INTUITIONISTIC PERMEABLE VALUES IN $BCK/BCI$-ALGEBRAS

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Abstract. Intuitionistic permeable values in $BCK/BCI$-algebras are introduced, and several properties are investigated. A relation between an intuitionistic permeable S-value and an intuitionistic permeable I-value is discussed. Conditions for the intuitionistic lower (upper) level set to be $S$-energetic and $I$-energetic are considered. Conditions for a couple of numbers to be an intuitionistic permeable S-value are studied.

Keywords: $S$-energetic subset, $I$-energetic subset, intuitionistic permeable $S$-value, intuitionistic permeable $I$-value.

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1. Introduction

The notions of energetic (resp. right vanished, right stable) subsets and (anti) permeable values in $BCK/BCI$-algebras are introduced by Jun et al. [2]. Using the notion of (anti) fuzzy subalgebras/ideals of BCK/BCI-algebras, they investigated relations among subalgebras/ideals, energetic subsets, (anti) permeable values, right vanished subsets and right stable subsets.

In this paper, we introduce the notion of intuitionistic permeable values in $BCK/BCI$-algebras, and investigate several properties. We provide a relation between an intuitionistic permeable S-value and an intuitionistic permeable I-value. We study conditions for the intuitionistic lower (upper) level set to be

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S-energetic and I-energetic. We consider conditions for a couple of numbers to be an intuitionistic permeable S-value. We show that if an intuitionistic fuzzy set $A = (f_A, g_A)$ in a BCK-algebra $X$ is an intuitionistic fuzzy ideal of $X$, then the nonempty intuitionistic upper level set of $A = (f_A, g_A)$ is a right stable subset of $X$ and the nonempty intuitionistic lower level set of $A = (f_A, g_A)$ is a right vanished subset of $X$.

2. Preliminaries

The BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and has been extensively investigated by many researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $(\forall x, y, z \in X) \left( (((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0) \right)$,

(II) $(\forall x, y \in X) \left( (x \ast (x \ast y)) \ast y = 0 \right)$,

(III) $(\forall x \in X) \left( x \ast x = 0 \right)$,

(IV) $(\forall x, y \in X) \left( x \ast y = 0, y \ast x = 0 \Rightarrow x = y \right)$.

If a BCI-algebra $X$ satisfies the following identity

(V) $(\forall x \in X) \left( 0 \ast x = 0 \right)$,

then $X$ is called a BCK-algebra. Any BCK/BCI-algebra $X$ satisfies the following axioms:

(2.1) $(\forall x \in X) \left( x \ast 0 = x \right)$,

(2.2) $(\forall x, y, z \in X) \left( x \leq y \Rightarrow x \ast z \leq y \ast z, z \ast y \leq z \ast x \right)$,

(2.3) $(\forall x, y, z \in X) \left( (x \ast y) \ast z = (x \ast z) \ast y \right)$,

(2.4) $(\forall x, y, z \in X) \left( (x \ast z) \ast (y \ast z) \leq x \ast y \right)$

where $x \leq y$ if and only if $x \ast y = 0$. A nonempty subset $S$ of a BCK/BCI-algebra $X$ is called a subalgebra of $X$ if $x \ast y \in S$ for all $x, y \in S$. A subset $I$ of a BCK/BCI-algebra $X$ is called an ideal of $X$ if it satisfies

(2.5) $0 \in I$,

(2.6) $(\forall x \in X) \left( \forall y \in I \right) \left( x \ast y \in I \Rightarrow x \in I \right)$.

We refer the reader to the books [1], [4] for further information regarding BCK/BCI-algebras.

An intuitionistic fuzzy set $A = (f_A, g_A)$ in $X$ is called an intuitionistic fuzzy subalgebra of $X$ (see [3]) if the following condition is valid.

(2.7) $(\forall x, y \in X) \left( f_A(x \ast y) \geq \min\{f_A(x), f_A(y)\} \right)$

\[ g_A(x \ast y) \leq \max\{g_A(x), g_A(y)\} \right).


