COPROXIMINALITY RESULTS IN KÖTHE BOCHNER SPACES

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Abstract. As a counterpart to best approximation in normed linear spaces, best co-approximation was introduced by Franchetti and Furi, [3], in 1972. If X is a Banach space, $E(X)$ the Köthe Bochner function space and G is a closed subspace of X, the problem is: under what conditions the subspace $E(G)$ is proximinal (coproximinal) in $E(X)$? In this paper we prove that if G is a coproximinal separable subspace of X, then $E(G)$ is coproximinal in $E(X)$. Some other results are presented.

Keywords: best coapproximation, coproximinal set, Köthe Bochner function spaces.

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1. Introduction

Let (T, μ) be a finite complete measure space and E denote the space of all equivalence classes of μ-measurable real-valued functions on T. For f and $g \in E$, $f \leq g$ means that $f(t) \leq g(t)$, μ-almost every where $t \in T$. For simplicity, we write: a.e $t \in T$. When E is equipped with a norm $\|\cdot\|_E$ and becomes a Banach space, it is then called (see [8] or [9]) a Köthe function space if the following conditions hold:

1. For each measurable subset A of T, with $\mu(A)$ is finite then the characteristic function $\chi_A \in E$.

2. For any two functions f and g such that $|f| \leq |g|$ and $g \in E$ imply $f \in E$ with $\|f\|_E \leq \|g\|_E$.

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A Köthe space E is said to be strictly monotone if $|f| > |g|$ then $\|f\|_E > \|g\|_E$, in other words: if $f \geq g \geq 0$ and $\|f\|_E = \|g\|_E$ imply $f = g$.

For a real Banach space $(X, \|\cdot\|_X)$ and a Köthe space E on a measure space (T, μ), let $E(X)$ be the space of all (equivalence classes of) strongly measurable functions $f : T \to X$ such that $\|f(\cdot)\|_X \in E$, equipped with the norm

$$|||f||| = \|\|f(\cdot)\|_X\|_E .$$

It is known, see [8], that $(E(X), |||\cdot|||)$ is a Banach space called the Köthe Bochner function space. The most important classes of Köthe Bochner function spaces are the Lebesgue-Bochner spaces $L^p(X)$, $(1 \leq p < \infty)$.

The concept of best coapproximation was initially introduced by Franchetti and Furi, [3], in order to study some characteristic properties of real Hilbert spaces. It was then considered by Papini and Singer, [13], and recently by many, see for example [12], [14] and [4]. This theory is largely concerned with the questions of existence, uniqueness and characterization of elements of best coapproximation.

For $x \in X$ and G a non empty subset of a normed space X, then $y_0 \in G$ is called best coapproximation point of G to x, if $\|y_0 - y\| \leq \|x - y\|, \forall y \in G$. The set G is called coproximinal in X, if for each $x \in X$, there exists at least one point of best coapproximation to x in G. The set-valued map $R_G : X \to 2^G$ defined by

$$R_G(x) = \{y_0 \in G : \|y_0 - y\| \leq \|x - y\|, \forall y \in G\},$$

is called the cometric projection. Let $R(G) = \{x \in X : R_G(x) \neq \phi\}$. Clearly, $G \subset R(G)$ and G is coproximinal in X iff $R(G) = X$. G is called densely coproximinal in X if the set $R(G)$ is dense in X. It is known [4] that if G is coproximinal subspace of X then it is closed and satisfies that $X = G \oplus \breve{G}$, where $\breve{G} = \{x \in X : \|y\| \leq \|y - x\|, \forall y \in G\}$. The set $\breve{G}$ is called the cometric complement of G.

The proximinality problem in vector valued function spaces was widely studied by many authors, see [1], [5], [7]. The problem of best coapproximation in the space of continuous functions was considered in [14, 15], while some few results on best coapproximation in $L^p(I, X), 1 \leq p < \infty$, can be found in [4], [10]. The problem of best coapproximation in the Köthe Bochner function space never been considered.

The object of this paper is to extend some of the results, on best coapproximation, in [4, 10] to Köthe Bochner function spaces and to get new ones, in order to give an answer to the following question: If G is coproximinal in X, under what conditions $E(G)$ is coproximinal in $E(X)$? Our main result is to prove that, Theorem 8, if G is a coproximinal separable subspace of X, then $E(G)$ is coproximinal in $E(X)$.

2. Coproximinality of $E(G)$ in $E(X)$

Throughout this section, (I, μ) is a finite measure space, X is a real Banach space and $E(X)$ is a Köthe Bochner function space.

