G on X. The pair (X, d) then defines a 2-metric space and (X, G), a G-metric space. These two notions were introduced by Gahler [8] and Mustafa and Sims [23] respectively as natural generalizations of a metric.

To begin with, we have

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Definition 1.1. Let X be a nonempty set and $d : X \times X \times X \to [0, \infty)$ such that

(2m1) Given a pair of distinct elements $x, y \in X$, there exists a $z \in X$ such that $d(x, y, z) > 0$, and $d(x, y, z) = 0$ whenever at least two of x, y and z are equal in X,

(2m2) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(z, x, y) = d(y, z, x) = d(z, y, x)$ for all $x, y, z \in X$,

(2m3) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

Then d is called a 2-*metric* on X and the pair (X, d), a 2-*metric space*, and (2m2) is usually known as the *axiom of symmetry* under a permutation on x, y and z. In view of the fact that the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining faces, (2m3) is referred to as the *tetrahedron inequality.*

Definition 1.2. The topology induced by a 2-metric consists of all open spheres with two centres of the form $B_r(x, y) = \{z \in X : G(x, y, z) < r, r > 0\}$, and is called a 2-*metric topology.*

Remark 1.1. Given any metric space which consists of more than two points, there always exists a 2-metric compatible with the topology of the space. But the converse is not true. That is one can find a 2-metric space which does not have a countable basis associated with one of its arguments [8].

Remark 1.2. In a metric space (X, ρ), the metric ρ is always a continuous function of both x and y. In a 2-metric space (X, d), d may not be continuous in all the three variables x, y and z, though it is continuous function in any one of them. A 2-metric d is said to be *continuous* [24] if it is continuous in any two and hence in all of x, y and z.

Definition 1.3. A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is said be to 2-*Cauchy* if $d(x_n, x_{n+k}, z) \to 0$ as $n \to \infty$ for each integer $k \geq 1$ and all $z \in X$.

Definition 1.4. A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is said to be 2-*convergent* with limit $p \in X$ if $d(x_n, p, z) \to 0$ as $n \to \infty$ for all $z \in X$ and is denoted by $x_n \to p$ as $n \to \infty$.

Definition 1.5. A 2-metric space X is said to be *complete* if every 2-Cauchy sequence in it is 2-convergent with limit in it [24].

Remark 1.3. It is well-known that every convergent sequence is Cauchy in a metric space. But a 2-convergent sequence may fail to be 2-Cauchy, as shown in [24]. However, every 2-convergent sequence is 2-Cauchy whenever the 2-metric d is continuous.

In view of Remarks 1.1-1.3, it is appropriate to state that there is no relation between a metric and 2-metric, unlike the claim by Gahler [8] that 2-metric space is a natural generalization of a metric. For more about fixed point theorems in 2-metric spaces, one can refer to [10], [15], [16], [25], [26], [29], [31], [32], [34] and [35].

Definition 1.6. Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}$ such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$ for all $x, y, z \in X$

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

Then the pair (X, G) is called a G-metric space with G-metric G on X. Axioms (G4) and (G5) are referred to as the *symmetry* and the *rectangle inequality* (of G) respectively.

Given a G-metric space (X, G), define

$$\rho_G(x, y) = G(x, y, y) + G(x, x, y) \text{ for all } x, y, z \in X. \tag{1.1}$$

Then it is seen in [23] that ρ_G is a metric on X, and that the family of all G-balls $\{B_G(x, r) : x \in X, r > 0\}$ is the base topology, called the G-metric topology $\tau(G)$ on X, where $B_G(x, r) = \{y \in X : G(x, y, y) < r\}$. Further, it was shown that the G-metric topology coincides with the metric topology induced by the metric ρ_G, which allows us to readily transform many concepts from metric spaces into the setting of G-metric space.

Definition 1.7. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-convergent with limit $p \in X$ if it converges to p in the G-metric topology, $\tau(G)$.

It is also known from [23] that $G(x, y, z)$ is jointly continuous in all the three variables x, y and z.

Definition 1.8. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-Cauchy if for every $\varepsilon > 0$ there is a positive integer N such that $G(x_n, x_m, x_n) < \varepsilon$ for all $l, m, n \geq N$.

By Corollary 1 of Proposition 9 of [23], it follows that every G-convergent sequence in a G-metric space (X, G) is G-Cauchy.

Definition 1.9. A G-metric space (X, G) is said to be G-complete (or complete) if every G-Cauchy sequence in X converges in it.

Definition 1.10. [The authors, [28]] A fixed point p of f on a G-metric space (X, G) is a G-contractive fixed point of it if the orbital sequence

$$O_f(x) = x, fx, ..., f^n x, ...$$

at each $x \in X$ converges to p.

Example 1.1. Let $X = [0, \infty)$ with $G(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$

Define $fx = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2}, \\ qx & \text{otherwise,} \end{cases}$ for all $x \in X$, where $0 \leq q < 1$. Then, we see that 0 is the unique fixed point of f and for each $x \in X$, the f-orbit $O_f(x) = x, qx, q^2x, ..., q^n, x, ...$ converges to 0. That is, 0 is a G-contractive fixed point of f.