An intuitionistic fuzzy set \( A = (f_A, g_A) \) in \( X \) is called an \textit{intuitionistic fuzzy ideal} of \( X \) (see [3]) if the following conditions are valid.

\begin{align}
(\forall x \in X) \left( f_A(0) \geq f_A(x), \ g_A(0) \leq g_A(x) \right),
\end{align}

\begin{align}
(\forall x, y \in X) \left( f_A(x) \geq \min \{f_A(x \ast y), f_A(y)\} \right. \\
\left. g_A(x) \leq \max \{g_A(x \ast y), g_A(y)\} \right).
\end{align}

Note that every intuitionistic fuzzy ideal is an intuitionistic fuzzy subalgebra in a \( BCK \)-algebra (see [3]).

Every intuitionistic fuzzy ideal \( A = (f_A, g_A) \) of \( X \) satisfies the following condition (see [3]).

\begin{align}
(\forall x, y \in X) \left( x \ast y = 0 \Rightarrow f_A(x) \geq f(y), \ g_A(x) \leq g_A(y) \right).
\end{align}

3. Intuitionistic permeable values

In what follows, let \( X \) denote a \( BCK/BCI \)-algebra unless otherwise specified.

**Definition 3.1.** ([2]) A non-empty subset \( A \) of \( X \) is said to be \textit{S-energetic} if it satisfies

\begin{align}
(\forall a, b \in X) \left( a \ast b \in A \Rightarrow \{a, b\} \cap A \neq \emptyset \right).
\end{align}

**Definition 3.2.** Let \( A = (f_A, g_A) \) be an intuitionistic fuzzy set in \( X \) and \( (t_A, s_A) \in [0, 1] \times [0, 1] \).

We say that \( (t_A, s_A) \) is an \textit{intuitionistic permeable S-value} for \( A = (f_A, g_A) \) if the following assertion is valid:

\begin{align}
(\forall x, y \in X) \left( f_A(x \ast y) \geq t_A \Rightarrow \max \{f_A(x), f_A(y)\} \geq t_A \right. \\
\left. g_A(x \ast y) \leq s_A \Rightarrow \min \{g_A(x), g_A(y)\} \leq s_A \right).
\end{align}

**Example 3.3.** (1) Let \( X = \{0, 1, 2, 3\} \) be a \( BCK \)-algebra with the following Cayley table:

\[
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
2 & 2 & 2 & 0 & 0 \\
3 & 3 & 2 & 1 & 0 \\
\end{array}
\]

Let \( A = (f_A, g_A) \) be an intuitionistic fuzzy set in \( X \) defined by

\[
f_A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.3 & 0.5 & 0.4 & 0.6 \end{pmatrix}, \quad g_A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.6 & 0.4 & 0.3 & 0.2 \end{pmatrix}.
\]

If we take \( (t_A, s_A) \in (0.3, 1] \times [0.6, 0.2) \), then it is easy to check that \( (t_A, s_A) \) is an intuitionistic permeable \( S \)-value for \( A = (f_A, g_A) \).
(2) Let $X = \{0, 1, 2, a, b\}$ be a $BCI$-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>a</th>
<th>b</th>
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<tr>
<td>0</td>
<td>0</td>
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<td>b</td>
<td>a</td>
<td>b</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in $X$ defined by

$$f_A = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.2 & 0.5 & 0.3 & 0.4 & 0.5 \end{pmatrix}$$

and

$$g_A = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.7 & 0.4 & 0.6 & 0.2 & 0.3 \end{pmatrix}.$$ 

If we take $(t_A, s_A) \in (0.2, 1] \times (0.3, 0.7)$, then we know that $(t_A, s_A)$ is an intuitionistic permeable $S$-value for $A = (f_A, g_A)$.