Theorem 1 *Let G be a closed subset of a Banach space X and let E be a Köthe space. For f in $E(X)$ and g in $E(G)$ such that $g(t)$ are best coapproximation points in G to $f(t)$ in X, a.e. $t \in I$, then g is a best coapproximation to f.*

Proof. For $t \in I$, let $g(t)$ be a best coapproximation element in G to $f(t) \in X$. Then

$$\|g(t) - y\| \leq \|f(t) - y\|, \ \forall y \in G, \text{ a.e. } t \in I.$$

Hence, in particular, for any function h in $E(G)$, we have

$$\|g(t) - h(t)\| \leq \|f(t) - h(t)\|, \text{ a.e. } t \in I.$$

This implies,

$$\left\|\|g(\cdot) - h(\cdot)\|_X\right\|_E \leq \left\|\|f(\cdot) - h(\cdot)\|_X\right\|_E, \forall h \in E(G).$$

So, we get

$$|||g - h||| \leq |||f - h|||, \forall h \in E(G).$$

Hence, g is a best coapproximation to f. ∎

Let $A_1, ..., A_n$ be a sequence of mutually disjoint measurable subsets of I, such that $\cup A_i = I$ and let $a_1, ..., a_n$ be a finite sequence of real or complex numbers. A simple function is a function $f : I \to \mathbb{C}$ of the form

$$f(t) = \sum_{k=1}^{n} a_k \chi_{A_k}(t), \quad t \in I,$$

where for each k, χ_{A_k} is the characteristic function on A_k. A simple function in $E(X)$ is a function $f : I \to X$ of the form

$$f(t) = \sum_{k=1}^{n} x_k \chi_{A_k}(t), \quad x_k \in X \text{ and } t \in I.$$

It is known that the collection of all simple functions on a given measurable space forms a commutative algebra over $\mathbb{C}$ that is dense in the function space $E(X)$ on that space.

Lemma 2 *Let G be coproximinal set in X, then every simple function in $E(X)$ has a best coapproximation (a simple function) in $E(G)$.*

Proof. Let $f = \sum_{i=1}^{n} x_i \chi_{A_i}$ be a simple function in $E(X)$. For each $g \in E(G)$ and a given $\epsilon > 0$, there exists a simple function $\varphi_g = \sum_{i=1}^{n} y_i \chi_{A_i}$ in $E(G)$, with $y_i \in G$, such that $|||g - \varphi_g||| < \epsilon$. Since G is coproximinal, let z_i, for each i, be the best coapproximation in G to $x_i \in X$. Hence,

$$\|x_i - y_i\|_X \geq \|z_i - y_i\|_X, \quad 1 \leq i \leq n.$$

Now,

$$
\begin{aligned}
|||f-g||| &\geq |||f-\varphi_g||| - |||\varphi_g - g||| \\
&> \left\| \sum_{i=1}^{n} \|x_i - y_i\|_X \, \chi_{A_i}(t) \right\|_E - \epsilon \\
&\geq \left\| \sum_{i=1}^{n} \|z_i - y_i\|_X \, \chi_{A_i}(t) \right\|_E - \epsilon \\
&\geq |||g_0 - \varphi_g||| - \epsilon
\end{aligned} \tag{1}
$$

by taking $g_0 = \sum_{i=1}^{n} z_i \chi_{A_i}$. This implies from (1) above

$$|||f-g||| > |||g_0 - g||| - 2\epsilon$$

and since ϵ arbitrary, we get

$$|||f-g||| \geq |||g_0 - g|||$$

for all $g \in E(G)$. ■

Theorem 3 *For any densely coproximinal set $G \subset X$, $E(G)$ is densely coproximinal in $E(X)$.*

Proof. Let $f = \sum_{i=1}^{n} x_i \chi_{A_i}$ be a simple function in $E(X)$. Since $R(G)$ is dense, given $\epsilon > 0$, we get a simple function $\varphi = \sum_{i=1}^{n} z_i \chi_{A_i}$, with $z_i \in R(G)$ and $|||f - \varphi||| < \epsilon$. For each i, let $y_i \in G$ be a best coapproximation point to z_i, then from the above Lemma, $\varphi_0 = \sum_{i=1}^{n} y_i \chi_{A_i}$ is a best coapproximation to φ in $E(G)$. ■

In what follows, we consider coproximinality when G is separable in the Banach space X. Let G be a closed subset in the Banach space X. For each $f \in E(X)$, define the map $\pi_f : I \to 2^G$ as

$$\pi_f(t) = \{g \in G : f(t) - g \in \breve{G}\}, \; t \in I,$$

which is a set-valued map, taking each element t of a measurable space I into a subset of G, precisely the set of best coapproximation points to $f(t)$. One can easily see that π_f can be written as the composition $R_G \circ f$. Certain conditions can be imposed on π_f so that it has a measurable selection. Recall that ([13]) a set-valued map on a measure space I, $\pi : I \to 2^X$ is said to be weakly measurable if the set $\{t \in I : \pi(t) \bigcap O \neq \phi\}$ is measurable, for every open subset O in X. By definition, such a mapping has a measurable selection if there exists a measurable function $h : I \to X$ such that $h(t) \in \pi(t)$ for each $t \in I$. The following lemma can be found in [13] (see also [10] or [4]).