An extensive research has been done in recent years in G-metric spaces. To mention a few, we have Abbas and Rhoades [1], Aydi et al [4], Choudary et al [5], Jleli & Samet [11], Karapinar & Paul [12], Meenakshi et al [17], Mahanta & Mohanta [18], Mustafa [19], Mustafa et al [22], Shatanawi [30] and others cited in references.

Many fixed point theorems established in the literature of 2-metric and G-metric structures followed the usual iterative procedure to obtain a fixed point. For the first time, the authors presented an elegant proofs of some fixed point theorems including the contraction mapping theorem in 2-metric space and G-metric space (See [27], [28]). In this paper, we present elegant proofs of some fixed point theorems in 2-metric space and G-metric space. In this sequel, the unique fixed points obtained are contractive fixed points in these spaces.

3. Main results

We begin with the infimum property of real numbers, as stated below:

Lemma 3.1. *Let $S \subset \mathbb{R}$ be nonempty and bounded below. Then $\alpha = \inf S$ exists.*

An immediate consequence of Lemma 3.1 is:

Lemma 3.2. *Let α be the infimum of $S \subset \mathbb{R}$. Then there exists a sequence $\langle p_n \rangle_{n=1}^{\infty}$ in S with $\lim_{n \to \infty} x_n = \alpha$.*

We have

Theorem 3.1. *Let f be a self-map on a 2-metric space (X, d) such that*

$$d(fx, fy, z) \leq \alpha d(x, fx, z) + \beta d(y, fy, z) + \gamma d(x, y, z) \text{ for all } x, y, z \in X, \tag{3.1}$$

where α, β and γ are nonnegative real numbers with at least one of them positive and $\alpha + \beta + \gamma < 1$. If X is 2-complete, then f will have a unique fixed point p, which will also be a 2-contractive fixed point of f.

Proof. Note that if $\alpha = \beta = \gamma = 0$, then from (3.1) with $y = fx$, we get $d(fx, f^2x, z) \leq 0$ for all $z \in X$ so that $f^2x = fx$ or that each fx is a fixed point of f. Given $x \in X$ with $fx \neq x$, by (2m1), there exists a point $z \in X$ such that $d(x, fx, z) > 0$. Therefore, we define $S = \{d(x, fx, z) > 0 : x, z \in X\}$. In view of

Lemma 3.1, $\inf S = a \geq 0$ exists. If possible suppose that $a > 0$. Then (3.1) with $y = fx$ gives

$$d(fx, f^2x, z) \leq \alpha d(x, fx, z) + \beta d(fx, f^2x, z) + \gamma d(x, fx, z)$$

or $(1-\beta)d(fx, f^2x, z) \leq (\alpha+\gamma)d(x, fx, z)$ so that $d(fx, f^2x, z) \leq cd(x, fx, z)$, where $c = \frac{\alpha+\gamma}{1-\beta}\cdot$ Now $\alpha+\beta+\gamma < 1$ implies that $c < 1$ so that from (3.1), it follows that $d(fx, f^2x, z) < a$. Since $d(fx, f^2x, z) \in S$, this shows that a cannot be a lower bound of S, which is a contradiction. Therefore, $\inf S = a = 0$.

Hence, by Lemma 3.2, we can choose a sequence $x_1, x_2, ..., x_n, ...$ of points in X such that

$$\lim_{n,\to\infty} d(x_n, fx_n, z) = 0 \text{ for each } z \in X. \tag{3.2}$$

To prove that $\langle x_n\rangle_{n=1}^{\infty}$ is 2-Cauchy, in this case, we use (2m2), (2m3) and (3.1) repeatedly to get

$$\begin{aligned} d(x_n, x_{n+k}, z) &\leq d(x_n, x_{n+k}, fx_n) + d(x_n, fx_n, z) + d(fx_n, x_{n+k}, z) \\ &\leq d(x_n, fx_n, x_{n+k}) + d(x_n, fx_n, z) + [d(fx_n, x_{n+k}, fx_{n+k}) \\ &\qquad +d(fx_n, fx_{n+k}, z) + d(fx_{n+k}, x_{n+k}, z)] \\ &= d(x_n, fx_n, x_{n+k}) + d(x_n, fx_n, z) + d(fx_n, fx_{n+k}, x_{n+k}) \\ &\qquad +d(fx_n, fx_{n+k}, z) + d(x_{n+k}, fx_{n+k}, z) \\ &\leq d(x_n, fx_n, x_{n+k}) + d(x_n, fx_n, z) \\ &\qquad +[\alpha d(x_n, fx_n, x_{n+k}) + \beta d(x_{n+k}, fx_{n+k}, x_{n+k}) \\ &\qquad\qquad +\gamma d(x_n, x_{n+k}, x_{n+k})] \\ &\qquad +[\alpha d(x_n, fx_n, z) + \beta d(x_{n+k}, fx_{n+k}, z) \\ &\qquad\qquad +\gamma d(x_n, x_{n+k}, z)] + d(x_{n+k}, fx_{n+k}, z)] \end{aligned}$$

so that for all $n, k \geq 1$ and all $z \in X$, we have

$$\begin{aligned} (1-\gamma)d(x_n, x_{n+k}, z) &\leq d(x_n, fx_n, x_{n+k}) + \beta d(x_{n+k}, fx_{n+k}, x_{n+k}) \\ &\qquad +\gamma[d(x_n, x_{n+k}, x_{n+k})\} \\ &\quad +\alpha[d(x_n, fx_n, z) + \beta d(x_{n+k}, fx_{n+k}, z) \\ &\qquad +\gamma d(x_{n+k}, fx_{n+k}, z). \end{aligned}$$