For an intuitionistic fuzzy set $A = (f_A, g_A)$ in $X$ and $(t_A, s_A) \in [0, 1] \times [0, 1]$, the intuitionistic upper (resp. lower) level sets are defined as follows:

$$U(f_A; t_A) := \{x \in X \mid f_A(x) \geq t_A\}, \quad L(g_A; s_A) := \{x \in X \mid g_A(x) \leq s_A\},$$

**Theorem 3.4.** If $(t_A, s_A)$ is an intuitionistic permeable $S$-value for an intuitionistic fuzzy set $A = (f_A, g_A)$ in $X$, then the intuitionistic upper level set $U(f_A; t_A)$ and the intuitionistic lower level set $L(g_A; s_A)$ are $S$-energetic subsets of $X$ when they are nonempty.

**Proof.** Assume that $U(f_A; t_A) \neq \emptyset \neq L(g_A; s_A)$ for $(t_A, s_A) \in [0, 1] \times [0, 1]$. Let $a, b, x, y \in X$ be such that $a * b \in U(f_A; t_A)$ and $x * y \in L(g_A; s_A)$. Then $f_A(a * b) \geq t_A$ and $g_A(x * y) \leq s_A$. It follows from (3.2) that

$$\max\{f_A(a), f_A(b)\} \geq t_A,$$

$$\min\{g_A(x), g_A(y)\} \leq s_A.$$ 

The fact (3.3) implies that $f_A(a) \geq t_A$ or $f_A(b) \geq t_A$. Thus $\{a, b\} \cap U(f_A; t_A) \neq \emptyset$. From the fact (3.4), we have $\{x, y\} \cap L(g_A; s_A) \neq \emptyset$. Therefore $U(f_A; t_A)$ and $L(g_A; s_A)$ are $S$-energetic subsets of $X$. 

Note from [2, Theorem 3.3] that if $A$ is an $S$-energetic subset of $X$, then $X \setminus A$ is a subalgebra of $X$. Hence we have the following corollary.

**Corollary 3.5.** If $(t_A, s_A)$ is an intuitionistic permeable $S$-value for an intuitionistic fuzzy set $A = (f_A, g_A)$ in $X$, then $X \setminus U(f_A; t_A)$ and $X \setminus L(g_A; s_A)$ are subalgebras of $X$ when they are nonempty.
Theorem 3.6. If $A = (f_A, g_A)$ is an intuitionistic fuzzy subalgebra of $X$, then $X \setminus U(f_A; t)$ and $X \setminus L(g_A; t)$ are S-energetic subsets of $X$ for all $t \in [0, 1]$ whenever they are nonempty.

Proof. Let $x, y \in X$ be such that $x \ast y \in X \setminus U(f_A; t)$ and $x \ast y \in X \setminus L(g_A; t)$ for $t \in [0, 1]$. If $(x, y) \cap (X \setminus U(f_A; t)) = \emptyset$, then $(x, y) \cap (X \setminus L(g_A; t)) = \emptyset$. Assume that $(x, y) \cap (X \setminus L(g_A; t)) = \emptyset$. Then $x, y \in L(g_A; t)$ and so $x \ast y \in L(g_A; t)$ since $L(g_A; t)$ is a subalgebra of $X$. This is a contradiction. Therefore $X \setminus U(f_A; t)$ and $X \setminus L(g_A; t)$ are S-energetic subsets of $X$ for all $t \in [0, 1]$.

Corollary 3.7. If $A = (f_A, g_A)$ is an intuitionistic fuzzy ideal of a BCK/BCI-algebra $X$, then $X \setminus U(f_A; t)$ and $X \setminus L(g_A; t)$ are S-energetic subsets of $X$ for all $t \in [0, 1]$ whenever they are nonempty.