Lemma 4 *Let $\pi : I \to 2^Y$ be a weakly measurable set-valued map. If Y is a separable Banach space then π has a measurable selection.*

Theorem 5 *Let G be a separable subspace of X such that π_f as defined above is weakly measurable. Then $E(G)$ is coproximinal in $E(X)$ if G is coproximinal in X.*

Proof. Suppose that G is coproximinal in X, and let f be in $E(X)$. The assumption that the map $\pi_f : I \to 2^G$, defined above, is weakly measurable implies that it has a measurable selection, say $g : I \to G$ such that $g(t) \in \pi_f(t)$, for all $t \in I$ (by the previous lemma), since its values are in a separable space G. Hence the result follows from Theorem 1 above if we show that $g \in E(G)$.

Now, $g : I \to G$ such that $g(t) \in \pi_f(t)$, for all $t \in I$, so $g(t) \in G$ is a best coapproximation point of $f(t)$ in X. Thus $f(t) - g(t) \in \breve{G}$, for all $t \in I$, which implies that

$$\|z\| \leq \|z - (f(t) - g(t))\|, \forall z \in G$$

taking $z = -g(t)$, we get

$$\|g(t)\| \leq \|-g(t) - (f(t) - g(t))\| = \|f(t)\|, \ a.e\ t \in I.$$

Hence, $g \in E(G)$ and $g(t)$ is a best coapproximation point to $f(t)$. ∎

3. Main result

Our main result of this paper concerns coproximanility of $E(G)$ in $E(X)$ for a separable closed subspace G in X: Theorem 7 below. For the proof, we will follow a technique used by Mendoza, [11], for proximinality. The following lemma is also needed in the proof (Lemma 3 of [16]).

Lemma 6 *Assume $\mu(I) < \infty$. Suppose (M, d) is a metric space and A is a subset of I such that $\mu^*(A) = \mu(I)$, where μ^* denotes the outer measure associated to μ. If g is a mapping from I to M with separable range, then for any $\epsilon > 0$, there exists a countable partition $\{E_n\}$ of I in measurable sets and $A_n \subset A \cap E_n$ such that $\mu^*(A_n) = \mu(E_n)$ and $diam(g(A_n)) < \epsilon$ for all n, where*

$$diam(g(A_n)) = \sup\{\|g(s) - g(t)\| : s, t \in A_n\}.$$

Theorem 7 *Let (I, μ) be a finite measure space, G be a separable coproximinal subspace of X and $f : I \to X$ be measurable function. Then there is a measurable function $g : I \to G$ such that $g(t)$ is a point of coapproximation to $f(t)$ in G, a.e. $t \in I$.*

Proof. Let $f : I \to X$ be a measurable function. So we may assume that $f(I)$ is a separable set in X. Using the fact that μ is finite we can find countable partitions $\{I_n\}_{n=1}^{\infty}$ of I in measurable sets such that $\text{diam}(f(I_n)) < \frac{1}{2}$ and $\mu(I_n) < \infty$, for all n.

Now, since G is coproximinal in X then for each $t \in I$, let $y_t \in G$ be a point of best coapproximation to $f(t)$. Define g_0 from I into G such that $g_0(t) = y_t$, as above. Applying Lemma 6 above in this section, to the mapping g_0 in each I_n, taking $\epsilon = \frac{1}{2}$ and $I = A = I_n$. We get a countable partition in each I_n and therefore a countable partition in the whole of I. Thus we get a countable partition $\{E_n\}_{n=1}^{\infty}$ of I in measurable sets and a sequence of subsets $\{A_n\}_{n=1}^{\infty}$ of I such that

$$\begin{aligned} A_n &\subseteq E_n,\ \mu^*(A_n) = \mu(E_n) < +\infty, \\ \operatorname{diam}(g_0(A_n)) &< \frac{1}{2},\ \operatorname{diam}(f(E_n)) < \frac{1}{2}. \end{aligned}$$