Proceeding the limit as $n \to \infty$ in this and then using (3.2), we see that

$$(1-\gamma)\lim_{n,\to\infty} d(x_n, x_{n+k}, z) = 0 \text{ or } \lim_{n,\to\infty} d(x_n, x_{n+k}, z) = 0$$

for all $k \geq 1$ and all $z \in X$, proving that $\langle x_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence in X.

Since X is 2-complete, $\langle x_n\rangle_{n=1}^{\infty}$ is 2-convergent with

$$\lim_{n,\to\infty} d(x_n, p, z) = p \text{ for all } z \in X. \tag{3.3}$$

To prove that $fp = p$, again by (2m2) and (2m3), (3.1), we see that

$$\begin{aligned} d(fp,p,z) &\leq d(fp,p,x_n)+d(fp,x_n,z)+d(x_n,p,z) \\ &= d(x_n,p,fp)+d(fp,x_n,z)+d(x_n,p,z) \\ &\leq d(x_n,p,fp)+[d(fp,x_n,fx_n)+d(fp,fx_n,z)+d(fx_n,x_n,z)]+d(x_n,p,z) \\ &= d(x_n,p,fp)+d(fx_n,fp,x_n)+d(fx_n,fp,z)+d(x_n,fx_n,z)+d(x_n,p,z) \\ &\leq d(x_n,p,fp)+[\alpha d(x_n,fx_n,x_n)+\beta d(p,fp,x_n)+\gamma d(x_n,p,x_n)] \\ &\qquad +[\alpha d(x_n,fx_n,z)+\beta d(p,fp,z)+\gamma d(x_n,p,z)] \\ &\qquad\qquad +d(x_n,fx_n,z)+d(x_n,p,z). \end{aligned}$$

As $n \to \infty$, this together with (3.3), we obtain that $d(fp,p,z) \leq 0$ for all $z \in X$ or $fp = p$. That is p is a fixed point of f.

Uniqueness. Let q be another fixed point of f so that$fq = q$. Then, from (3.1), we get

$$d(p,q,z = d(fp,fq,z) \leq \beta[d(p,fp,z)+d(q,fq,z)] \text{ or } (1-\alpha-\beta-\gamma)d(p,q,z) \leq 0$$

for all $z \in X$ and hence $p = q$. That is, the fixed point of f is unique.

Further, all the f-orbits converge to the fixed point p.

In fact, taking $y = p = fp$ in (3.1) and simplifying, it follows that

$$\begin{aligned} d(f^n x,p,z) &= d(f^n x,f^n p,z) \\ &\leq \alpha d(f^{n-1}x,f^n x,z)+\beta d(f^{n-1}p,f^n p,z)+\gamma d(f^{n-1}x,f^{n-1}p,z) \\ &\leq \alpha d(f^{n-1}x,f^n x,z)+\gamma d(f^{n-1}x,f^{n-1}p,z) \\ &= \alpha d(f^{n-1}x,f^n x,z)+\gamma d(f^{n-1}x,p,z) \\ &= \alpha d(f^{n-1}x,f^n x,z)+\gamma[d(f^{n-1}x,p,f^n x)+d(f^{n-1}x,f^n x,z)+d(f^n x,p,z)] \end{aligned}$$

or

$$(3.4) \quad d(f^n x,p,z) \leq \left(\frac{\alpha+\gamma}{1-\gamma}\right) d(f^{n-1}x,f^n x,z)+\left(\frac{\gamma}{1-\gamma}\right) d(f^{n-1}x,f^n x,p).$$

Again by (3.1), we have

$$d(f^{n-1}x,f^n x,z) \leq \alpha d(f^{n-2}x,f^{n-1}x,z)+\beta d(f^{n-1}x,f^n x,z)\gamma d(f^{n-2}x,f^{n-1}x,z)$$

or

$$d(f^{n-1}x,f^n x,z) \leq \left(\frac{\alpha+\gamma}{1-\beta}\right) d(f^{n-2}x,f^{n-1}x,z),$$

which, by induction gives

$$(3.5) \qquad d(f^{n-1}x,f^n x,z) \leq \left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-1} d(x,fx,z) \text{ for all } z \in X.$$

In particular,

$$d(f^{n-1}x, f^n x, p) \le \left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-1} d(x, fx, p). \tag{3.6}$$

Inserting (3.5) and (3.6) in (3.4), it follows that

$$\begin{aligned} d(f^n x, p, z) \le & \left(\frac{\alpha+\gamma}{1-\gamma}\right)\left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-1} d(x, fx, z) \\ & + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-1} d(x, fx, p) \text{ for all } z \in X. \end{aligned}$$