Definition 3.8. An intuitionistic fuzzy set $A = (f_A, g_A)$ in $X$ is called an anti-intuitionistic fuzzy subalgebra if the following condition is valid.

\[
(3.5) \quad (\forall x, y \in X) \left( f_A(x \ast y) \leq \max \{f_A(x), f_A(y)\} \right) \land \left( g_A(x \ast y) \geq \min \{g_A(x), g_A(y)\} \right).
\]

Theorem 3.9. For an intuitionistic fuzzy set $A = (f_A, g_A)$ in $X$, let $(t_A, s_A) \in [0, 1] \times [0, 1]$ be such that $U(f_A; t_A) \neq \emptyset \neq L(g_A; s_A)$. If $A = (f_A, g_A)$ is an anti-intuitionistic fuzzy subalgebra of $X$, then $(t_A, s_A)$ is an intuitionistic permeable S-value for $A = (f_A, g_A)$.

Proof. Let $x, y \in X$ be such that $f_A(x \ast y) \geq t_A$ and $g_A(x \ast y) \leq s_A$. Using (3.5), we have

\[
t_A \leq f_A(x \ast y) \leq \max \{f_A(x), f_A(y)\}
\]

and

\[
s_A \geq g_A(x \ast y) \geq \min \{g_A(x), g_A(y)\}.
\]

Therefore $(t_A, s_A)$ is an intuitionistic permeable S-value for $A = (f_A, g_A)$.

Theorem 3.10. If $A = (f_A, g_A)$ is an anti-intuitionistic fuzzy subalgebra of $X$, then $U(f_A; t_A)$ and $L(g_A; s_A)$ are S-energetic subsets of $X$ for all $(t_A, s_A) \in [0, 1] \times [0, 1]$ with $U(f_A; t_A) \neq \emptyset \neq L(g_A; s_A)$.

Proof. Assume that $U(f_A; t_A) \neq \emptyset \neq L(g_A; s_A)$ for $(t_A, s_A) \in [0, 1] \times [0, 1]$. Let $x, y \in X$ be such that $x \ast y \in U(f_A; t_A)$. Then $t_A \leq f_A(x \ast y) \leq \max \{f_A(x), f_A(y)\}$, and so $f_A(x) \geq t_A$ or $f_A(y) \geq t_A$. Thus $(x, y) \cap U(f_A; t_A) \neq \emptyset$. If $a \ast b \in L(g_A; s_A)$ for $a, b \in X$, then $s_A \geq g_A(a \ast b) \geq \min \{g_A(a), g_A(b)\}$. It follows that $g_A(a) \leq s_A$ or $g_A(b) \leq s_A$, that is, $a \in L(g_A; s_A)$ or $b \in L(g_A; s_A)$. Hence $(a, b) \cap L(g_A; s_A) \neq \emptyset$. This completes the proof.

The following example shows that the converse of Theorem 3.10 is not true in general, that is, there exist an intuitionistic fuzzy set $A = (f_A, g_A)$ in $X$ and $(t_A, s_A) \in [0, 1] \times [0, 1]$ such that
1. $U(f_A; t_A)$ and $L(g_A; s_A)$ are S-energetic subsets of $X$.

2. $A = (f_A, g_A)$ is not an anti-intuitionistic fuzzy subalgebra of $X$.

**Example 3.11.** Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
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<td>b</td>
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<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
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<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in $X$ defined by

$$f_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.3 & 0.4 & 0.4 & 0.2 & 0.4 \end{pmatrix}$$

and

$$g_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.5 & 0.4 & 0.4 & 0.3 & 0.2 \end{pmatrix}.$$AD

Let $(t_A, s_A) \in (0.3, 0.4] \times [0.2, 0.4)$. Then $U(f_A; t_A) = \{a, b, d\}$ and

$$L(g_A; s_A) = \begin{cases} \{c, d\} & \text{if } s_A \in [0.3, 0.4), \\ \{d\} & \text{if } s_A \in [0.2, 0.3) \end{cases}$$

are S-energetic subsets of $X$. But $A = (f_A, g_A)$ is not an anti-intuitionistic fuzzy subalgebra of $X$.