Let us apply again the same argument in each E_n with $\epsilon = \frac{1}{2^2}$, $I = E_n$ and $A = A_n$. For each n we get a countable partition $\{E_{n_k} : 1 \leq k < \infty\}$ of E_n in measurable sets and a sequence $\{A_{n_k} : 1 \leq k < \infty\}$ of subsets of I such that

$$\begin{aligned} A_{n_k} &\subseteq E_{n_k} \cap A_n,\ \mu^*(A_{n_k}) = \mu(E_{n_k}), \\ \operatorname{diam}(g_0(A_{n_k})) &< \frac{1}{2^2} \text{ and } \operatorname{diam}(f(E_{n_k})) < \frac{1}{2^2}, \end{aligned}$$

for all n and k. Now for each natural number k, let $\triangle_k$ be the set of k-tuples of natural numbers and let $\triangle = \bigcup_{k=1}^{\infty} \triangle_k$. On this $\triangle$ consider the partial order defined by $(m_1, m_2, ..., m_i) \leq (n_1, n_2, ..., n_j)$ if and only if $i \leq j$ and $m_k = n_k$ for $k = 1, 2, ..., i$. Then by induction for each natural number k, we can take a partition $\{E_\alpha : \alpha \in \triangle_k\}$ of measurable subsets of I and a collection $\{A_\alpha\}_{\alpha \in \triangle_k}$ of subsets of I such that

(1) $A_\alpha \subseteq E_\alpha$ and $\mu^*(A_\alpha) = \mu(E_\alpha)$ for each α.

(2) $A_\alpha \subseteq A_\beta$ and $E_\alpha \subseteq E_\beta$ if $\beta \leq \alpha$.

(3) $\operatorname{diam}(f(E_\alpha)) < \frac{1}{2^k}$ and $\operatorname{diam}(g_0(A_\alpha)) < \frac{1}{2^k}$ if $\alpha \in \triangle_k$.

We may assume that $A_\alpha \neq \emptyset$ for all α (forget the $\alpha' s$ for which $A_\alpha = \emptyset$). For each $\alpha \in \triangle$, take $t_\alpha \in A_\alpha$ and define g_k from I into G by

$$g_k(\cdot) = \sum_{\alpha \in \triangle_k} g_0(t_\alpha) \chi_{E_\alpha}(\cdot).$$

Then, for each $t \in I$ and $n \leq k$, we have

$$\|g_n(t) - g_k(t)\| = \left\| \sum_{\alpha \in \triangle_n} g_0(t_\alpha) \chi_{E_\alpha}(t) - \sum_{\beta \in \triangle_k} g_0(t_\beta) \chi_{E_\beta}(t) \right\|$$

By (1) and (2) above and since $n \leq k$, we have:

$$\begin{aligned} \|g_n(t) - g_k(t)\| &\leq \left\| \sum_{\beta \in \triangle_k} \left(g_0(t_\alpha) - g_0(t_\beta)\right) \chi_{E_\beta}(t) \right\| \\ &\leq \sum_{\beta \in \triangle_k} \left\|g_0(t_\alpha) - g_0(t_\beta)\right\| \mu(E_\beta) \\ &\leq \frac{1}{2^n}. \end{aligned}$$

Therefore, $(g_k(t))$ is a Cauchy sequence in X for each $t \in I$, hence convergent. Let $g : I \to G$ be the pointwise limit of (g_k). Since g_k is measurable for each k, then g is measurable. Now, let $t \in I$ and n a natural number.

Suppose $t \in E_\alpha$. Then, for any $y \in G$, we have:

$$\begin{aligned} \|g_n(t) - y\| &= \|g_0(t_\alpha) - y\| \\ &\leq \|f(t_\alpha) - y\| \\ &\leq \|f(t) - y\| + \|f(t) - f(t_\alpha)\| \\ &\leq \|f(t) - y\| + \frac{1}{2^n}. \end{aligned}$$

Letting $n \to \infty$ we get

$$\|g(t) - y\| \leq \|f(t) - y\|$$

for all $y \in G$ and so $g(t)$ is a best coapproximation point of $f(t)$ in G. ∎

Theorem 8 *Let G be a separable subspace of X and E a Köthe space. Then $E(G)$ is coproximinal in $E(X)$ if G is coproximinal in X.*

Proof. Suppose that G is a separable coproximinal subspace in X. Let f be a function in $E(X)$, then Theorem 7 guarantees that there exists a measurable function g defined on I with values in G such that $g(t)$ is a point of best coapproximation to $f(t)$ in G for almost all t. It follows from Theorem 1 that g is a point of best coapproximation to f in $E(G)$. ∎

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