Then, applying the limit as $n \to \infty$ and using the choice of $\alpha+\beta+\gamma<1$ in this, finally we get

$$d(f^n x, p, z) \to 0 \text{ for all } x, z \in X,$$

that is p is a 2-contractive fixed point of f. ∎

Writing $\gamma = 0$ and $\alpha = \beta = k$ in Theorem (3.1), we get Kannan's type result as follows:

Corollary 3.1. *Let f be a self-map on a 2-metric space (X, d) such that*

$$d(fx, fy, z) \le k[d(x, fx, z) + d(y, fy, z)] \text{ for all } x, y, z \in X \tag{3.7}$$

where $0 \le k < \frac{1}{2}\cdot$ If X is 2-complete, then f will have a unique fixed point.

From this definition of a G-metric space, it immediately follows that

$$\text{If } x, y \in X \text{ are such that } G(x, y, y) = 0, \text{ then } x = y. \tag{3.8}$$

and that

$$G(x, y, y) \le 2G(x, x, y) \text{ for all } x, y \in X. \tag{3.9}$$

Lemma 3.3. [Mustafa and Sims [23]] *The following statements are equivalent in a G-metric space (X, G):*

(a) $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *is G-convergent with limit $p \in X$,*

(b) $\lim\limits_{n\to\infty} G(x_n, x_n, p) = 0$,

(c) $\lim\limits_{n\to\infty} G(x_n, p, p) = 0$.

The authors [28] presented an analytical proof of the following result of Mustafa et al [21]:

Theorem 3.2. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \le aG(x, fx, fx) &+ bG(y, fy, fy) + cG(z, fz, fz) \\ &+ eG(x, y, z) \quad \textit{for all } x, y, z \in X, \end{aligned} \tag{3.10}$$

where a, b, c and e are nonnegative real numbers with $a+b+c+e<1$. Then, f will have a unique fixed point p and f is continuous at p.

Then, writing $a = b = c = q$ and $e = 0$ in Theorem 3.2, we get

Corollary 3.2. *Suppose that* (X, G) *is a complete* G*-metric space and* f*, a self-map on* X *satisfying*

$$G(fx, fy, fy) \leq q[G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz)] \quad \text{for all } x, y, z \in X, \tag{3.11}$$

where $0 \leq q < \frac{1}{3}\cdot$ *Then* f *will have a unique fixed point* p *and* f *is continuous at* p.

Recently, Vats et al [33] established the following theorem:

Theorem 3.3. *Suppose that* (X, G) *is a complete* G*-metric space and* f*, a self-map on* X *satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq k \max\{&G(x, fx, fx), G(x, fy, fy), \\ &G(x, fz, fz), G(y, fy, fy), G(y, fx, fx), \\ &G(y, fz, fz), G(z, fz, fz), G(z, fx, fx), \\ &G(z, fy, fy)\} \text{ for all } x, y, z \in X, \end{aligned} \tag{3.12}$$

where $0 \leq k < \frac{1}{2}\cdot$ *Then* f *will have a unique fixed point* p *and* f *is continuous at* p.

It may be noted that condition (3.12) is partially weaker than (3.11). In fact, since the arithmetic average of three real numbers cannot exceed their maximum, (3.11) can be written as

$$\begin{aligned} G(fx, fy, fy) &\leq 3q \left[\frac{G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz)}{3}\right] \\ &\leq k' \max\{G(x, fx, fx), G(y, fy, fy), G(z, fz, fz)\} \\ &\leq k' \max\{G(x, fx, fx), G(x, fy, fy), \\ &\qquad G(x, fz, fz), G(y, fy, fy), G(y, fx, fx), \\ &\qquad G(y, fz, fz), G(z, fz, fz), G(z, fx, fx), G(z, fy, fy)\}, \end{aligned} \tag{3.13}$$

which is (3.12) with $0 \leq k' = 3q < 1$. In other words, under the restricted range $\left[0, \frac{1}{2}\right)$ for k', (3.11) implies (3.12). Thus, Theorem 3.3 is a partial generalization of Theorem 3.2.

Remark 3.4. If $k = 0$, then with $z = y = fx$, (3.12) or (3.11) gives $G(fx, f^2x, f^2x) = 0$ or $f^2x = fx$ for each $x \in X$. That is, every fx is a fixed point of f or the fixed point is not unique. Therefore, the choice of $k = 0$ is excluded n this paper.

We shall give an analytical proof of Theorem 3.3, which does not need iterations to obtain a fixed point p of f. Then, we shall establish that p is, in fact, a G-contractive fixed point of f.

Proof. Let $S = \{G(x, fx, fx) : x \in X\}$. Then, S being a nonempty set of nonnegative numbers, which is bounded below, has an infimum, say $a \geq 0$, by Lemma 3.1.