**Definition 3.12.** ([2]) A non-empty subset $A$ of $X$ is said to be $I$-energetic if it satisfies:

$$\forall x, y \in X \ (y \in A \Rightarrow \{x, y \ast x\} \cap A \neq \emptyset).$$

**Definition 3.13.** Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in $X$ and $(t_A, s_A) \in [0, 1] \times [0, 1]$. We say that $(t_A, s_A)$ is a permeable I-value for $A = (f_A, g_A)$ if the following assertion is valid:

$$\forall x, y \in X \left( f_A(x) \geq t_A \Rightarrow \max\{f_A(x \ast y), f_A(y)\} \geq t_A \right) \land \left( g_A(x) \leq s_A \Rightarrow \min\{g_A(x \ast y), g_A(y)\} \leq s_A \right).$$

**Example 3.14.** In Example 3.3(2), the number $(t_A, s_A) \in (0.2, 1] \times (0.3, 0.7)$ is an intuitionistic permeable I-value for $A = (f_A, g_A)$.

**Theorem 3.15.** If $A = (f_A, g_A)$ is an anti-intuitionistic fuzzy subalgebra of a BCK-algebra $X$, then every intuitionistic permeable I-value for $A = (f_A, g_A)$ is an intuitionistic permeable S-value for $A = (f_A, g_A)$. 
Proof. Note that $f_A(0) \leq f_A(x)$ and $g_A(0) \geq g_A(x)$ for all $x \in X$.

Let $(t_A, s_A) \in [0, 1] \times [0, 1]$ be an intuitionistic permeable I-value for $A = (f_A, g_A)$. Assume that $f_A(x * y) \geq t_A$ and $g_A(x * y) \leq s_A$ for all $x, y \in X$. Using (3.7), (2.3), (III) and (V), we have

$$t_A \leq \max \{f_A((x * y) * x), f_A(x)\} = \max \{f_A((x * x) * y), f_A(x)\} = \max \{f_A(0 * y), f_A(x)\} = \max \{f_A(0), f_A(x)\} = f_A(x)$$

and

$$s_A \geq \min \{g_A((x * y) * x), g_A(x)\} = \min \{g_A((x * x) * y), g_A(x)\} = \min \{g_A(0 * y), g_A(x)\} = \min \{g_A(0), g_A(x)\} = g_A(x).$$

It follows that

$$\max \{f_A(x), f_A(y)\} \geq f_A(x) \geq t_A \quad \text{and} \quad \min \{g_A(x), g_A(y)\} \leq g_A(x) \leq s_A.$$

Therefore $(t_A, s_A)$ is an intuitionistic permeable S-value for $A = (f_A, g_A)$.

Corollary 3.16. If $A = (f_A, g_A)$ is an anti-intuitionistic fuzzy ideal of a BCK-algebra $X$, then every intuitionistic permeable I-value for $A = (f_A, g_A)$ is an intuitionistic permeable S-value for $A = (f_A, g_A)$.

The converse of Theorem 3.15 is not true in general as seen in the following example.

Example 3.17. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
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<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in $X$ defined by

$$f_A = \begin{pmatrix}
0 & a & b & c \\
0.2 & 0.4 & 0.5 & 0.5
\end{pmatrix}, \quad g_A = \begin{pmatrix}
0 & a & b & c \\
0.3 & 0.2 & 0.4 & 0.2
\end{pmatrix}.$$

If we take $(t_A, s_A) \in (0.4, 0.5] \times [0.2, 0.3)$, then $U(f_A; t_A) = \{b, c\}$ and $L(g_A; s_A) = \{a, c\}$. It is easy to check that $(t_A, s_A)$ is an intuitionistic permeable S-value for $A = (f_A, g_A)$. Note that $f_A(b) = 0.5 \geq t_A$ and $g_A(a) = 0.2 \leq s_A$. Hence $\max \{f_A(b * a), f_A(a)\} = 0.4 < t_A$ and $\min \{g_A(a * b), g_A(b)\} = 0.3 > s_A$. This shows that $(t_A, s_A)$ is not an intuitionistic permeable I-value for $A = (f_A, g_A)$.