If possible, let $a > 0$. From (3.12) with $y = fx$ and $z = fx$, we have

$$\begin{aligned} G(fx, f^2x, f^2x) \leq k \max \{&G(x, fx, fx), G(x, f^2x, f^2x), G(x, f^2x, f^2x), \\ &G(fx, f^2x, f^2x), G(fx, fx, fx), G(fx, f^2x, f^2x), \\ &G(fx, f^2x, f^2x), G(fx, fx, fx), G(x, f^2x, f^2x)\}, \end{aligned}$$

which implies

$$G(fx, f^2x, f^2x) \leq k \max\{G(x, fx, fx), G(x, f^2x, f^2x), G(fx, f^2x, f^2x)\}.$$

But by the rectangle inequality (G5),

$$G(x, f^2x, f^2x) \leq G(fx, f^2x, f^2x) + G(x, fx, fx).$$

Using this in the previous inequality, we get

$$G(fx, f^2x, f^2x) \leq k[G(fx, f^2x, f^2x) + G(x, fx, fx)]$$

or

$$G(fx, f^2x, f^2x) \leq \left(\frac{k}{1-k}\right) G(x, fx, fx) \text{ for all } x \in X. \tag{3.14}$$

Since $0 \leq k < \frac{1}{2}$, we see that $\frac{k}{1-k} < 1$. So, from (3.14), it would follow that $G(fx, f^2x, f^2x) < a$, where $G(fx, f^2x, f^2x) \in S$. This implies that a cannot be a lower bound of S, contradicting the choice of a.

Therefore, $\inf S = a = 0$.

By Lemma 3.2, we can choose the points $x_1, x_2, ..., x_n, ...$ in X such that

$$G(x_n, fx_n, fx_n) \in S \text{ for } n = 1, 2, 3, ...$$

and

$$\lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0. \tag{3.15}$$

Now, repeatedly using the rectangle inequality (G5) and (3.12), we see that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, fx_n, fx_n) + G(fx_n, x_m, x_m) \\ &\leq G(x_n, fx_n, fx_n) + [G(fx_n, fx_m, fx_m) + G(fx_m, x_m, x_m)] \\ &\leq G(x_n, fx_n, fx_n) + G(fx_n, fx_m, fx_m) + 2G(x_m, fx_m, fx_m) \end{aligned}$$

Now

$$\begin{aligned}G(fx_n, fx_m, fx_m) &\leq k\max\{G(x_n, fx_n, fx_n), G(x_n, fx_m, fx_m), G(x_n, fx_m, fx_m),\\ &\quad G(x_m, fx_m, fx_m), G(x_m, fx_n, fx_n), G(x_m, fx_m, fx_m),\\ &\quad G(x_m, fx_m, fx_m), G(x_m, fx_n, fx_n), G(x_m, fx_m, fx_m)\}\\ &= k\max\{G(x_n, fx_n, fx_n), G(x_n, fx_m, fx_m),\\ &\qquad G(x_m, fx_m, fx_m), G(x_m, fx_n, fx_n)\}\\ &\leq k\max\{G(x_n, fx_n, fx_n) + G(x_m, x_n, x_n),\\ &\qquad G(x_m, fx_m, fx_m) + G(x_n, x_m, x_m)\}\\ &\leq k\max\{G(x_n, fx_n, fx_n) + 2G(x_n, x_m, x_m)],\\ &\qquad G(x_m, fx_m, fx_m) + G(x_n, x_m, x_m)\}\\ &= k[G(x_n, fx_n, fx_n) + G(x_m, fx_m, fx_m)] + 2G(x_n, x_m, x_m)]\end{aligned}$$

or

$$G(x_n, x_m, x_m) \leq \left(\frac{1+k}{1-2k}\right) G(x_n, fx_n, fx_n) + \left(\frac{2+k}{1-2k}\right) G(x_m, fx_m, fx_m).$$

Employing the limit as $m, n \to \infty$ in this and using (3.15), we get

$$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0,$$

proving that $\langle x_n\rangle_{n=1}^{\infty}$ is G-Cauchy.

Since X is G-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = p. \tag{3.16}$$

Again using (G5) and (3.9) in (3.12), we have

$$\begin{aligned}G(p, fp, fp) &\leq G(p, fx_n, fx_n) + G(fx_n, fp, fp)\\ &\leq [G(p, x_n, x_n) + G(x_n, fx_n, fx_n)] + G(fx_n, fp, fp).\end{aligned} \tag{3.17}$$

Also

$$\begin{aligned}G(fx_n, fp, fp) &\leq k\max\{G(x_n, fx_n, fx_n), G(x_n, fp, fp), G(x_n, fp, fp),\\ &\qquad G(p, fp, fp), G(p, fx_n, fx_n), G(p, fp, fp),\\ &\qquad G(p, fp, fp), G(p, fx_n, fx_n), G(p, fp, fp)\}\\ &= k\max\{G(x_n, fx_n, fx_n), G(x_n, fp, fp),\\ &\qquad G(p, fp, fp), G(p, fx_n, fx_n)\}\\ &\leq k\max\{G(x_n, fx_n, fx_n), G(x_n, fp, fp), G(p, fp, fp),\\ &\qquad G(x_n, fx_n, fx_n) + G(p, x_n, x_n)\}.\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we get

$$\begin{aligned}G(p, fp, fp) &\leq G(p, x_n, x_n) + G(x_n, fx_n, fx_n)\\ &\quad + k\max\{G(x_n, fx_n, fx_n), G(x_n, fp, fp), G(p, fp, fp),\\ &\qquad G(x_n, fx_n, fx_n) + G(p, x_n, x_n)\}.\end{aligned}$$

Proceeding the limit as $n \to \infty$ in this, and then using (3.15) and (3.16),

$$G(p, fp, fp) \leq [0 + 0 + k \max\{0, G(p, fp, fp), G(p, fp, fp), [0 + 0]\}]$$

which implies that $(1 - \beta)d(p, fp, fp) \leq 0$ or $fp = p$.