Theorem 3.18. If $A = (f_A, g_A)$ is an intuitionistic fuzzy ideal of $X$, then $X \setminus U(f_A; t_A)$ and $X \setminus L(g_A; s_A)$ are empty or I-energetic subsets of $X$ for all $(t_A, s_A) \in [0, 1] \times [0, 1]$. 
Proof. Assume that \(X \setminus U(f_A; t_A)\) and \(X \setminus L(g_A; s_A)\) are nonempty for all \((t_A, s_A) \in [0, 1] \times [0, 1]\). If \(y \in X \setminus U(f_A; t_A)\) and \(b \in X \setminus L(g_A; s_A)\), then

\[
(3.8) \quad t_A > f_A(y) \geq \min \{f_A(y \ast x), f_A(x)\}
\]

\[
(3.9) \quad s_A < g_A(b) \leq \max \{g_A(b \ast a), g_A(a)\}
\]

for all \(a, x \in X\). From (3.8), we have \(f_A(y \ast x) < t_A\) or \(f_A(x) < t_A\), that is, \(y \ast x \in X \setminus U(f_A; t_A)\) or \(x \in X \setminus U(f_A; t_A)\). Hence \(\{x, y \ast x\} \cap (X \setminus U(f_A; t_A)) \neq \emptyset\). From (3.9), we get \(g_A(b \ast a) > s_A\) or \(g_A(a) > s_A\), that is, \(b \ast a \in X \setminus L(g_A; s_A)\) or \(a \in X \setminus L(g_A; s_A)\). Thus \(\{a, b \ast a\} \cap (X \setminus L(g_A; s_A)) \neq \emptyset\). Therefore \(X \setminus U(f_A; t_A)\) and \(X \setminus L(g_A; s_A)\) are \(I\)-energetic subsets of \(X\) for all \((t_A, s_A) \in [0, 1] \times [0, 1]\).

Theorem 3.19. If \((t_A, s_A)\) is an intuitionistic permeable \(I\)-value for an intuitionistic fuzzy set \(A = (f_A, g_A)\) in \(X\), then the intuitionistic upper level set \(U(f_A; t_A)\) and the intuitionistic lower level set \(L(g_A; s_A)\) are empty or \(I\)-energetic subsets of \(X\).

Proof. Assume that \(U(f_A; t_A)\) and \(L(g_A; s_A)\) are nonempty for all \((t_A, s_A) \in [0, 1] \times [0, 1]\). Let \(y \in U(f_A; t_A)\). Then \(f_A(y) \geq t_A\), and so

\[\max \{f_A(y \ast x), f_A(x)\} \geq t_A\]

for all \(x \in X\). It follows that \(f_A(y \ast x) \geq t_A\) or \(f_A(x) \geq t_A\), that is, \(y \ast x \in U(f_A; t_A)\) or \(x \in U(f_A; t_A)\). Hence \(\{x, y \ast x\} \cap U(f_A; t_A) \neq \emptyset\). If \(b \in L(g_A; s_A)\), then \(g_A(b) \leq s_A\) and thus

\[\min \{g_A(b \ast a), g_A(a)\} \leq s_A\]

for all \(a \in X\). It follows that \(g_A(b \ast a) \leq s_A\) or \(g_A(a) \leq s_A\) and so that

\[\{a, b \ast a\} \cap L(g_A; s_A) \neq \emptyset\]

Therefore \(U(f_A; t_A)\) and \(L(g_A; s_A)\) are \(I\)-energetic subsets of \(X\).

Theorem 3.20. Let \(A = (f_A, g_A)\) be an intuitionistic fuzzy subalgebra of \(X\) which satisfies the following condition:

\[
(x \ast y) \ast z = 0 \Rightarrow \left( f_A(x) \geq \min \{f_A(y), f_A(z)\}, \quad g_A(x) \leq \max \{g_A(y), g_A(z)\} \right)
\]

for all \(x, y, z \in X\). Then \(X \setminus U(f_A; t_A)\) and \(X \setminus L(g_A; s_A)\) are empty or \(I\)-energetic subsets of \(X\) for all \((t_A, s_A) \in [0, 1] \times [0, 1]\).

Proof. Assume that \(A = (f_A, g_A)\) is an intuitionistic fuzzy subalgebra of \(X\) satisfying the condition (3.10). Note that \(f_A(0) \geq f_A(x)\) and \(g_A(0) \leq g_A(x)\) for all \(x \in X\). Since \((x \ast (x \ast y)) \ast y = 0\) for all \(x, y \in X\), it follows from (3.10) that \(f_A(x) \geq \min \{f_A(x \ast y), f_A(y)\}\) and \(g_A(x) \leq \max \{g_A(x \ast y), g_A(y)\}\). Hence \(A = (f_A, g_A)\) is an intuitionistic fuzzy ideal of \(X\). Using Theorem 3.18, we know that \(X \setminus U(f_A; t_A)\) and \(X \setminus L(g_A; s_A)\) are \(I\)-energetic subsets of \(X\) for all \((t_A, s_A) \in [0, 1] \times [0, 1]\) whenever they are nonempty.
Definition 3.21. ([2]) Let $Q$ be a non-empty subset of $X$. Then $Q$ is said to be right vanished if it satisfies:

\[(\forall a, b \in X) \ (a \ast b \in Q \Rightarrow a \in Q).\]

(3.11)

$Q$ is said to be right stable if $Q \ast X := \{a \ast x \mid a \in Q, \ x \in X\} \subseteq Q$.

Theorem 3.22. If $A = (f_A, g_A)$ is an intuitionistic fuzzy ideal of a $BCK$-algebra $X$, then

(1) The nonempty intuitionistic upper level set of $A = (f_A, g_A)$ is a right stable subset of $X$.

(2) The nonempty intuitionistic lower level set of $A = (f_A, g_A)$ is a right stable subset of $X$.

Proof. (1) Let $y \in U(f_A; t_A)$ Then $f_A(y) \geq t_A$. Since $(y \ast x) \ast y = 0$ for all $x, y \in X$, it follows from (2.10) that $f_A(y \ast x) \geq f_A(y) \geq t_A$, that is, $y \ast x \in U(f_A; t_A)$ for all $x \in X$. Thus $U(f_A; t_A)$ is a right stable subset of $X$.

(2) If $b \in L(g_A; s_A)$, then $g_A(b) \leq s_A$ and so $g_A(b \ast x) \leq g_A(b) \leq s_A$ for all $x \in X$ by (2.10). Hence $b \ast x \in L(g_A; s_A)$ for all $x \in X$, and therefore $L(g_A; s_A)$ is a right stable subset of $X$.

4. Conclusion

We have introduced the notion of intuitionistic permeable values in $BCK/BCI$-algebras, and have investigated several properties. We have discussed the relation between an intuitionistic permeable S-value and an intuitionistic permeable I-value, and have presented conditions for the intuitionistic lower (upper) level set to be S-energetic and I-energetic. We have considered conditions for a couple of numbers to be an intuitionistic permeable S-value, and have shown that if an intuitionistic fuzzy set $A = (f_A, g_A)$ in a $BCK$-algebra $X$ is an intuitionistic fuzzy ideal of $X$, then the nonempty intuitionistic upper level set of $A = (f_A, g_A)$ is a right stable subset of $X$ and the nonempty intuitionistic lower level set of $A = (f_A, g_A)$ is a right vanished subset of $X$. Based on this paper, we will apply the notion of this article to other algebraic structures, for example, semigroup theory, (pseudo) $MV$-algebras, (pseudo) $BL$-algebra, etc.

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References


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