That is, p is a fixed point of f.

Uniqueness. Suppose q is another fixed point of f so that $fq = q$. Then from (3.12) with $x = p$ and $y = z = q$, we have

$$\begin{aligned} G(p, q, q) &= G(fp, fq, fq) \\ &\leq k \max\{G(p, fp, fp), G(p, fq, fq), G(p, fq, fq), \\ &\qquad G(q, fq, fq), G(q, fp, fp), G(q, fq, fq), \\ &\qquad G(q, fq, fq), G(q, fp, fp), G(q, fq, fq)\} \\ &= k \max\{0, G(p, q, q), G(p, q, q), 0, G(q, p, p), 0, 0, G(q, p, p), 0\} \\ &= k \max\{G(p, q, q), G(q, p, p)\} \\ &\leq 2kG(p, q, q), \end{aligned}$$

which implies that $(1 - 2k)G(p, q, q) \leq 0$ or $p = q$. That is p is unique fixed point of f. The G-continuity of f at p is obtained as in [33]. ∎

Remark 3.5. An exciting feature of Theorem 3.3 is that p is a G-contractive fixed point of f, whenever $0 \leq k < \frac{1}{3}\cdot$ In fact, we write $y = z = p$ in (3.12) and then use (G5) to obtain

$$\begin{aligned} G(f^n x, fp, fp) &=\leq k \max\{G(f^{n-1}x, fx, fx), G(f^{n-1}x, fp, fp), G(f^{n-1}x, fp, fp), \\ &\qquad G(p, fp, fp), d(p, f^{n-1}x, f^{n-1}x), G(p, fp, fp), \\ &\qquad G(p, fp, fp), G(p, f^{n-1}x, f^{n-1}x), G(p, fp, fp)\} \\ &= k \max\{G(f^{n-1}x, f^n x, f^n x), G(f^{n-1}x, p, p), 0, \\ &\qquad G(p, f^n x, f^n x), 0, G(p, f^n x, f^n x)\} \\ &= k \max\{G(f^{n-1}x, f^n x, f^n x), G(f^{n-1}x, p, p), G(p, f^n x, f^n x)\} \\ &= k\{G(p, f^n x, f^n x) + G(f^{n-1}x, p, p)\} \\ &\leq k\{2G(f^n x, p, p) + G(f^{n-1}x, p, p)\}. \end{aligned}$$

or

$$G(f^n x, p, p) \leq c \cdot G(f^{n-1}x, p, p),$$

where $\frac{k}{1-2k} = c$. By induction, we have

$$G(f^n x, p, p) \leq c^n G(fx, p, p),$$

which as $n \to \infty$ implies that $G(f^n x, p, p) \to 0$ for each $x \in X$, since $c < 1$. Thus p is a G-contractive fixed point of f.

Taking $z = y$ in Theorem 3.3, we have

Corollary 3.3. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fy) \leq k \max\{G(x, fx, fx), G(x, fy, fy), G(y, fy, fy), G(y, fx, fx)\} \text{ for all } x, y, z \in X, \tag{3.19}$$

where $0 \leq k < \frac{1}{2}\cdot$ Then f will have a unique fixed point p and f is continuous at p.

Proof. We set

$$\rho_G(x, y) = \max\{(G(x, y, y), G(x, x, y)\} \text{ for all } x, y \in X, \tag{3.20}$$

where the exchange of x and y yields the symmetry of ρ_G. It can be seen that ρ_G defines a metric on X.

Now, (3.19) can be written as

$$\begin{aligned} &G(fx, fy, fy) \\ &\leq 2k \max\left\{\frac{G(x, fx, fx) + G(x, x, fx)}{2}, \frac{G(x, fy, fy) + G(x, x, fy)}{2}, \right. \\ &\left. \frac{G(y, fx, fx) + G(y, y, fx)}{2}, \frac{G(y, fy, fy) + G(y, y, fy)}{2}\right\}. \end{aligned} \tag{3.21}$$

Interchanging the roles of x and y in this, we get

$$\begin{aligned} &G(fy, fx, fx) \\ &\leq 2k \max\left\{\frac{G(y, fy, fy) + G(y, y, fy)}{2}, \frac{G(y, fx, fx) + G(y, y, fx)}{2}, \right. \\ &\left. \frac{G(x, fy, fy) + G(x, x, fy)}{2}, \frac{G(x, fx, fx) + G(x, x, fx)}{2}\right\}. \end{aligned} \tag{3.22}$$

Taking the maximum of (3.21) and (3.22), and then using (3.20), it follows that

$$\begin{aligned} \rho_G(x, y) &\leq 2k \max\left\{\frac{G(x, fx, fx) + G(x, x, fx)}{2}, \frac{G(x, fy, fy) + G(x, x, fy)}{2}, \right. \\ &\left. \frac{G(y, fx, fx) + G(y, y, fx)}{2}, \frac{G(y, fy, fy) + G(y, y, fy)}{2}\right\} \\ &\leq 2k \max\{\rho_G(x, fx), \rho_G(x, fy), \rho_G(y, fx), \rho_G(y, fy)\} \text{ for all } x, y \in X, \end{aligned}$$

which is a special case of Ciric's quasi-contraction, and a unique fixed point follows from [7] for complete metric space. In general, if any two of the three variables in the the contraction type condition (3.12) are the same, a unique fixed point can be obtained from Ciric's result in a complete metric space [7]. ∎

Writing $z = fy$ in Theorem 3.3, we have

Corollary 3.4. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, f^2y) \leq k \max\{&G(x, fx, fx), G(x, fy, fy),\\ &G(x, f^2y, f^2y), G(y, fy, fy), G(y, fx, fx),\\ &G(y, f^2y, f^2y), G(fy, f^2y, f^2y), G(fy, fx, fx)\},\\ &\text{for all } x, y, z \in X, \end{aligned} \tag{3.23}$$

where $0 < k < \frac{1}{2}$. Then f will have a unique fixed point p and f is continuous at p.

Remark 3.6. Write

$$\rho_G(x, y) = \max\{(G(x, fx, f^2y), G(y, fy, f^2x)\} \text{ for all } x, y \in X. \tag{3.24}$$

It is interesting to remark that $x = y$ does not imply that $\rho_G(x, y) = 0$. In fact, from (3.24) with $y = x$, we get $\rho_G(x, x) = G(x, fx, f^2x)$, which will be positive for each $x \in X$ with $fx \neq x$. That is, ρ_G is not a metric on X. In other words, Corollary 3.4 cannot be characterized in terms of metric space to determine a fixed point of f.

With an altered form of the condition (3.12), we now prove

Theorem 3.4. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq k \max\{&G(x, x, fx), G(x, x, fy), G(x, x, fz),\\ &G(y, y, fy), G(y, y, fx), G(y, y, fz), G(z, z, fz),\\ &G(z, z, fx), G(z, z, fy)\} \text{ for all } x, y, z \in X, \end{aligned} \tag{3.25}$$

where $0 < k < \frac{1}{2}\cdot$ Then f will have a unique fixed point p and f is continuous at p. Further if $0 < k < 1/3$, then p will be its G-contractive fixed point.

Proof. Let $a \geq 0$ be the infimum of the set

$$S = \{G(x, x, fx) : x \in X\},$$

in view of Lemma 3.1.

If possible, let $a > 0$. From (3.25) with $y = x$ and $z = fx$, we have

$$\begin{aligned} G(fx, fx, f^2x) \leq k \max\{&G(x, x, fx), G(x, x, fx), G(x, x, f^2x),\\ &G(x, x, fx), G(x, x, fx), G(x, x, f^2x), G(fx, fx, f^2x),\\ &G(fx, fx, fx), G(x, fx, fx)\}, \end{aligned}$$

which implies

$$G(fx, fx, f^2x) \leq k \max\{G(x, x, fx), G(x, x, f^2x), G(fx, fx, f^2x)\}.$$

But, by rectangle inequality,

$$G(x,x,f^2x) \le G(x,x,fx) + G(fx,fx,f^2x).$$

Using this in the previous inequality,

$$G(fx,fx,f^2x) \le k[G(x,x,fx) + G(fx,fx,f^2x)] \text{ for all } x \in X.$$

or

$$d(fx,fx,f^2x) \le \left(\frac{k}{1-k}\right) G(x,x,fx) \text{ for all } x \in X. \tag{3.26}$$

Since $\frac{k}{1-k} < 1$, from (3.26), it would follow that $G(fx,fx,f^2x) < a$ for some $x \in X$. This contradicts the choice of a. Therefore, $a = \inf S = 0$.

By Lemma 3.2, we can choose the points $x_1, x_2, ..., x_n, ...$ in X such that

$$G(x_n, x_n, fx_n) \in S \text{ for } n = 1, 2, 3, ...$$

and

$$\lim_{n\to\infty} G(x_n, x_n, fx_n) = 0. \tag{3.27}$$

Now repeatedly using the rectangle inequality (G5) and (3.25), we see that

$$\begin{aligned} G(x_m, x_m, x_n) &\le G(x_m, fx_m, fx_m) + G(fx_m, fx_m, x_n) \\ &\le G(x_m, x_m, fx_m) + [G(fx_m, fx_m, fx_n) \\ &\qquad +G(fx_n, x_n, x_n)] \\ &\le G(x_m, x_m, fx_m) + G(fx_m, fx_m, fx_n) \\ &\qquad +2G(x_n, fx_n, fx_n). \end{aligned} \tag{3.28}$$

Now,

$$\begin{aligned} G(fx_m, fx_m, fx_n) \le k \max\{&G(x_m, x_m, fx_m), G(x_m, x_m, fx_m), \\ &G(x_m, x_m, fx_n), G(x_m, x_m, fx_m), G(x_m, x_m, fx_m), \\ &G(x_m, x_m, fx_n), G(x_n, x_n, fx_n), \\ &G(x_n, x_n, fx_m), G(x_n, x_n, fx_m)\} \end{aligned}$$

which can be written as

$$\begin{aligned} G(fx_m, fx_m, fx_n) &\le k \max\{G(x_m, x_m, fx_m), G(x_m, x_m, fx_n), \\ &\qquad G(x_n, x_n, fx_n), G(x_n, x_n, fx_m)\} \\ &\le k \max\{[G(x_m, x_m, x_n) + G(x_n, x_n, fx_n)], \\ &\qquad [G(x_n, x_n, x_m) + G(x_m, x_m, fx_m)]\} \\ &\le k \max\{[G(x_m, x_m, x_n) + G(x_n, x_n, fx_n)], \\ &\qquad [2G(x_m, x_m, x_n) + G(x_m, x_m, fx_m)]\} \\ &= k[2G(x_m, x_m, x_n) + G(x_n, x_n, fx_n)] \\ &\qquad +G(x_m, x_m, fx_m)]. \end{aligned} \tag{3.29}$$

Substituting (3.29) in (3.28), we get

$$\begin{aligned} G(x_m,x_m,x_n) \leq\ & G(x_m,x_m,fx_m)+2G(x_n,x_n,fx_n) \\ & +k[2G(x_m,x_m,x_n)+G(x_m,x_m,fx_m)+G(x_n,x_n,fx_n)] \end{aligned}$$

or

$$G(x_m,x_m,x_n) \leq \left(\frac{1+k}{1-2k}\right) G(x_m,x_m,fx_m)+\left(\frac{2+k}{1-2k}\right) G(x_n,x_n,fx_n).$$

Employing the limit as $m,n \to \infty$ in this and using (3.27), we get

$$\lim_{n,m\to\infty} G(x_m,x_m,x_n)=0,$$

proving that $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy.

Since X is G-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = p. \tag{3.30}$$

Again, by repeated application of rectangle inequality (G5), from (G4) and (3.25), we have

$$\begin{aligned} G(p,p,fp) &\leq G(p,p,fx_n)+G(fx_n,fx_n,fp) \\ &\leq [G(p,p,x_n)+G(x_n,x_n,fx_n)]+G(fx_n,fx_n,fp). \end{aligned} \tag{3.31}$$

Now,

$$\begin{aligned} G(fx_n,fx_n,fp) \leq\ & k\max\{G(x_n,,x_n,fx_n),G(x_n,,x_n,fx_n), \\ & G(x_n,,x_n,fp),G(x_n,,x_n,fx_n), \\ & G(x_n,,x_n,fx_n),G(x_n,,x_n,fp), \\ & G(p,p,fp),G(p,p,fx_n),G(p,p,fx_n)\} \\ =\ & k\max\{G(x_n,x_n,fx_n),G(x_n,x_n,fp), \\ & G(p,p,fp),G(p,p,fx_n)\} \\ \leq\ & k\max\{G(x_n,x_n,fx_n),G(x_n,x_n,fp),G(p,p,fp), \\ & [G(p,p,x_n)+G(x_n,x_n,fx_n)]\}. \end{aligned} \tag{3.32}$$

Substituting (3.32) in (3.31), we get

$$\begin{aligned} G(p,p,fp) \leq\ & G(p,p,x_n)+G(x_n,x_n,fx_n)+k\max\{G(x_n,fx_n,fx_n), \\ & G(x_n,x_n,fp),G(p,p,fp),[G(p,p,x_n)+G(x_n,x_n,fx_n)]\}. \end{aligned}$$

Proceeding the limit as $n \to \infty$ in this, we get

$$G(p,p,fp) \leq [0+0+k\max\{0,G(p,p,fp),G(p,p,fp),[0+0]\}],$$

which implies that

$$(1-k)G(p,p,fp) \leq 0 \text{ or } fp = p.$$

That is, p is a fixed point of f. It can be easily established as in the proof of Theorem 3.3 and Remark 3.5 that the fixed point is unique, which will also be a G-contractive fixed point of f. ∎

Restricting the the terms in 3.12, we have

Corollary 3.5. *Let* (X, G) *is a complete G-metric space satisfying,*

$$G(fx, fy, fz) \leq k[G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx)], \quad \text{for all } x, y, z \in X. \tag{3.33}$$

Then f *has a unique fixed point.*

While, restricting the terms in (3.25), we have

Corollary 3.6. *Let* (X, G) *is a complete G-metric space satisfying,*

$$G(fx, fy, fz) \leq k[G(x, x, fy) + G(y, y, fz) + G(z, z, fx)], \quad \text{for all } x, y, z \in X. \tag{3.34}$$

Then f *has a unique fixed point.*

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